PAIRINGS OF BINARY REFLEXIVE RELATIONAL STRUCTURES

By

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Thesis submitted in fulfillment of the requirements for
the degree of
MASTER OF SCIENCE
in the
Department of Mathematics and Applied Mathematics,
University of the Western Cape

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Keywords and phrases

Category

Functor

Homotopy

Weak homotopy equivalence

Partially ordered set

Barycentric subdivision

Binary reflexive relational structure
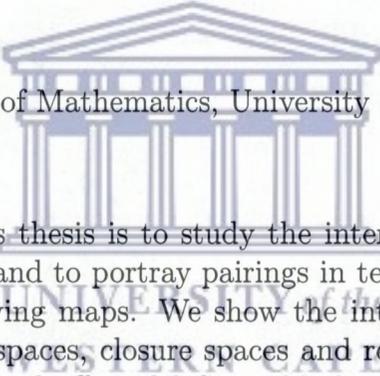
Pairing

Hopf construction

Finite model

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Abstract

PAIRINGS OF BINARY REFLEXIVE RELATIONAL STRUCTURES

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The main purpose of this thesis is to study the interplay between relational structures and topology, and to portray pairings in terms of some finite poset models and order preserving maps. We show the interrelations between the categories of topological spaces, closure spaces and relational structures. We study the 4-point non-Hausdorff model S_4 weakly homotopy equivalent to the circle $\mathbb{S}^1$. We study pairings of some objects in the category of relational structures, similar to the multiplication of Hopf spaces in topology. The multiplication $S_4 \times S_4 \to S_4$ fails to be order preserving for posets. Nevertheless, applying a single barycentric subdivision on S_4 to get S_8, an 8-point model of the circle enables us to define an order preserving poset map $S_8 \times S_8 \to S_4$. Restricted to the axes, this map yields weak homotopy equivalences $S_8 \to S_4$. Hence it is a pairing. Further, using the non-Hausdorff join $S_8 \circledast S_8$, we obtain a version of the Hopf map $S_8 \circledast S_8 \to \mathbb{S}S_4$. This model of the Hopf map is in fact a map of non-Hausdorff double mapping cylinders.

July 2007

Declaration

I declare that *Pairings of Binary Reflexive Relational Structures* is my work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.
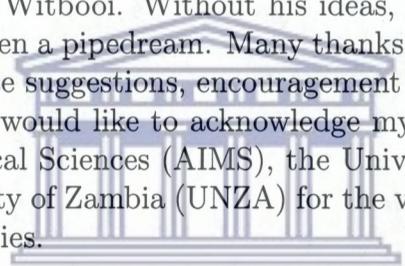
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Signed.....................................

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Acknowledgments

I would like to thank my father and my first mathematics teacher: my mother for giving me life in the first place and for being very supportive during the times when I needed it most. They always believed in me and readily gave unconditional support for me to pursue my interests. I am indebted to my supervisor Prof Peter J. Witbooi. Without his ideas, guidance and patience, this thesis would have been a pipedream. Many thanks to my co-supervisor Dr Eric C. Mwambene whose suggestions, encouragement and timely advice kept me on course. Finally I would like to acknowledge my sponsors: the African Institute for Mathematical Sciences (AIMS), the University of Western Cape (UWC) and the University of Zambia (UNZA) for the various forms of support rendered during my studies.
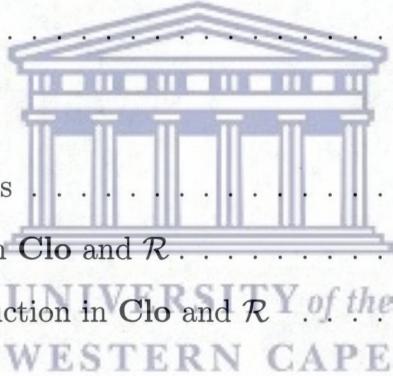
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Chapter 1

Introduction

1.1 Introduction and historical background
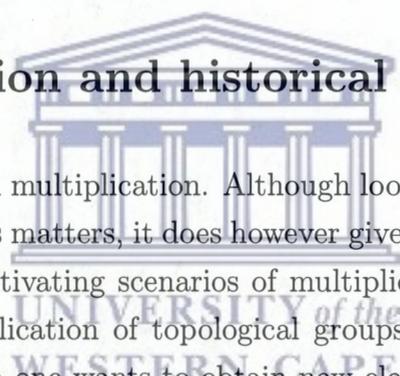
A pairing is a generalized multiplication. Although looking at a pairing in this perspective oversimplifies matters, it does however give the spirit and intricacy of the matter. Some motivating scenarios of multiplication include multiplication of groups, multiplication of topological groups and group action. In classical homotopy, when one wants to obtain new elements of the homotopy groups of spheres, one of the possible things do is to apply the Hopf's construction to a pairing. P.S.Alexandroff in [3] supplemented by M.McCord [33] establishes a connection between finite topological spaces and finite posets. In particular, [33] shows the interrelation between the homotopy theory in the context of T_0 spaces and compact polyhedra: that one can work directly with the T_0 space instead of the associated polyhedra. In this thesis, we are particularly interested in pairings of some objects in the category $\mathcal{R}$ of relational structures. By a binary reflexive relational structure (X, θ), we mean an arbitrary set X with some reflexive relation θ defined on it. In general, the relation θ might not be symmetric or transitive. Binary reflexive relational structures have been considered for a while, for instance in by A. Pultr and V. Trnkov in [40]. In a recent study [29], B. Larose and C. Tardif define homotopy groups

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for relational structures to study complexity class problems. With an intention to find poset models of some representative maps in homotopy theory, Hardie *et al.* in [26], [24] and [23] for instance, replace Hausdorff topological spaces by some finite posets.

1.2   Overview of the thesis

In this section, we describe an overall perspective of the thesis. We present the main contributions of each chapter of the thesis.

In Chapter 2 we introduce some category theoretic preliminaries necessary for the presentation of what follows in the latter chapters of the thesis.

Chapter 3 is a brief look on basic homotopy. Further, we recall the functorial association from a pointed topological space to some group structure. This is essential because if you need to talk about symmetry, then group theoretic concepts come in very handy. We give a result (Theorem 3.2.5) similar to that by B. Gray [21] on weak homotopy equivalence in terms of open covers. The result in [21] has a defect similar to that by J.P. May in [30] and the correction can be found in the paper [48] by P.J. Witbooi. The homotopy equivalences discussed here, thus, give us a tool to construct the models. In Section 4.5 we discuss a well known four point model which we denote by S_4. We explicitly describe S_4. By the implication of Theorem 3.2.5, we show that this model is weakly homotopic to the Hausdorff space S^1. This may be considered as one of the possible models weakly equivalent to S^1. In Section 2.11.3, we discuss a well known isomorphism between the category of T_0 spaces and the subcategory **LFPos** (locally finite posets) of **Pos**. The weak homotopy equivalence enables us to study the space by way of finite models that admits a T_0 separation.

In Chapter 4, we consider the category $\mathcal{R}$ whose class of objects consists of binary reflexive relational structures and maps are $\mathcal{R}$-morphisms. We give some examples of binary reflexive relational structures. Further, when one takes quotients on a given relational structure, new ones may be obtained. We

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show that by taking the barycentric subdivision of a relational structure, one obtains a functorial association $B^* : \mathcal{R} \to \mathbf{Pos}$ from the category of relational structures to $\mathbf{Pos}$ the category of posets. We show that for relational structures X and Y, the construction $\sigma_k(X \times Y)$ is isomorphic to $\sigma_k(X) \times \sigma_k(Y)$ (here, $\sigma_k(X)$ for a relational structure X and $k \in \mathbb{N}$ is understood to mean the $\mathcal{R}$ version of the usual kth homotopy group).

In Chapter 5, we discuss closure spaces. We prove a result similar to Lemma 1 of [44] but in our case, restricted to binary reflexive relational structures. We study the functorial association between $\mathbf{Clo}$, the category of closure spaces and continuous maps and $\mathbf{Top}$, the category of topological spaces and continuous functions. There is a Galois correspondence between the category of binary relational structures $\mathcal{R}$ and $\mathbf{Clo}$. In Chapter 6, we describe a non-Hausdorff version of the cylinder object. Here, the unit interval is replaced by a suitable three point poset.

In Chapter 7, some basic definitions for the construction of a pairing in a category are considered. Further, we give a map $\times : S_4 \times S_4 \to S_4$ following [24]. Though it fails to be order preserving, by applying a single barycentric subdivision on each of S_4 on $S_4 \times S_4$, we obtain an order preserving map $\mu : S_8 \times S_8 \to S_4$. We describe a model (Hopf Construction) $\Gamma(\mu) : S_8 \times S_8 \to \mathbb{S}S_4$ of the Hopf map $S^3 \to S^2$. We show that the non-Hausdorff join $S_8 \circledast S_8$ and the non-Hausdorff suspension $\mathbb{S}S_8$ are special cases of the non-Hausdorff double mapping cylinder. Furthermore, we describe a model of a generalised Whitehead product (GWP), and how it relates to the Hopf construction.

Chapter 2

Basic categorical constructions

The study of mathematical structures entails the deployment of tools which can show how the different structures relate. Category theory serves this purpose well. In this chapter, we formulate some basics of category theory that will be important in what follows in the latter chapters of this thesis. The main references are J. Adamek [1], J. Adamek *et al.* [2], N. Chishwashwa [11] and H. Herrlich *et al.* [27].

Definition 2.0.1. (cf. [1], [27]) A *category* $\mathcal{C}$ consists of

(i) a collection $ob(\mathcal{C})$ whose members are the *objects* of $\mathcal{C}$;

(ii) for each $A, B \in ob(\mathcal{C})$, a collection $\mathcal{C}(A, B)$ whose members are maps or morphisms or simply arrows from A to B,

(iii) for each $A, B, C \in ob(\mathcal{C})$, and some morphisms $\mathcal{C}(A, B)$, $\mathcal{C}(B, C)$ a function
$\circ : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)$, called the composition,

(iv) for each object $A \in ob(\mathcal{C})$, an element $1_A \in \mathcal{C}(A, A)$ called the *identity* on A satisfying *unit Laws*: $f \circ 1_A = f = 1_B \circ f$ for all $A, B \in ob(\mathcal{C})$ and $f \in \mathcal{C}(A, B)$.

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(v) *Associativity* : $(h \circ g) \circ f = h \circ (g \circ f)$ for all $A, B, C, D \in ob(\mathcal{C})$ $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$ and $h \in \mathcal{C}(C, D)$.

From the definition, one sees that the notion of a category is that of a class of sets together with a class of functions between them. In the case where we have a class of structured sets and structure preserving morphisms between them, then the category is said to be a *construct*. We give examples of categories that will serve as our reference points in the thesis.

(1) **Set**: Consider the class of sets and set functions between sets. This forms the category **Set** whose objects are sets and morphisms are set functions. We show that **Set** satisfies the conditions of Definition 2.0.1. Let A and B be objects of $\mathcal{C}$ and $f : A \to B$ a function defined on A with values in B. Let $g : B \to C$ be another function, then their composite will be $(g \circ f)(a) = g(f(a))$ for some $a \in A$. The operation $\circ$ is associative because when we consider another function $h : C \to D$, one can form the composites $h \circ (g \circ f)$ and $(h \circ g) \circ f$. But $((h \circ g) \circ f)(a) = h(g(f(a))) = (h \circ (g \circ f))(a)$, i.e, $(h \circ g) \circ f = h \circ (g \circ f)$. For each set A we have the identity map $1_A : A \to A$ defined as $1_a : a \mapsto a$ for all $a \in A$. These identity maps are units under the operation $\circ$ and have $f \circ 1_A = 1_B \circ f$.

(2) **Pos**: Consider the class of partially ordered sets, and order preserving functions between these sets. This forms the category **Pos** whose objects are posets and a morphism $f : A \to B$ is such that for $a', a \in A$,

$$a \leq_A a' \Rightarrow f(a) \leq_B f(a').$$

The identity $1_A : A \to A$ is trivially order preserving since $a \leq_A a$ implies $1_A(a) \leq_A 1_A(a)$. It remains to check composition of these order preserving (monotone) functions. Let $g : B \to C$ where C is another poset. We have that $f \circ g : A \to C$ is monotone since

$$a \leq_A a' \Rightarrow f(a) \leq_B f(a') \Rightarrow g(f(a)) \leq_C g(f(a')) \Rightarrow (g \circ f)(a) \leq_C (g \circ f)(a')$$

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Composition is associative hence we have a category of posets and maps as order preserving functions.

(3) **Top** is the category whose objects are topological spaces and the maps are continuous functions. In this thesis, we are particularly interested in the objects which are pairs (X, x_0) where x_0 is a point in the topological space X. In some cases, the point x_0 will be denoted by $*$. A morphism from (X, x_0) to (Y, y_0) is defined as the continuous function $f : X \to Y$ where $f(x_0) = y_0$. The class of such pairs of topological spaces with a privileged point and the respective morphisms between them forms the category **Top*** (note that **Top*** denotes the category of pointed topological spaces). Furthermore, there is a category denoted as **Top**^{2} whose objects consists of pairs (X, A) where X is a topological space and $A \subset X$. A morphism from the object (X, A) to another object (Y, B) being defined by the continuous function $f : X \to Y$ where $f(A) \subseteq B$.

(4) **Grp** is the category whose objects are groups and morphisms are group homomorphisms. For objects (groups) G and H of **Grp** the set $\text{Hom}(G, H)$ consists of group homomorphisms from G to H. In the usual manner, if $f \in \text{Hom}(G, H)$, and $g, g' \in G$ then $f(gg') = f(g)f(g')$.

2.1 Product category

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, then the product category of $\mathcal{C}$ and $\mathcal{D}$ denoted as $\mathcal{C} \times \mathcal{D}$ consists of objects in the form of pairs (C, D) where C and D are objects of $\mathcal{C}$ and $\mathcal{D}$ respectively. Morphisms are

$$(f, g) : (C, D) \to (C', D')$$

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where $f : C \to C' \in \mathcal{C}$ and $g : D \to D' \in \mathcal{D}$. Composition of morphisms and units are defined componentwise, as follows:

$$(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$$

$$1_{(C,D)} = (1_C, 1_D).$$

2.2 Subcategory

A category $\mathcal{B}$ is said to be a subcategory of $\mathcal{C}$ if the class of objects and morphisms of $\mathcal{B}$ are contained in those of $\mathcal{C}$ and that $\mathcal{B}$ is closed under the category operations of domain, codomain, composition and identity. Formally, we have the following.

Definition 2.2.1. (cf. [1], [27]) The category $\mathcal{B}$ is a *subcategory* of $\mathcal{C}$ provided the following conditions hold.

(1) $ob(\mathcal{B}) \subset ob(\mathcal{C})$,

(2) $Mor(\mathcal{B}) \subset Mor(\mathcal{C})$,

(3) the domain, codomain and composition of morphisms of $\mathcal{B}$ are restrictions of the corresponding functions of $\mathcal{C}$,

(4) every $\mathcal{B}$-identity is a $\mathcal{C}$-identity.

In the instance where the morphisms of a subcategory $\mathcal{B}$ of $\mathcal{C}$ are exactly the same as with respect to the "larger" category $\mathcal{C}$ then the subcategory $\mathcal{B}$ is said to be full.

Definition 2.2.2. (cf. [1], [27]) A subcategory $\mathcal{B}$ of $\mathcal{C}$ is said to be a *full subcategory* of $\mathcal{B}$ if for all objects $A, B \in o\mathcal{B})$ the morphisms $mor_\mathcal{B}(A, B) = mor_\mathcal{C}(A, B)$.

2.3 Locally finite objects in Pos

In this section, we discuss a subcategory of **Pos** which is of particular interest in our study. We first recall the following definition.

Definition 2.3.1. A *poset* is a binary reflexive structure $P = (X, \leq)$ where $\leq$ is an order relation defined on X which is reflexive, antisymmetric and transitive.

Consider all those objects X of **Pos** in which for every $x \in X$, we have that the subsets $\{y \in X | y \leq x\}$ and $\{y \in X | y \geq x\}$ of X are finite. These objects together with the morphisms between them forms a subcategory of **Pos**. We shall denote this subcategory by **LFposet**. It is easy to observe that **LFPoset** is a full subcategory of **Pos** since the morphisms between any pair of objects in **LFPoset** are exactly the same morphisms as with respect to **Pos**.

2.4 Final and initial objects

Definition 2.4.1. An object A in a category $\mathcal{C}$ is said to be an *initial* object if for any other object X of $\mathcal{C}$ there exist a unique morphism $i : A \to X$.

Dually, a final (terminal) object in a category is defined as follows:

Definition 2.4.2. An object A' in a category $\mathcal{C}$ is said to be a *final* object in $\mathcal{C}$ if for any other X in $\mathcal{C}$ there exist a unique arrow $j : A' \leftarrow X$ in $\mathcal{C}$.

In some cases, an object in a category is both final and initial. In such a case, then we say that the object is null.

Definition 2.4.3. A category with a null object is said to be a pointed category.

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2.5 Product and coproduct

Products of objects in a category generalize the Cartesian products of objects in a structured set. The underlying principle being that the product is not just an object, but is should consist of an object together with a system of projection morphisms. The uniqueness of a product in a category just goes up to isomorphism.

Definition 2.5.1. The *product* of two objects A and B in a category $\mathcal{C}$ if it exists is a triad

$$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$$

satisfying the following conditions. For any other object X and a pair of morphisms $f_1 : X \to A$ and $f_2 : X \to B$ there exists a unique morphism $\langle f_1, f_2 \rangle : X \to A \times B$ such that $\pi_1 \circ \langle f_1, f_2 \rangle = f_1$ and $\pi_2 \circ \langle f_1, f_2 \rangle = f_2$. The latter two identities is equivalent to saying that the diagram

commutes.

Definition 2.5.2. The *coproduct* of two objects A and B in a category $\mathcal{C}$ if it exists is a triad

$$A \xrightarrow{j_1} A \vee B \xleftarrow{j_2} B$$

satisfying the following conditions. For any other object X and a pair of morphisms $g_1 : X \leftarrow A$ and $g_2 : X \leftarrow B$, there exists a unique morphism $[g_1, g_2] : A \vee B : X \to X$ such that $j_1 \circ [g_1, g_2] = g_1$ and $j_2 \circ [g_1, g_2] = g_2$. This is equivalent to saying that the diagram

commutes.

2.6 Pullback and pushout

Definition 2.6.1. The *pullback* of morphisms f and g in a category $\mathcal{C}$, with

$$
\begin{array}{ccc}
 & & A \\
 & & \downarrow f \\
B & \xrightarrow{g} & C
\end{array} ,
$$

consists of morphisms

$$
\begin{array}{ccc}
P & \xrightarrow{p_2} & A \\
\downarrow p_1 & & \\
B & &
\end{array}
$$
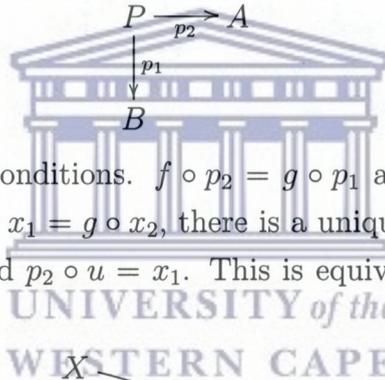
satisfying the following conditions. $f \circ p_2 = g \circ p_1$ and that given any triad $A \xleftarrow{x_1} X \xrightarrow{x_2} B$ with $f \circ x_1 = g \circ x_2$, there is a unique morphism $u : X \to P$ such that $p_1 \circ u = x_2$ and $p_2 \circ u = x_1$. This is equivalent to saying that the diagram

$$
\begin{array}{ccccc}
X & & & & \\
 & \searrow u & \searrow x_1 & & \\
x_2 & & P & \xrightarrow{p_2} & A \\
 & \searrow & \downarrow p_1 & & \downarrow f \\
 & & B & \xrightarrow{g} & C
\end{array}
$$
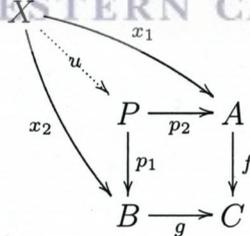
commutes.

Definition 2.6.2. The *pushout* of morphisms f and g in a category $\mathcal{C}$ where

$$
\begin{array}{ccc}
C & \xrightarrow{f} & A \\
\downarrow g & & \\
B & &
\end{array}
$$

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consists of

$$
\begin{array}{c}
A \\
\downarrow {\scriptstyle p_1} \\
B \xrightarrow[p_2]{} P
\end{array}
$$

satisfying the following conditions. $p_1 \circ f = p_2 \circ g$ and that given any triad $A \xrightarrow{x_1} X \xleftarrow{x_2} B$ with $x_1 \circ f = x_2 \circ g$, there is a unique arrow $u : P \to X$ such that $u \circ p_1 = x_1$ and $u \circ p_2 = x_2$. This is equivalent to saying that the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{f} & A \\
{\scriptstyle g}\downarrow & & \downarrow {\scriptstyle p_1} \\
B & \xrightarrow{p_2} & P
\end{array}
$$

commutes.

2.7 Functors

Having discussed categories, their objects and morphisms between them, it is natural to consider maps from one category to another.

Definition 2.7.1. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A *functor* $F : \mathcal{A} \to \mathcal{B}$ consists of

(i) a function

$$
\begin{array}{ccc}
ob(\mathcal{A}) & \xrightarrow{F} & ob(\mathcal{B}) \\
A & \xrightarrow{F} & FA
\end{array}
$$

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(ii) for each $A, A' \in ob(\mathcal{A})$, a function

$$\mathcal{A}(A, A') \xrightarrow{F} \mathcal{B}(FA, FA')$$
$$f \xrightarrow{F} Ff,$$

such that:

(iii) $F(f' \circ f) = Ff' \circ Ff$ for all $A \xrightarrow{f} A' \xrightarrow{f'} A'$ in $\mathcal{A}$,

(iv) $F1_A = 1_{FA}$ for all $A \in ob(\mathcal{A})$.

With each object A of $\mathcal{A}$ is associated an object FA of $\mathcal{B}$ and every string $A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} A_n$ (of objects and morphism) in $\mathcal{A}$ gives precisely one morphism $FA_0 \to FA_n$ in $\mathcal{B}$. A functor defined in this way is sometimes called a covariant functor. On the other hand, a contravariant functor assigns objects to objects but reverses the arrows. In this thesis, by functor (without prefix) we mean a covariant functor.

Proposition 2.7.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor and suppose that A and A' are isomorphic objects of $\mathcal{A}$. Then FA and FA' are isomorphic objects of $\mathcal{B}$.*

Proof. If A and A' are isomorphic objects of $\mathcal{A}$ then there exist an isomorphism $f : A \to A'$. Hence there are maps $FA \underset{F(f^{-1})}{\overset{Ff}{\rightleftarrows}} FA'$ in $\mathcal{B}$. Clearly Ff and $F(f^{-1})$ are mutually inverse, since

$$(F(f^{-1})) \circ (Ff) = F(f^{-1} \circ f) = F1_A = 1_{FA}.$$

In a dual sense, $FA \cong FA'$. $\square$

Remark 2.7.3. This shows that functors preserve isomorphism in that if f is an isomorphism, then so is Ff.

2.8 The functor $T_0 : \mathbf{Pos} \to T_0$

One of the usefulness of category theory come in through the maps from one category to the other. Some functors introduce a new structure on the *image* category whilst others do not. As an example, we give some functors of interest to our work in this thesis. There is a covariant functor from the category of topological spaces and continuous maps to the category of sets and functions which assigns each topological space its underlying set. This functor forgets some of the structure of the topological space. The functor we describe below turns every poset not only into a topological space, but one which satisfies the T_0 separation axiom. Consider the subsets U_x ($x \in X$) of X where $U_x = \{y \in X \mid y \leq x\}$ Viewing these sets as open sets and taking them to be the basis, we have that P is a topological space. We show that this topology satisfies the T_0 separation axiom. For any distinct points $x, x' \in X$, we have the following possibilities:

a. x and x' are not related, that is they are not comparable in any way i.e $x \not\leq x'$ and $x \not\geq x'$ or,

b. x and x' are comparable that is $x \leq x'$ or $x \geq x'$.

If **a** is the case, then one can construct an open set $U_{x'} = \{y : y \leq x'\}$ containing the point x' but not x. In the case of **b** we have that $U_x = \{y : y \leq x\}$ does not contain x' since $x \neq x'$. This association is a functor $T_0 :$ **Poset** $\to T_0$ from the category of posets and order preserving functions to the category of topological spaces and continuous functions.

Definition 2.8.1. Locally finite T_0-spaces (**LFT$_0$**), is the category whose class of objects are those T_0 spaces X having the property that for every $x \in X$, x has a finite neighborhood and the closure $\overline{\{x\}}$ is finite.

Definition 2.8.2. A functor $F : \mathcal{A} \to \mathcal{B}$ is said to be full if for each hom set $mor_{\mathcal{A}}(A, A')$ where $A, A' \in ob(\mathcal{A})$, we have that the hom-set restriction $F : mor_{\mathcal{A}}(A, A') \to mor_{\mathcal{B}}(FA, FA')$ is surjective.

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In the case where this hom-set restriction is injective, then F is said to be faithful.

2.9 Natural transformation

We have seen that functors can be regarded as morphisms between categories, we now consider transformations between functors.

Definition 2.9.1. Let $\mathcal{A}$, $\mathcal{B}$ be categories and F, G functors such that $F, G : \mathcal{A} \to \mathcal{B}$. A *natural transformation* from F to G assigns to an object A of $\mathcal{A}$ a $\mathcal{B}$-morphism $\mu_A : FA \to GA$ such that if $\alpha : A \to A'$ is an $\mathcal{A}$-morphism, we have $\mu_{A'} \circ F_\alpha = G_\alpha \circ \mu_A$, that is, the diagram

$$
\begin{array}{ccc}
FA & \xrightarrow{F_\alpha} & FA' \\
\mu_A \downarrow & & \downarrow \mu_{A'} \\
GA & \xrightarrow{G_\alpha} & GA'
\end{array}
$$

commutes.

2.10 Limits and colimits

To define a limit, we need the notion of a *diagram* and that of a *cone*. A diagram $D : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ and $\mathcal{C}$ are categories is some functor of the form:
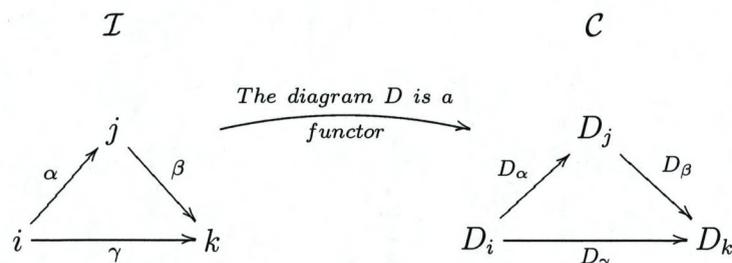
The functor D takes the objects i, j, k of $\mathcal{I}$ to the objects D_i, D_j, D_k of $\mathcal{C}$ and the maps α , β and γ of I to the maps D_α D_β and D_γ of $\mathcal{C}$.
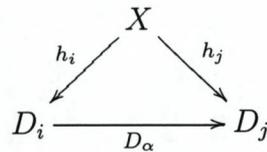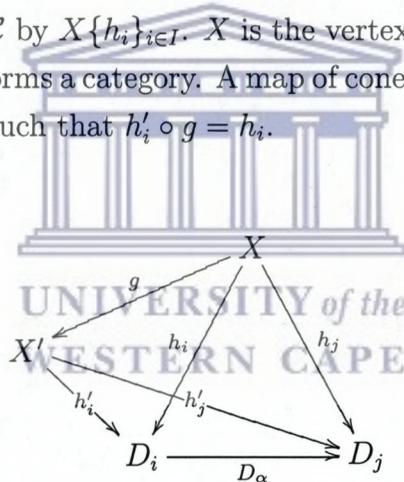
Definition 2.10.1. Let D_α be maps between the objects D_i and D_j, then the object X together with the triad $D_i \leftarrow X \rightarrow D_j$ form a *cone* of the diagram D if

$$
\begin{array}{ccc}
 & X & \\
{}^{h_i}\swarrow & & \searrow^{h_j} \\
D_i & \xrightarrow[D_\alpha]{} & D_j
\end{array}
$$

commutes.

We denote this cone in $\mathcal{C}$ by $X\{h_i\}_{i\in I}$. X is the vertex of the cone. The cones for a diagram $A_i \rightarrow A_j$ forms a category. A map of cones $g : X\{h_i\}_{i\in I} \longrightarrow X'\{h'_i\}_{i\in I}$ is a map $g : X \longrightarrow X'$ such that $h'_i \circ g = h_i$.

$$
\begin{array}{ccc}
 & X & \\
{}^{g}\swarrow \quad {}_{h_i}\downarrow & & \searrow^{h_j} \\
X' & & \\
{}_{h'_i}\searrow \quad \downarrow^{h'_j} & & \searrow \\
 & D_i \xrightarrow[D_\alpha]{} D_j &
\end{array}
$$

Definition 2.10.2. A cone (as in definition 2.10.1) is a *limit* in $\mathcal{C}$ if for any other object X' of $\mathcal{C}$ with maps h'_i from X' to D_i, there exists a unique map K from X' to X such that $h_i \circ k = h'_i$ and $h_j \circ k = h'_j$. That is, the diagram

$$
\begin{array}{ccc}
 & X' & \\
{}^{h'_i}\swarrow & {}_{k}\big\downarrow & \searrow^{h'_j} \\
 & X & \\
{}_{h_i}\swarrow & & \searrow^{h_j} \\
D_i & \xrightarrow[D_\alpha]{} & D_j
\end{array}
$$
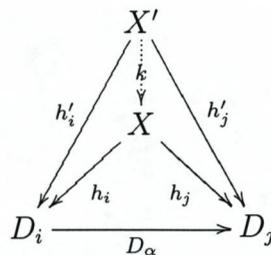
commutes.

When constructing a limit, one has to use a functor that picks out the relevant objects or morphisms. In most cases, the category $\mathcal{I}$ as used in the definition of the limit of a diagram is small that is it has fewer elements than $\mathcal{C}$.

Definition 2.10.3. A category $\mathcal{C}$ is said to be *complete* if every functor $F : \mathcal{I} \to \mathcal{C}$ has a limit.

Remark 2.10.4. The categories **Set**, **Grp** and **Top** are complete. In general, categories that are not complete are those which have some restriction on their size. One of the most obvious category which is not complete is that of finite-dimensional vector spaces over a fixed field.

Dually, reversing the direction of the morphisms in Definitions 2.10.1 and 2.10.2 gives the notion of a **co-cone** and **colimit** respectively.

2.11 Equivalence of categories

A natural question one might ask is: when are two categories essentially the same? That is, when is it possible to infer results and properties in a category one is working with into the other. The equivalence of categories is deployed to describe such an instance. The equivalence of categories is given by appropriate functors between two categories.

Definition 2.11.1. Given functors $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation $\alpha : F \to G$ is a *natural isomorphism* if there exists another natural transformation $\beta : G \to F$ such that $\alpha\beta = 1_F$ and $\beta\alpha = 1_G$.

From the definition, it follows that a natural transformation $\alpha : F \to G$ is a natural isomorphism if for every object A of $\mathcal{C}$, every morphism $\alpha_A : FA \to GA$ is invertible.

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Definition 2.11.2. A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if it has a weak inverse, i.e, there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that there exists natural isomorphisms $\alpha : FG \to 1_C$ and $\beta : GF \to 1_D$.

Theorem 2.11.3. *Restricted to the subcategory* **LFPoset***, the functor* T_0 *as in Subsection 2.8 is an isomorphism of the category* **LFPoset** *onto the category* **LFT$_0$***.*

Proof. We need to construct a functorial association $P :$ **LFT$_0$** $\to$ **LFPos** such that $PT_0 = 1_{\mathbf{LFT_0}}$ and $T_0 P = 1_{\mathbf{LFT_0}}$. For $x, y \in X$ where X is a locally finite T_0 space, we declare that $y \leq x$ if and only if $x \in \overline{\{y\}}$. Note that $\{y | y \leq x\}$ is the smallest open set that contains x. In this way, we have constructed exactly the same finite poset that T_0 associates on **LFT$_0$**. $\square$

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Chapter 3

Basic Homotopy

In this chapter, we present some homotopy concepts that are pertinent to the sequel. Homotopy can be introduced in two different ways: by the use of the cylinder, or the path space. For the work in this thesis, we employ the cylinder approach. Two functions from one topological space to the other are said to be homotopic if one can be deformed continuously into the other. For our purposes, the topological spaces will be *pointed* topological spaces, i.e, we consider pairs $(X, *)$ where $*$ is a distinguished point in a topological space X. By abuse of notation, as is common, we shall in some cases for convenience suppress the base point. The main references are J.P. May [31], J.R. Munkres [34], H. Sato [42], M.C. McCord [33] and E.H. Spanier [45].

3.1 Homotopy relation

In this section, we define homotopy and in Proposition 3.1.2, show that it is an equivalence relation.

Definition 3.1.1. Let $(X, *)$ be a pointed topological space, I^k a k-dimensional unit square with boundary ∂I^k. Fix $k \in \mathbb{N}$. Consider the set $F_k(X, *)$ of all

continuous functions

$$f : (I^k, \partial I^k) \to (X, *).$$

For $f_1, f_2 \in F_k$, we say that f_1 is *homotopic* to f_2 if and only if there exists a continuous map $H : I^k \times I \to X$ such that:

$$
\begin{aligned}
H(x, 0) &= f_1(x), & \forall\, x; \\
H(x, t) &= *, & \forall\, x \in \partial I^k, \forall\, t \in I; \\
H(x, 1) &= f_2(x), & \forall\, x.
\end{aligned}
$$

Let $[f] \subseteq F_k(X, *)$ be a set of all functions homotopic to f, then we have the following proposition.

Proposition 3.1.2. *The homotopy relation on $F_k(X, *)$ is an equivalence relation.*

Proof. Reflexivity: This is obvious by taking for a given $f \in F_k(X, *)$, the constant homotopy $H(x, t) = f(x) \; \forall \; x$.

Symmetry: Suppose $f_0 \simeq f_1$ via the homotopy $H(x, t)$.

Then $f_1 \simeq f_0$ via the inverse homotopy $H(x, 1 - t)$ because

$$
\begin{aligned}
H(x, 1 - t)\,|_{t=0} = H(x, 1) &= f_1(x) \; \forall \; x \\
H(x, 1 - t)\,|_{0 \leq t \leq 1} &= * \qquad \forall \; x \in \partial I^k \\
H(x, 1 - t)\,|_{t=1} = H(x, 0) &= f_0(x) \; \forall \; x.
\end{aligned}
$$

Transitivity: Suppose $f_0 \simeq f_1$ via the homotopy $H(x, t)$ and $f_1 \simeq f_2$ via the homotopy $G(x, t)$, then $f_0 \simeq f_2$ via the homotopy

$$
F(x, t) = \begin{cases} H(x, 2t), & 0 \leq t \leq \frac{1}{2} \\ G(x, 2t - 1), & \frac{1}{2} < t \leq 1. \end{cases}
$$

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Further, we have

$$H(x, 2t) \mid_{t=\frac{1}{2}} = f_1(x) = G(x, 2t - 1) \mid_{t=\frac{1}{2}},$$

that is, the two homotopies H and G agree on $t = 1/2$. Thus continuity of F follows by the pasting lemma (see Munkres [34]). $\qquad\square$

Denote this collection of equivalence classes by $\pi_k(X, *)$ and define a binary operation $\star$ on the later set as follows. For $f, g, h \in F$, $[f] \star [g] = [h]$ where

$$h(x, t) = \begin{cases} g(x, 2t), & 0 \leq t \leq \frac{1}{2} \\ f(x, 2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Note that $x \in I^{k-1}$. This gives a group structure on $\pi_k(X, *)$ with the identity being the equivalence class consisting of constant maps. The inverse $[f]^{-1}$ for an element $[f]$ of $\pi_k(X, *)$ is $[g]$ where $g : (x, t) \mapsto f(x, 1 - t)$.

Definition 3.1.3. For a pointed topological space $(X, *)$ and natural numbers k, the group $\pi_k(X, *)$ is called the kth *homotopy group* of X for $k=1,2,3,\ldots$

In some instances, it is convenient to use S^k in the place of I^k in Definition 3.1.1. Suppose $f : (X, *) \to (Y, *)$ is a map taking base point $* \in X$ to basepoint $* \in Y$. We show that the induced map $f_* : \pi_k(X, *) \to \pi_k(Y, *)$ is a group homomorphism. We define f_* in terms of composition of maps as follows: given $g : I^k \to X$, we let $f_*[g] = [f \circ g]$. The map f_* is well defined because if we consider the homotopy $H : I \times X \to X$ of loops based at $*$, $f \circ H$ gives a composed homotopy of loops based at $f(*)$. And for $g_0 \simeq g_1$ via H,

$$f_*[g_0] = [f \circ g_0] = [f \circ g_1] = f_*[g_1].$$

That is, if $f_0, f_1 : (X, *) \to (Y, f_i(*))$ $i = 0, 1$ are pointed homotopic then we have $\pi_k(f_0) = \pi_k(f_1)$. This shows that f_* is well defined. Furthermore,

$$f_*[g \circ h] = [f \circ (g \star h)] = [(f \circ g) \star (f \circ h)] = f_*[g]f_*[h].$$

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Hence the induced map f_* is indeed a group homomorphism. The continuous maps between the spaces $(X, *)$ and $(Y, f(*))$ are projected onto homomorphisms between their algebraic images. Thus topologically related objects will have algebraically related images,

$$(X, x) \xrightarrow{\ f\ } (Y, f(x)) \quad .$$

$$\pi_k(X, x) \xrightarrow{\ \pi_k(f)\ } \pi_k(Y, f(x))$$

In this way, for each k, the association $\pi_k : \mathbf{Top}^* \to \mathbf{Grp}$ defines a functor from the category of pointed topological spaces and continuous maps to the category of groups and group homomorphisms.

Definition 3.1.4. The map $f : (X, x) \to (Y, y)$ is a *homotopy equivalence* if there exists a map $g : (Y, y) \to (X, x)$ such that $f \circ g \simeq \mathbf{1_Y}$ and $g \circ f \simeq \mathbf{1_X}$.

Remark 3.1.5. In the case where $f : (X, x) \to (Y, y)$ is a homotopy equivalence, then the induced homomorphism $\pi_k(f) : \pi_k(X, x) \to \pi_k(Y, y)$ is an isomorphism for each k. To show this, consider (cf. [31]) the diagram

$$(Y, y) \xrightarrow{\ g\ } (X, x) \xrightarrow{\ f\ } (Y, y) \xrightarrow{\ g\ } (X, x)$$

with 1_Y and 1_X indicated.

$$\pi_k(Y, y) \xrightarrow[\pi_k(g)]{} \pi_k(X, x) \xrightarrow[\pi_k(f)]{} \pi_k(Y, y) \xrightarrow[\pi_k(g)]{} \pi_k(X, x)$$

with $1_{\pi_k(Y,y)}$ and $1_{\pi_k(X,x)}$ indicated.

Since $1_{\pi_k(X,x)}$ and $1_{\pi_k(Y,y)}$ are isomorphisms, we have that $\pi_k(g)$ is an injection and $\pi_k(f)$ is a surjection, $\pi_k(f)$ is an injection and $\pi_k(g)$ is a surjection thus $\pi_k(f)$ is a bijection for each k.

Definition 3.1.6. A map $f : X \to Y$ is a *weak homotopy equivalence* if the induced homomorphism $f_* : \pi_k(X, x) \to \pi_k(Y, f(x))$ is a bijection for each

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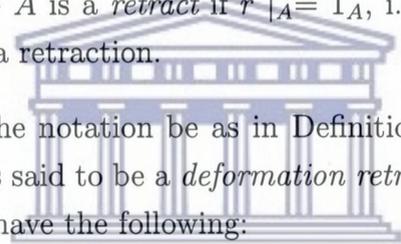$k \in \mathbb{N} \cup \{0\}$, for all $x \in X$.

From the previous remark and the definition of weak homotopy equivalence, we have the following implication.

Proposition 3.1.7. *If $f : (X, x) \to (Y, y)$ is a homotopy equivalence, then it is weak homotopy equivalence.*

Proof. This follows from the fact that if f is a homotopy equivalence, the induced homomorphism $\pi_k(f) : \pi_k(X, x) \to \pi_k(Y, y)$ is an isomorphism for each k. Thus it is a bijection for $k \in \mathbb{N} \cup \{0\}$. $\square$

Definition 3.1.8. Let (X, A) be a topological space where $A \subset X$. Then a continuous map $r : X \to A$ is a *retract* if $r \mid_A = 1_A$, i.e $r(a) = a$ $\forall a \in A$. In this case, r is said to be a retraction.

Definition 3.1.9. Let the notation be as in Definition 3.1.8. A continuous map $d : X \times [0, 1] \to X$ is said to be a *deformation retraction* if, for all $x \in X$, $a \in A$, and $t \in [0, 1]$, we have the following:

$$
\begin{aligned}
d(x, 0) &= x \\
d(a, t) &= a \\
d(x, 1) &\in A
\end{aligned}
$$

In relation to Definition 3.1.1, we have that a deformation retraction is a homotopy between the maps:

$$
\begin{aligned}
i_X : X &\to X \text{ the identity map on } X \text{ and} \\
r : X &\to A \text{ a retraction of } X \text{ onto } A, \text{ where } r(x) = d(x, 1) \text{ for all } x \in X.
\end{aligned}
$$

Deformation retraction is related to the notion of homotopy equivalence, in that two spaces are homotopy equivalent if and only if they are both deformation retracts of a single space.

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Definition 3.1.10. A topological space X is said to be *contractible* if it is homotopy equivalent to a point.

This implies that the identity map $1_X : x \mapsto x$ on each point $x \in X$ is homotopic to the map $f : X \to X$ for which $f(X) = *$. Thus we have that $\pi_k(X) \cong \pi_k(*) = 0$. Any topological space which deformation retracts to a point is contractible. Contractibility, however, is a weaker condition, as contractible spaces exist which do not deformation retract to a point [45].

3.2   Cone, suspension and join of spaces

Let X be any topological space. Consider the space Y obtained as follows. $Y = X \times I$ where I is the usual unit interval $[0, 1]$, and let $X_0 = X \times \{0\}$. Shrinking the subspace X_0 to a point on Y, we denote the resultant space by Y/X_0. The space Y/X_0 is called the unreduced cone on X. The space X is identified to the closed subspace $X \times \{1\}$ in Y/X_0.

Definition 3.2.1. Let $x_0 \in X$, the *reduced cone* (or simply *cone*) over the pointed space (X, x_0) is the space

$$
\begin{aligned}
\mathbb{C}X &= Y/(X \times \{0\} \cup (\{x_0\} \times [0, 1])) \\
&= Y/(X \times \{0\} \cup (\{x_0\} \times I)).
\end{aligned}
$$

The space $\mathbb{C}X$ is contractible to the point x_0. Let $X_1 = X \times \{1\}$, then the space $Y/X_0 \cup X_1$ is the unreduced suspension of X. In this construction, the subspace space $X \times \{0\}$ is shrunk to a single point say α and all the points of $X \times \{1\}$ are shrunk to another point say β. Keeping the same notation for the subspaces X_0 and X_1, we define the suspension of a space with base point as follows.

Definition 3.2.2. The space $Y/(X_0 \cup \{x_0\} \times I \cup X_1)$ is called the *reduced suspension* (or simply *suspension*) of the space X and denoted as $\mathbb{S}X$.

Note that the points α and β are identified in case of reduced suspension. From this construction, we see that there is a natural map

$$\phi_X : \mathbb{C}X \to \mathbb{S}X \text{ such that } \phi_X(x,t) = (x,t).$$

Taking x_0 to be the basepoint of X, we choose $x_0 = (x_0, 0)$ to be the basepoint in the cone $\mathbb{C}X$ and the suspension $\mathbb{S}X$.

Suppose X and Y are some arbitrary pointed spaces. Let $x \in X$ and $y \in Y$, and consider the product space $Z = X \times Y \times [0,1]$. We make a partition of Z consisting of the following sets.

1. the sets $\{x\} \times Y \times \{0\}$ for all $x \in X$,

2. the sets $X \times \{y\} \times \{1\}$ for all $y \in Y$ and

3. the singletons $\{(x,y,t)\}$ for $x \in X, y \in Y, t \in (0,1)$.

Definition 3.2.3. The *join* of two spaces X and Y denoted by $X * Y$, is defined to be the quotient space $(X \times Y \times I)/R$, where I is the interval $[0,1]$ and R is the relation defined by

$(x,y_1,0) \sim (x,y_2,0) \ \forall \ x \in X$ and $y_1, y_2 \in Y$,
$(x_1,y,1) \sim (x_2,y,1) \ \forall \ x_1, x_2 \in X$ and $y \in Y$.

Remark 3.2.4. Thus the natural map $f : X \times Y \times [0,1] \to X * Y$ sends $X \times Y \times \{0\}$ onto some closed subspace which is homeomorphic to X, and sends $X \times Y \times \{1\}$ onto some closed subspace homeomorphic to Y.

It is interesting to note that, given a weak homotopy equivalence defined on two open sets, there exist some conditions for the map to be a weak homotopy equivalence when defined on their union. We give a result in this direction.

The following result is very similar to that in [21], Lemma 16.24. The more general result in terms of excisive triads can be deduced from 3.2.5 below as shown in P. Witbooi, [48].

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Theorem 3.2.5. (cf. B. Gray [21]) *Suppose that X_1, X_2 are open sets in X and Y_1, Y_2 are open sets in Y, and that $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. Let $\phi : X \to Y$ with $\phi(X_1) \subset Y_1$ and $\phi(X_2) \subset Y_2$. If $\phi\mid_{X_1}: X_1 \to Y_1, \phi\mid_{X_2}: X_2 \to Y_2$ and $\phi\mid_{X_1 \cap X_2}: X_1 \cap X_2 \to Y_1 \cap Y_2$ are weak homotopy equivalences, then so is ϕ.*

Proof. To prove this, we need to show that the induced homomorphism $\phi_* : \pi_n(X, *) \to \pi_n(Y, *)$ is an isomorphism. To show the isomorphism, it is sufficient to prove the following:

(A) Given $f : \triangle^n \to Y, g : \partial\triangle^n \to X$ with $\phi g = f\mid_{\partial\triangle^n}$, there exists $F : \triangle^n \to X$ with $F\mid_{\partial\triangle^n} = g$ and $\phi \circ F \sim f$(relative to $\partial\triangle^n$). We further show that (A) implies the following.

(B)   [(i)] $\phi_* : \pi_{n-1}(X, *) \to \pi_{n-1}(Y, *)$ is mono,

     [(ii)] $\phi_* : \pi_n(X, *) \to \pi_n(Y, *)$ is epi.

We first show that indeed $A \Rightarrow B$;

An arbitrary element α of $\pi_n(X, *)$ can be represented by a function

$$f : (\triangle^n, \partial\triangle^n) \to (X, *).$$

Let $g : \partial\triangle^n \to * \in Y$ be the constant map, then by condition (A), a map $F : (\triangle^n, \partial\triangle^n) \to (Y, *)$ exists such that $\phi_*[F] = [f]$, i.e, $\phi_* : \pi_n(X, *) \to (Y, *)$ is an epimorphism. Hence (B) (ii) is proved. Suppose that $\beta \in \pi_{n-1}(X, *)$ and $\phi_*\beta = 0$. We must prove that $\beta = 0$. Now β can be represented by a map $g' : \partial\triangle^n \to X$. Since $\phi_*\beta = 0$, it means that there exists $f : \triangle^n \to Y$ such that $f\mid_{\partial\triangle^n} \sim \phi \circ g'$. Let $g = f\mid_{\partial\triangle^n}$. The F which exists by (A) guarantees that $\beta = 0$, thus (B) (i)

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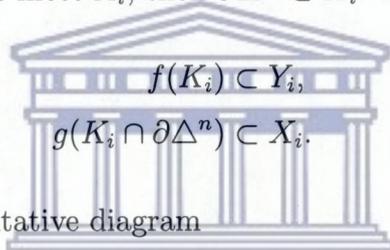
We now show the sufficiency condition that (A) holds.

$$\text{Let } A_1 = g^{-1}(X - X_1) \cup f^{-1}(Y - Y_1) \text{ and}$$
$$A_2 = g^{-1}(X - X_2) \cup f^{-1}(Y - Y_2),$$

then A_1 and A_2 are disjoint closed sets. We subdivide $\triangle^n$ in such a way that no simplex ∂ meets both A_1 and A_2. We then define

$$K_1 = \{\delta \mid g(\delta \cap \partial\triangle^n) \subset X_1, f(\delta) \subset Y_1\} \text{ and}$$
$$K_2 = \{\delta \mid g(\delta \cap \partial\triangle^n) \subset X_2, f(\delta) \subset Y_2\}.$$

Then K_1 and K_2 are subcomplexes and $\triangle^n = K_1 \cup K_2$. This is because if $\partial\triangle^n$ is a simplex that does not meet A_i, then $\partial\triangle^n \subset K_i$. Furthermore,

$$f(K_i) \subset Y_i,$$
$$g(K_i \cap \partial\triangle^n) \subset X_i.$$

Thus we have the commutative diagram

$$
\begin{array}{ccc}
X_1 \cap X_2 & \xrightarrow{\phi} & Y_1 \cap Y_2 \\
\uparrow{\scriptstyle g} & & \uparrow{\scriptstyle f} \\
\partial\triangle^n \cap K_1 \cap K_2 & \longrightarrow & K_1 \cap K_2
\end{array}
$$

There exists $F : K_1 \cap K_2 \to X_1 \cap X_2$ with

$$F\mid_{K_1 \cap K_2 \cap \partial\triangle^n} = g \text{ and } \phi F \sim f(\text{relative to } \partial\triangle^n \cap K_1 \cap K_2).$$

Now we define $G_1 : K_1 \cap (\partial\triangle^n \cup K_2) \to X_1$ by $G_1\mid_{K_1 \cap \partial\triangle^n} = g\mid_{k_1 \cap \partial\triangle^n}$ and $G_1\mid_{K_1 \cap K_2} = F$, then $\phi G_1 \sim f\mid_{K_1 \cap (\partial\triangle^n \cup K_2)}$.

We extend this homotopy to a homotopy $H : K_1 \times I \to Y_1$ of f to a map $f_1 : K_1 \to Y_1$ with $f_1\mid_{K_1 \cap (\partial\triangle^n \cup K_2)} = \phi G_1$.

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Thus the diagram

$$X_1 \xrightarrow{\phi} Y$$
$$G_1 \uparrow \qquad \uparrow f_1$$
$$K_1 \cap (\partial\triangle^n \cup K_2) \longrightarrow K_1$$

commutes and we may find $F_1 : K_1 \to Y_1$ where $F_1 \mid_{K_1 \cap (\partial\triangle^n \cup K_2)} = G_1$ and $\phi F_1 \sim f_1$ (relative to $(K_1 \cap (\partial\triangle^n \cup K_2))$). This implies that $\phi F_1 \sim f$ (relative to $K_1 \cap \partial\triangle^n$). In a similar manner, we construct $F_2 : K_2 \to X_2$ with $F_2 \mid_{K_2 \cap (\partial\triangle^n \cup K_1)} = G_2$ and $\phi F_2 \sim f$ (relative to $K_2 \cap \partial\triangle^n$). In this way F_1 and F_2 agree on $K_1 \cap K_2$, hence we define a map $\tilde{F} : I^n \to Y$ as follows,

$$\tilde{F} = \begin{cases} F_1, & \text{on } K_1 \\ F_2, & \text{on } K_2. \end{cases}$$

Then $\tilde{F} \mid_{\partial\triangle^n} = g$, the homotopies $\phi F_1 \sim f_1 \sim f$ and $\phi F_2 \sim f_2 \sim f$ agree on $(K_1 \cup K_2) \times I$ because $\phi F_1 \sim f_1$, $\phi F_2 \sim f_2$ relative to $K_1 \cap K_2$ and the homotopies $f_1 \sim f_2$ when restricted to $(K_1 \cap K_2) \times I$ both yield the homotopy $\phi F \sim f$. Thus $\phi \tilde{F} \sim f$ is the homotopy relative to $(K_1 \cap \partial\triangle^n) \cup (K_2 \cap \partial\triangle^n) = \partial\triangle^n$. □

3.3 Exactness in a sequence of homotopy groups

Let $\phi : G_0 \to G_1$ is a group homomorphism. Recall that ϕ is monomorphic if Ker $\phi = \{0\}$ and it is epimorphic if the quotient $G_1/\text{Im}\phi = \{0\}$. For a pair (X, A) of topological spaces, there is a homotopy sequence of homotopy groups and group homomorphisms

$$\pi_k(X, *) \xrightarrow{j} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \xrightarrow{i} \pi_n(X) \to$$

The homomorphisms j and i arise out of the inclusions $(X, *) \hookrightarrow (X, A)$ and $A \hookrightarrow X$ respectively. The homomorphism ∂ arise from the boundary operator.

Definition 3.3.1. A sequence

$$G_0 \xrightarrow{h} G_1 \xrightarrow{q} G_2$$

of groups and group homomorphism is said to be *exact* at G_1 if $\operatorname{Im}(h) = \operatorname{Ker}(q)$.

Remarks A longer sequence

$$\cdots G_{-2} \to G_{-1} \to G_0 \to G_1 \to G_2 \cdots$$

is exact if any two consecutive homomorphisms form an exact sequence. From the definition of exactness, it follows that it is not defined on the first or last group of a sequence if at all the groups do exist.

Definition 3.3.2. An exact sequence of the form

$$0 \to G_0 \to G_1 \to G_2 \to 0$$

is called a short exact sequence.

For example, suppose H is a normal subgroup of a group G, then we have the following

$$0 \to H \xrightarrow{i} G \xrightarrow{\pi} G/H \to 0$$

Where i is the inclusion of H in G and π is the projection of G on G/H.

3.4 Cofibrations and fibrations

In this section, we give the definition of a fibration and cofibration.

Definition 3.4.1. Let $f : X \to Y$ and $h : A \times I \to Y$. A map $i : A \to X$ is a *cofibration* if it satisfies the homotopy extension property (HEP) with respect

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to any space. This means that if $h \circ i_0 = f \circ i$ in the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{i_0} & A \times I \\
\downarrow{\scriptstyle i} & & \downarrow \\
X & \longrightarrow & X \times I \\
\end{array}
$$

Then there exists $\tilde{h}$ that makes the diagram commute. The map $\tilde{h}$ does not necessarily have to be unique.

Remark 3.4.2. In [31], May shows by way of some "diagram chasing" that the pushout of two cofibrations is itself a cofibration.

Dually, we have the following:

Definition 3.4.3. A surjective map $p : E \to B$ is a *fibration* if it satisfies the homotopy lifting property (HLP) with respect to any space Y. That is, for each diagram of unbroken arrows
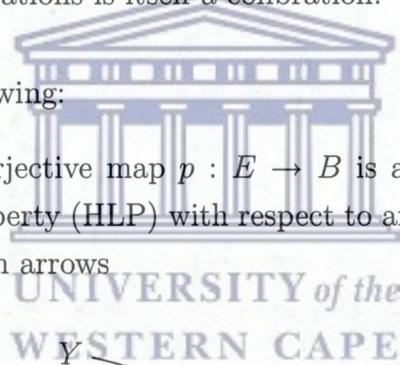
$$
\begin{array}{ccc}
 & E^I & \xrightarrow{q_0} E \\
Y & \downarrow{\scriptstyle p^I} & \downarrow{\scriptstyle p} \\
 & B^I & \xrightarrow{q_0} B \\
\end{array}
$$

there exists an $\tilde{h}$ that makes the diagram above commute.

Taking the natural map $i_0 : Y \to Y \times I$, the commutative diagram in Definition 3.4.3 can be displayed in a more visually helpful way as: $p : E \to B$ has HLP

if $h \circ i_o = p \circ f$ in the diagram,

$$\begin{array}{ccc} Y & \xrightarrow{\ f\ } & E \\ {\scriptstyle i_0}\downarrow & \overset{\tilde{h}}{\nearrow} & \downarrow{\scriptstyle p} \\ Y \times I & \xrightarrow[\ h\ ]{} & B \end{array}$$

there exists an $\tilde{h}$ that makes the diagram commutative.

Remark 3.4.4. Suppose $A \subset X$, then we say that the space X is obtained by attaching a cell to A if there exists the following pushout diagram:

$$\begin{array}{ccc} D^n & \longrightarrow & X \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle p} \\ S^{n-1} & \longrightarrow & A \end{array}$$

where the inclusions $S^{n-1} \subset D^n$ and $A \subset X$ are inclusions in **Top**. In the category **Top**, a map $p : A \to X$ is said to be a Serre-fibration if it has the homotopy lifting property with respect to the n-dimensional disk, $(n \geq 0)$.

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Chapter 4

Relational structures

In this chapter, we discuss relational structures. We define a relational structure and the concept of a relational morphism. In Section 4.1, we give some examples of relational structures. Furthermore, we show that one may obtain a new relational structure by taking quotients on a given one. In Section 4.2, we discuss the product of relational structures. We give a result that the projection map from a product of two relational structures into one of the relational structures is a relational morphism. In Section 4.3, we discuss the functorial association of the category of relational structures to that of partially ordered sets. In light of the weak equivalence between an object X and its barycentric subdivision X', one may regard (as in Section 4.4) objects of $\mathcal{R}$ as models. We illustrate by an example in Section 4.5 a 4-point model which is weakly homotopy equivalent to the 1-circle S^1. The main references are K.A Hardie *et al.* [23], [24], K.A. Hardie and Witbooi [26], and the paper of B. Larose and C. Tardiff [29]. We begin by giving a formal definition of a relational structure following [29].

Definition 4.0.5. (cf. [29]) A *binary reflexive relational structure* $\mathbf{X} = (X, \theta)$ is a set X together with a reflexive relation

$$\theta \subseteq X \times X.$$

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If there is no possibility of ambiguity, we will call a binary reflexive relational structure a relational structure.

Definition 4.0.6. A *morphism* of relational structures $f : (X, \theta_X) \to (Y, \theta_Y)$, in short an $\mathcal{R}$-morphism, is a function $f : X \to Y$ satisfying the condition that $(x_1, x_2) \in \theta_X \Rightarrow (fx_1, fx_2) \in \theta_Y$.

It is easy to see that we obtain a category of relational structures and $\mathcal{R}$-morphisms which we denote by $\mathcal{R}$. Elsewhere in the literature, this category is sometimes denoted by **Rere**, as in the book by J. Adamek *et al.* [2] for instance.

Remark 4.0.7. In this case, we adopt the following notation. For a binary relation θ on a set X, we write $x_1 \to x_2$ as meaning $(x_1, x_2) \in \theta$. In some instances where it is convenient, we denote $(x_1, x_2) \in \theta$ by $x_1 \theta x_2$.

In the following section, we give examples of relational structures.
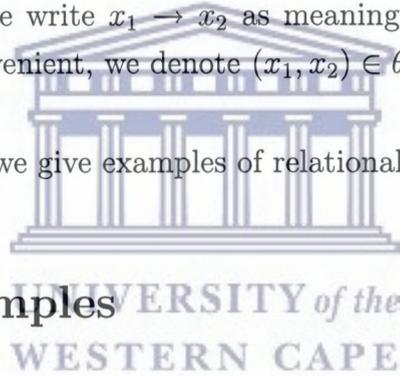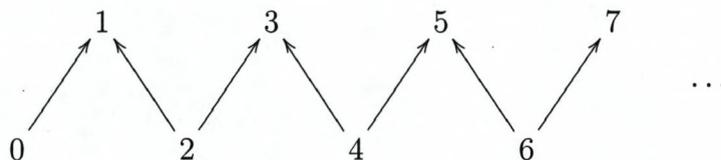
4.1 Some examples

(a) $\cdots -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \cdots$ The directed ray on $\mathbb{Z}$. In this case, $x \to y$ whenever $x = y$ or $y - x = 1$.

(b) $\cdots -2 \rightleftarrows -1 \rightleftarrows 0 \rightleftarrows 1 \rightleftarrows 2 \cdots$ In this case, we have a relation on $\mathbb{Z}$ whenever $x = y$ or $|y - x| = 1$.

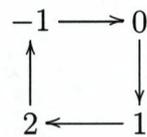(c) The one way infinite fence $\mathbb{F}$ is as below.

The relation is as follows: $x \to y$ if $x = y$ or x is even and $|x - y| = 1$.

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New structures may be constructed from a given one. For example, from the infinite ray on $\mathbb{Z}$, we can obtain the finite 5 point directed ray

$$-2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2.$$

We denote this by $\mathbb{F}_{-2,2}$. Identifying the point -2 with 2 on $\mathbb{F}_{-2,2}$ gives the relational structure below:
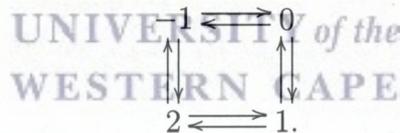
(d)

$$
\begin{array}{ccc}
-1 & \longrightarrow & 0 \\
\uparrow & & \downarrow \\
2 & \longleftarrow & 1
\end{array}
$$

(e) Similarly, from the relational structure in (b) we can obtain

$$-2 \rightleftarrows -1 \rightleftarrows 0 \rightleftarrows 1 \rightleftarrows 2,$$

and identifying the point -2 with 2 gives

$$
\begin{array}{ccc}
1 & \rightleftarrows & 0 \\
\updownarrow & & \updownarrow \\
2 & \rightleftarrows & 1.
\end{array}
$$

(f) From the fence $\mathbb{F}$ in (c), identify the points $\bar{0}$ and $\bar{4}$ to obtain $\mathbb{F}_4/_{(0=4)}$

$$
\begin{array}{ccccc}
 & & \bar{2} & & \\
 & \swarrow & & \searrow & \\
\bar{3} & & & & \bar{1} \\
 & \nwarrow & & \nearrow & \\
 & & \bar{4} & &
\end{array}
$$
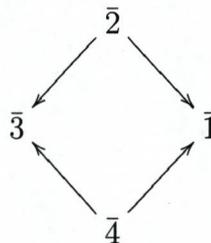
We now discuss the concept of a product of relational structures.

4.2 Product of binary relational structures

Given two relational structures, we can, in a natural way, extend the notion of relational structure to the product of the underlying sets.

Definition 4.2.1. The *product* of two binary relational structures $\mathbf{X}$ and $\mathbf{Y}$ is the structure $X \times Y$ on the product of the base sets of X and Y and we define $(x, y) \to (x', y')$ if $x \to x'$ and $y \to y'$.

Consider the product $\mathbf{X} \times \mathbf{Y}$ of the relational structures $\mathbf{X} = (X, R)$ and $\mathbf{Y} = (Y, S)$. R and S are the respective relations on X and Y. We have that

$$\mathbf{X} \times \mathbf{Y} = (X \times Y, T).$$

For the relation T on $X \times Y$, $(x_1, y_1) T (x_2, y_2)$ if and only if $x_1 R x_2$ and $y_1 S y_2$. The relation T is reflexive, hence $\mathbf{X} \times \mathbf{Y}$ is a relational structure.

Proposition 4.2.2. *Let $\mathbf{X}$ and $\mathbf{Y}$ be relational structures and $\mathbf{X} \times \mathbf{Y}$ their product. Then the projection map $f : \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$ is a $\mathcal{R}$-morphism.*

Proof. Let X and Y be the underlying sets for $\mathbf{X}$ and $\mathbf{Y}$ respectively. Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, and suppose $(x_1, y_1) \to (x_2, y_2)$ in the $\mathcal{R}$-product $X \times Y$. Then $f(x_1, y_1) = x_1$ and $f(x_2, y_2) = x_2$. But $f(x_1, y_1) = x_1 \to x_2 = f(x_2, y_2)$ hence $f(x_1, y_1) \to f(x_2, y_2)$ and so f is an $\mathcal{R}$-morphism. $\square$

It is interesting to note that we can extend the notion of relational structure to the corresponding set of morphisms from one relational structure to the other. Let $\mathbf{X} = (X, R)$ and $\mathbf{Y} = (Y, S)$ be binary structures. We define a binary structure $\mathrm{Hom}(X, Y)$ whose base set consists of all homomorphisms from X to Y. Here, we have that if f and g are two such homomorphisms, we define $f \to g$ if $f(x) \to g(y)$ whenever $x \to y$.

Proposition 4.2.3. *For an arbitrary object Y in $\mathcal{R}$ we have the following: $\mathrm{Hom}(Y, X_1 \times X_2) \cong \mathrm{Hom}(Y, X_1) \times \mathrm{Hom}(Y, X_2)$.*

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Proof. To the map $p : Y \to X_1 \times X_2$ in $\mathrm{Hom}(Y, X_1 \times X_2)$ we assign

$$(\pi_1 p, \pi_2 p) \in \mathrm{Hom}(Y, X_1) \times \mathrm{Hom}(Y, X_2).$$

Here, the projections $\pi_i : X_1 \times X_2 \to X_i$, $i = 1, 2$ are $\mathcal{R}$-morphisms by Proposition 4.2.2. Let $y \in Y$, then to a pair of $\mathcal{R}$-morphisms $(f, g) \in \mathrm{Hom}(Y, X_1) \times \mathrm{Hom}(Y, X_2)$ we assign a map $y \mapsto (f(y), g(y)) \in X_1 \times X_2$. Furthermore we have the maps $p \mapsto \pi_1 p$ and $p \mapsto \pi_2 p$ where $p \in \mathrm{Hom}(Y, X_1 \times X_2), \pi_1 p \in \mathrm{Hom}(Y, X_1)$ and $\pi_2 p \in \mathrm{Hom}(Y, X_2)$. But these maps are transformed by the evident bijection above into the maps; $(f, g) \mapsto f \in \mathrm{Hom}(Y, X_1)$ and $(f, g) \mapsto g \in \mathrm{Hom}(Y, X_2)$.

$\square$

This isomorphism of relational structures can be expressed equivalently as follows: For each pair of maps $X_1 \xleftarrow{f} Y \xdashrightarrow{g} X_2$ there exists exactly one map $p : Y \to X_1 \times X_2$ such that $f = \pi_1 f$ and $g = \pi_2 g$. This is a categorical product. Larose and Tardiff in [29] construct an analogue of the classical homotopy groups using pointed objects in $\mathcal{R}$ and $\mathcal{R}$-morphisms. For each object $(X, *)$ in $\mathcal{R}$, they define a sequence $\sigma_k(X, *)$, $k \in \mathbb{N} \cup \{0\}$ of homotopy sets. A loop based at x_0 in the structure X can be regarded as a sequence $x_1, x_2, \dots, x_n, x_0$ of elements of X such that $x_0 \theta x_1, x_2 \theta x_1, x_2 \theta x_3, \dots$ as displayed in fig. 4.1 below.
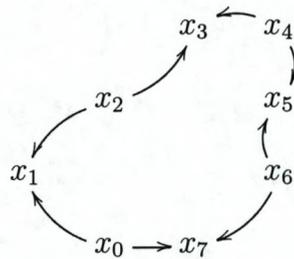

Figure 4.1:

The loop in fig. 4.1 represents the sequence

$$x_0 \theta x_1, x_2 \theta x_1, x_2 \theta x_3, \ldots, x_7 \theta x_0,$$

where each arrow belongs to the relation θ_X. The concept of homotopy is in this case mimicked by the connectivity of the class of loops based at x_0. The one way infinite fence $\mathbb{F}$ acts as the unit interval in this case. Note that there exists a path from x_i to x_j if and only if there exists a homomorphism $f : \mathbb{F} \to X$ and an integer $N \geq 0$ such that $f(0) = x_i$ and $f(n) = x_j$. Loops of different lengths are accommodated by defining a homomorphism from the one way infinite fence which is equal to x_0 almost everywhere. To define the kth homotopy, one simply replaces I^k with $\mathbb{F}^k$. Thus, the elements of $\sigma_k(X, x_0)$ are essentially of the form $\mathbb{F}^k \to (X, x_0)$.

Theorem 4.2.4. *For each k, we have $\sigma_k(X \times Y) \cong \sigma_k(X) \times \sigma_k(Y)$.*

Let X and Y be relational structures with base point x_0 and y_0. Consider any $f : \mathbb{F}^k \to X \times Y$ representing an arbitrary element of $\sigma_k(X \times Y)$. Let $\pi_1 \circ f = f_1$ and $\pi_2 \circ f = f_2$ where π_1 and π_2 are projection maps. Suppose that $g : \mathbb{F}^k \to X \times Y$ represents the same element of $\sigma_k(X \times Y)$ as f, i.e.,

$$[f] = [g] \in \sigma_k(X \times Y),$$

Then there is an $\mathcal{R}$-morphism (cf. Larose and Tardif [29]) $H : \mathbb{F}^k \times \mathbb{F}_m \to X \times Y$ for some m such that

$$
\begin{aligned}
H(t, 0) &= f(t) \ \forall \ t \\
H(t, m) &= g(t) \ \forall \ t \\
H(t, s) &= (x_0, y_0) \in X \times Y \text{ whenever } \prod_{i-1}^{k} t_i(t_i - N) = 0.
\end{aligned}
$$

Then we have that

$$f_1 \sim g_1 \quad \text{via the homotopy } \pi_1 \circ H \text{ and}$$
$$f_2 \sim g_2 \quad \text{via the homotopy } \pi_2 \circ H.$$

Therefore, there is a well defined map

$$\phi : \sigma_k(X \times Y) \to \sigma_k(X) \times \sigma_k(Y)$$

defined by

$$\phi([f]) = (\pi_1 \circ [f], \pi_2 \circ [f]) = ([\pi_1 \circ f], [\pi_2 \circ f]) = ([f_1], [f_2])$$

We prove that this map is an isomorphism.

Proof. That ϕ is a group homomorphism if $k \geq 1$ is obvious. To the map $f : \mathbb{F}^k \to X \times Y$ a representative of one of the components of $\sigma_k(X \times Y)$, we assign $(\pi_1 \circ f, \pi_2 \circ f)$. Note that $\pi_1 \circ f : \mathbb{F}^k \to X$, hence $[\pi_1 \circ f] \in \sigma_k(X)$. Similarly, $[\pi_2 \circ f] \in \sigma_k(Y)$. Hence we have $([\pi_1 \circ f], [\pi_2 \circ f]) \in \sigma_k(X) \times \sigma_k(Y)$. To the pair of maps $(f_1, f_2) : (\mathbb{F}^k, \mathbb{F}^k) \to (X, Y)$, i.e., $([f_1], [f_2]) \in \delta_k(X) \times \delta_k(Y)$, we assign a map $j : t \longmapsto (f_1(t), f_2(t))$ where $t \in \mathbb{F}^k$. Note that $[j]$ may be regarded as a component of $\sigma_k(X \times Y)$. Consider the maps $[f] \longmapsto [\pi_1 \circ f]$ and $[f] \longmapsto [\pi_2 \circ f]$. These maps are transformed by the evident bijection above into the maps $[(f_1, f_2)] \longmapsto [f_1]$ and $[(f_1, f_2)] \longmapsto [f_2]$. Note that these are maps from $\sigma_k(X \times Y)$ onto $\sigma_k(X)$ and $\sigma_k(Y)$ respectively. This completes the proof. $\square$

4.3 Barycentric subdivision

Given a relational structure (X, θ), an object in $\mathcal{R}$. By an ordered k-tuple of (X, θ), we mean a $\mathcal{R}$ morphism $x : A_k \to X$ where $A_k = \{1, 2, .., k\}$ for $k \in \mathbb{N}$ with the usual order in $\mathbb{N}$. The map *"point"* x is an increasing ordered k-tuple

and can be written as a *"cordinate"* on X, thus

$$x = (x_1, x_2, ..., x_k).$$

Note that the set $\{x_1, x_2, ..., x_k\}$ is a totally ordered subset of X and that $x_i \to x_k$ for all $i \in A_k$. In general, the relation θ is not antisymmetric, hence some entries may be repeated.

Definition 4.3.1. Let X be a relational structure, then its *barycentric subdivision* X' is the set of all finite chains in X.

Then X' is a poset, the relation being given by subset inclusion. Here, a finite chain is the image of an $\mathcal{R}$-morphism $h : A \to X$, where A is a finite subset of the poset $\mathbb{N}$

Proposition 4.3.2. (cf. [26]) *The association $X \to X'$ is a functor from the category $\mathcal{R}$ to* **Pos**.

Proof. For an $\mathcal{R}$-morphism $g : X \to Y$ we define a morphism of posets $g' : X' \to Y'$ as follows. For a finite chain C in X, we let $g'(C) = \{g(x) : x \in C\}$. Then $g'(C)$ is a chain in Y, moreover, if $C, D \in X'$ and $C \subset D$, then $g'(C) \subset g'(C)$. Thus g' is well defined and is a morphism of posets. $\square$

Larose and Tardif [29] further show that for each poset X, there is a natural isomorphism

$$\Delta_k : \sigma_k(X, *) \to \pi_k(TX, *)$$

where TX is the associated topological space. In this way, we have that the associated concept of weak homotopy equivalence can be constructed in $\mathcal{R}$. They give a proof that for (X, θ) a connected structure, and X' its barycentric subdivision, there is a weak $\mathcal{R}$-homotopy equivalence

$$\beta : X' \to X.$$

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Hence [26] if $f : (X, \theta) \to (Y, \theta)$ is a map of pointed relational structures, there is a diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
{\scriptstyle \beta X}\downarrow & & \downarrow{\scriptstyle \beta Y} \\
X & \xrightarrow{f} & Y
\end{array}
$$

which in general might not be commutative at the level of maps. But the corresponding diagram

$$
\begin{array}{ccc}
\sigma(X') & \xrightarrow{\sigma(f')} & \sigma(Y') \\
{\scriptstyle \sigma(\beta X)}\downarrow & & \downarrow{\scriptstyle \sigma(\beta Y)} \\
\sigma(X) & \xrightarrow[\sigma(f)]{} & \sigma(Y)
\end{array}
$$

of induced homotopy groups and group homomorphism is commutative. With regard to this scenario, one may take a further step as what follows in the next section.
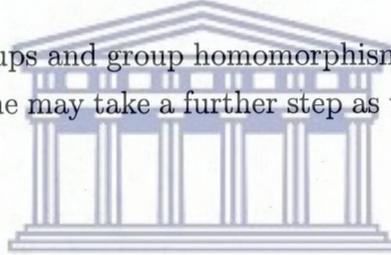
4.4 Objects of $\mathcal{R}$ as models

Hardie and Witbooi [25], Hardie *et. al.* [24], [23] use objects in $\mathcal{R}$ to model some topological concepts which would have been otherwise difficult to visualise using their classical definitions. We give a formal definition.

Definition 4.4.1. (cf. [6]) Let A be a topological space. A finite $\mathcal{R}$-object X is a *model* of A if it is weakly homotopy equivalent to X.

Let A_0 and A_1 be topological spaces and X_0 and X_1 some relational structures. Then an $\mathcal{R}$-morphism $f : X_0 \to X_1$ is a model of the **Top**-morphism $g : A_0 \to A_1$ if there are weak homotopy equivalences h_0 and h_1 making the following

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square commutative.

$$
\begin{array}{ccc}
|X'_0| & \xrightarrow{\;|f'|\;} & |X'_1| \\
{\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle h_1} \\
A_0 & \xrightarrow{\;g\;} & A_1
\end{array}
$$

The underlying reasoning is that ([26]), an object X of $\mathcal{R}$ becomes a model of a polyhedron $|\mathcal{K}X'|$ via the homotopy equivalence $|\mathcal{K}X'| \xrightarrow{qX'} X' \xrightarrow{\beta X} X$. Likewise, a map of polyhedron $|\mathcal{K}f'| : |\mathcal{K}X'| \to |\mathcal{K}Y'|$ can be modelled by the $\mathcal{R}$ morphism $f : X \to Y$ where f' is the barycentric subdivision of the map f.

4.4.1 Notation

In what follows in the thesis, we adopt the following notation. Recall that in Section 3.2, we used the letter $\mathbb{S}$ to mean the suspension of a space. To avoid some confusion, we write S^n to mean the n-sphere, whereas S_n represents an n-point model of the circle.
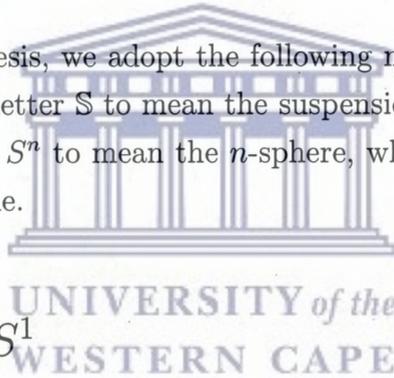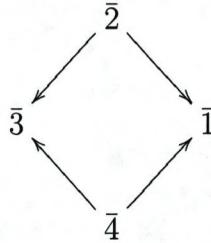
4.5 Model of S^1

As an example, we illustrate that there is a weak homotopy equivalence between a 4-point model and the 1-circle S^1. We construct the 4-point model as follows: From the fence $\mathbb{F}_4$ below,
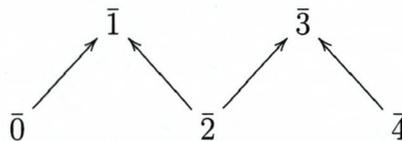
we identify the points $\bar{0}$ and $\bar{4}$ to obtain $\mathbb{F}_4/_{(0=4)}$

$$
\begin{array}{ccc}
 & \bar{2} & \\
\swarrow & & \searrow \\
\bar{3} & & \bar{1} \\
\nwarrow & & \nearrow \\
 & \bar{4} & \\
\end{array}
$$

Denote this set by $Y = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. This model is a poset and recall that in Section 2.8, we can functorially associate a topological space that admits a T_0 separation to it. We define a map f from the Hausdorff space $S^1 = X$ onto the finite model Y as follows.

$$
f(\cos t, \sin t) = \begin{cases} \bar{1} & , t = 0 \\ \bar{2} & , 0 < t < \pi \\ \bar{3} & , t = \pi \\ \bar{4} & , \pi < t < 2\pi \end{cases}
$$

We show that $f : X \to Y$ defined in this way is a weak homotopy equivalence. To illustrate this, we consider open sets

$$
\begin{aligned}
X_1 &= \{(\cos t, \sin t) : 0 < t < \frac{3\pi}{2}\} \text{ and} \\
X_2 &= \{(\cos t, \sin t) : 0 < t < \frac{\pi}{2} \cup \pi < t \leq 2\pi\} \quad \text{on} \quad S^1.
\end{aligned}
$$

We have $X_1 \cup X_2 = X$ and

$$
\begin{aligned}
X_1 \cap X_2 &= \{(\cos t, \sin t); 0 < t < \frac{\pi}{2} \cup \pi < t < \frac{3\pi}{2}\} \\
&= \{(\cos t, \sin t); 0 < t < \frac{\pi}{2}\} \cup \{(\cos t, \sin t); \pi < t < \frac{3\pi}{2}\}.
\end{aligned}
$$

On the finite model Y, we consider the open sets

$$Y_1 = \{\bar{2}, \bar{3}, \bar{4}\} \text{ and}$$
$$Y_2 = \{\bar{2}, \bar{1}, \bar{4}\}.$$

$$\text{Then } Y_1 \cup Y_2 \;=\; Y \text{ and}$$
$$Y_1 \cap Y_2 \;=\; \{\bar{2}, \bar{4}\}.$$

The open sets X_1, X_2, Y_1 and Y_2 are contractible, hence $f \mid_{X_1} \colon X_1 \to Y_1$ and $f \mid_{X_2} \colon X_2 \to Y_2$ are weak homotopy equivalences. The subset $X_1 \cap X_2$ is a disjoint union of two contractible sets, each mapped onto some contractible subset of $Y_1 \cap Y_2$, hence $f \mid_{X_1 \cap X_2} \colon X_1 \cap X_2 \to Y_1 \cap Y_2$ is a weak homotopy equivalence. We note that X_1, X_2 are open sets in X and Y_1, Y_2 are open sets in Y and that $X_1 \cup X_2 = X$ and $Y_1 \cup Y_2 = Y$.

By Theorem 3.2.5, we have that the map $f : S^1 = X \to Y$ as defined on the entire X is a weak homotopy equivalence. In this way, we have shown that there is a weak homotopy equivalence from S^1 to our four point model $\mathbb{F}_4 /_{(0=4)}$. It should be noted as elaborated in the context of minimal finite models in [6] that this construction is one of the many possible models weakly homotopy equivalent to S^1.

Chapter 5

Closure spaces

In this Chapter, we discuss closure spaces. Closure spaces have been studied by many authors including E.Čech [10]. Our interest is particularly in the relation between closure spaces and binary relational structures. In Section 5.1, we give a brief note on closure operators. Similar to J. Slapal in [44], we further show that a binary relation induces a closure on a relational structure. Conversely. There is [44] a Galois correspondence between closure spaces and α-nary relational structures. In this line, we give (Section 5.2) some mutually inverse functors between the category of closure spaces and that of relational structures. In Section 5.3 we study the notion of a quotient on a closure space. Here, the quotient is constructed by way of the relation induced by the closure operator.

5.1 Closure operators

Let U be some set of interest. Recall that 2^U denotes the set of all subsets of U. We define a closure operator.

Definition 5.1.1. A map $\alpha : 2^U \to 2^U$ is said to be a *closure operator* if it satisfies the following axioms:

(i) $A \subseteq \alpha A$, (extensionality),

(ii) $A \subseteq B$ implies $\alpha A \subseteq \alpha B$ (monotonicity).

Definition 5.1.2. The pair (X, α) where α is a closure operator defined on the set X is said to be *closure space*.

A closure space is a generalisation of topological space. If in addition, α satisfies the following axioms,

(iii) $\alpha\alpha A = \alpha^2 A = \alpha A$ (idempotence),

(iv) $\alpha\emptyset = \emptyset$

(v) $\alpha(A \cup B) = \alpha(A) \cup \alpha(B)$,

then the five axioms collectively are known as the Kuratowski axioms. In this case, this defines a unique topology on U. Let a subset A of U be said to be closed if and only if $\alpha A = A$. We have that each Kuratowski closure operator α gives a unique topology $\tau = \{A' \subseteq U | A - A' \text{ is closed } \}$ on U.

We denote by **Clo** the category of closure spaces and continuous maps. Here, continuity of a map $f : (X, \alpha) \to (Y, \beta)$ is in the sense that $f(\alpha A) \subseteq \beta f(A)$ whenever $A \subset X$. When a closure operator satisfies axiom (iii) of Definition 5.1.1, declaring the open sets to be the collection $\tau = \{A' \in U | A - A' \text{ is closed } \}$ is a functorial association from the category **Clo** of closure spaces and continuous maps to the category **Top** of topological spaces and continuous maps.

Definition 5.1.3. Let α and β be closure operators on A, then $\alpha \leq \beta$ if and only if $\alpha A \subseteq \beta A$.

In the usual way, take a binary relation θ on a set U with $\theta \subset U \times U$ to mean the pair (U, θ). Let α be a closure operator on U. Similar to J. Slapal in [44], we define a binary operator on U as follows:

$$b(\alpha) = \{(x, y) \in U^2 : y \in \alpha\{x\}\}$$

Clearly, the construction $b(\alpha)$ is a binary relation on U with respect to α. This is because $xb(\alpha)y$ always implies x is in the closure of y. For a binary relation

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θ on U, let $f_\theta : 2^U \longrightarrow 2^U$ be the map

$$f_\theta A = A \cup \{y \in U : \text{there exist } x \in A \text{ such that } x\theta y\}.$$

Proposition 5.1.4. *The map $f_\theta A = A \cup \{y \in U; \exists\, x \in A : x\theta y\}$ defines a closure operation on U.*

Proof. We show that f_θ satisfies all the closure axioms.

(i) $f_\theta \emptyset = \emptyset \cup \{y \in U; \exists x \in \emptyset \text{ such that } x\theta y\} = \emptyset \cup \emptyset = \emptyset$

(ii) Let $x \in A$, then by definition of $f_\theta A$, if $x \in A$, then $A \subseteq f_\theta A$

(iii) Suppose $A \subseteq B \subseteq U$, then by (ii) we have $A \subseteq f_\theta A$ and $B \subseteq f_\theta B$. $x \in f_\theta A$ implies $x \in A \cup \{y \in U; \exists x \in A : x\theta y\}$ but $A \subseteq B$, hence $x \in B \cup \{x \in U; \exists x \in B \ni x\theta y\}$ but this simply means $x \in f_\theta B$. Thus we have $f_\theta A \subseteq f_\theta B$.

$\square$
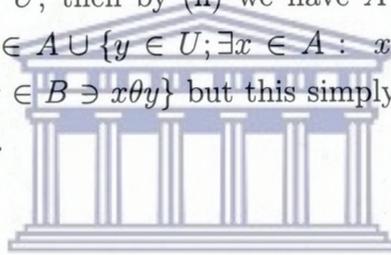
We have a result following [44].

Proposition 5.1.5. *For a set U, we have the following.*

(i) $f_{b(\alpha)} \leq \alpha$ for each closure operation α on U and that,

(ii) $\theta \subseteq b(f_\theta)$ for any binary operation θ on U.

Proof. To prove (i), we need to show that for $A \subseteq U$, $f_{b(\alpha)} \subseteq \alpha A$. Let $a \in f_{b(\alpha)}A$, then by definition of $f_{b(\alpha)}A$, we have $a \in A \cup \{y \in U; \exists x \in A : xb(\alpha)y\}$. That is either $a \in A$ or $a \in \{y \in U : \exists x \in A : xb(\alpha)y\}$. Since $A \subseteq \alpha A$ and $\{y \in U; \exists x \in A : xb(\alpha)y\} \subseteq \alpha A$ then their union $f_b(\alpha)$ must be contained in αA. Thus $a \in \alpha A \Rightarrow f_{b(\alpha)}A \subseteq \alpha A$ and that $f_{b(\alpha)} \leq \alpha$. We prove (ii) as follows, let θ be a binary operation on U. Suppose $(x, y) \in \theta \subseteq U^2$. We have that

$$bf_\theta = \{(x,y) \in U^2; xR_{f_\theta}y \Leftrightarrow y \in f_\theta\{x\}\}$$
$$= \{(x,y) \in U^2; xR_{f_\theta}y \Leftrightarrow y \in \{\{x\} \cup \{p \in U; \exists x \in \{x\} : x\theta p\}\}$$
$$= \{(x,y) \in U^2; xR_{f_\theta}y \Leftrightarrow y = x\}$$
$$\cup \{(x,y) \in U^2; xR_{f_\theta}y \Leftrightarrow y \in \{p \in U; \exists x : x\theta p\}\}$$

Thus $(x,y) \in b(f_\theta)$ and we have that $\theta \subseteq b(f_\theta)$. $\qquad\square$

5.2   Relation between Clo and $\mathcal{R}$

Galois correspondences in the context of sets with an order are well known and have a frequent occurrence not only in mathematics but have found some diverse applications in fields such as quantum physics [9], computer science [43] and systems biology [20]. In this section, we discuss the Galois correspondence between the categories **Clo** and $\mathcal{R}$. For each object $(X, \alpha) \in$ **Clo**, put $G(X, \alpha) = (X, b(\alpha))$, where $b(\alpha)$ is as already defined. For each object (X, θ) in $\mathcal{R}$, put $F(X, \theta) = (X, f_\theta)$ where f_θ is kept as in Proposition 5.1.4. Thus F so defined is a functor from the category of binary relational structures to the category of closure spaces. The association G is a functor from the category of closure spaces to the category of binary relational structures. We first recall the concept of Galois correspondence in the context of posets.

Definition 5.2.1. Let $(M, \leq_\gamma)$ and $(N, \leq_\varphi)$ be posets. A *Galois correspondence* between them consist of two order preserving functions $F^* : M \rightarrow N$, $G^* : N \rightarrow M$ such that $F^*(m) \leq_\varphi n$ if and only if $m \leq_\gamma G(n)$ for $m \in M, n \in N$.

These two functions uniquely determine each other. Actually, one can go further and consider these functions as specifications of one and the same object.

Proposition 5.2.2. *The functors $F : \mathcal{R} \rightarrow$ Clo and $G :$ Clo $\rightarrow \mathcal{R}$ preserve the underlying sets of objects and the underlying morphisms.*

Proof. For each object $A = (X, \alpha)$ of **Clo**, the object $GA = G(X, \alpha)$ is of the form (X, α^*), with $\alpha^* = b(\alpha)$ in $\mathcal{R}$. Let $f : (X, \alpha) \to (Y, \beta)$ be a continuous map, then $Gf : (X, \alpha^*) \to (Y, \beta^*)$ with $\alpha^* = b(\alpha)$, $\beta^* = b(\beta)$ in $\mathcal{R}$. For each object $B = (X, \theta)$ in $\mathcal{R}$, the object $FB = F(X, \theta)$ is of the form (X, θ^*) with $\theta^* = f_\theta$. Each relational homomorphism $f : (X, \theta) \to (Y, S)$, $Ff : (X, \theta^*) \to (Y, S^*)$ with $\theta^* = f_\theta$ and $S^* = f_S$. This establishes a Galois correspondence between **Clo** and $\mathcal{R}$. $\qquad\square$

5.3 Quotient construction in Clo and $\mathcal{R}$

It is interesting that one can extend in a natural way the construction of quotients on sets to that of closure spaces and relational structures. We first recall some of the basic concepts of quotient construction on sets.

Definition 5.3.1. Let $q : X \to Y$ be a surjective map between topological spaces X and Y. Then q is said to be a *quotient map* if and only if a subset B of Y is open whenever $q^{-1}(B)$ is open in X.

Equivalently this is true when the open sets are replaced by closed sets. The relation between quotient maps and continuous maps is as follows:

Suppose $q : X \to Y$ is a quotient map and $f : X \to Z$ is a continuous map constant on each set $q^{-1}(B)$ with $B \subset Y$ and that

$$
\begin{array}{ccc}
X & & \\
{\scriptstyle q}\downarrow & \searrow{\scriptstyle f} & \\
Y & \xrightarrow[g]{} & Z
\end{array}
$$

then there is a unique continuous map $g : Y \to Z$ such that $g \circ q = f$, and we say that g is induced by f.

The notion of a quotient map helps us construct a quotient space. Let the space $X^\sharp$ be the partition of some space X into disjoint subsets whose union

is X. Let $\pi : X \to X^\sharp$ map each point of X to some subset containing it in $X^\sharp$. We turn X into a topology by declaring some subset B of $X^\sharp$ to be open if and only if $\pi^{-1}(B)$ is open in X, hence π is a quotient map. The space so defined is a quotient topology and is the finest topology for which the map π is continuous. $X^\sharp$ together with this topology is called the quotient space.

Definition 5.3.2. Let $q : (X, \mu) \to (Y, v)$ be a **Clo** morphism. Then the map q is said to be a quotient map in **Clo** if it satisfies the following property. Given any $A \subseteq Y$, then $v(A) = q(u(q^{-1}(E)))$.

We have the following result.

Theorem 5.3.3. (cf. Hardie and Witbooi [26]) *Let (X, u) be any closure space and Y any set. Consider any surjective function $f : X \to Y$ of sets. For any subset B of Y, let $vB = fuf^{-1}(B)$. Then the following holds.*

(a) *The operation v is a closure operation on Y.*

(b) *The induced map $f : (X, u) \to (Y, v)$ is continuous.*

(c) *Given any closure space (Z, w) and a function $g : Y \to Z$ for which $g \circ f$ is a continuous $f \circ g : (X, u) \to (Z, w)$, then g is continuous.*

Proof. Let B be a subset of Y, then $f^{-1}(B) \subseteq X$ for every $f : X \to Y$ and $B \subseteq Y$. That u is a closure operator on X implies that $f^{-1}(B) \subseteq uf^{-1}(B)$. Let $y \in f^{-1}(B) \Rightarrow y = f(x) \in f[uf^{-1}(B)]$ for some $x \in uf^{-1}(B)$. That is, $f[f^{-1}(B)] \subseteq f[uf^{-1}(B)]$. Note that this implies $B \subseteq vB$, hence, the closure axiom (i) is satisfied. Let B_1, B_2 be subsets of Y with $B_1 \subseteq B_2$. Then $f^{-1}(B_1) \subseteq X$ and for each $x \in f^{-1}(B_1)$, there is a $y \in B_2$ such that $x = f^{-1}(y) \subseteq f^{-1}(B_2)$. Thus $f^{-1}(B_1) \subseteq f^{-1}(B_2)$. Since u is a closure operation on X, we have $uf^{-1}(B_1) \subseteq uf^{-1}(B_2)$. For each $y \in f[uf^{-1}(B_1)]$, we have some $x \in uf^{-1}(B_2)$ such that $y = f(x) \subseteq f[uf^{-1}(B_2)]$. Thus $[fuf^{-1}(B_1)] \subseteq f[uf^{-1}(B_2)]$ and we have that $B_1 \subseteq B_2 \Rightarrow vB_1 \subseteq vB_2$, this satisfies closure axiom (ii). We show that the induced map f is continuous as follows; Let $A \subset X$, then $vf(A) = fuf^{-1}[f(A)] = fu(f^{-1}(f(A))) = f(uA)$ thus (b) holds.

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To prove part (c), consider any set $B \subset X$. We have $g(vB) = g(fuf^{-1}(B)) = g \circ f(uf^{-1}(B)) = g \circ f(uf^{-1}(B)) \subseteq w(g \circ f(f^{-1}(B)))$. The last inclusion is as a result of the continuity of $g \circ f$. Furthermore we have $w(g \circ f(f^{-1}(B))) = wg(f(f^{-1}(B))) = wg(B)$. Hence, for every $B \subset X$, we have $g(vB) \subset wg(B)$. Therefore g is continuous. $\square$

Definition 5.3.4. Let $q : (X, \theta_X) \to (Y, \theta_Y)$ be a $\mathcal{R}$- morphism. The map q is a *quotient map* (in $\mathcal{R}$) if for any points $y_1, y_2 \in Y$, $(x_1, x_2) \in \theta_Y$ if and only if there exists $x_1 \in q^{-1}(y_1)$ and $x_2 \in q^{-1}(y_2)$ such that $(x_1, x_2) \in \theta_X$.

Consider a closure space X with u a closure operator on X. Let an equivalence relation $\sim$ on X be defined as follows;

$$x \sim y \Leftrightarrow x \in u\{y\} \text{ for each } x, y \in X^*.$$

Then the quotient $X = (X^*, \sim)$ is called an equiclosure space. The collection of equiclosure spaces and maps of objects $X \to Y$ being functions $X/\sim \to Y/\sim$ such that $X^* \to Y^*$ is a continuous map of closure spaces forms a category **Eclo** of closure spaces.

In the case where $X = (X^*, \sim)$ with X^* a topological space, we say that X is an equilogical space. The object X is in the category **Eql** of the equilogical spaces. **Eql** forms a full subcategory of **Eclo**. This is because, by definition of the morphisms between X and Y, we have $\text{Hom}_{\mathbf{Eql}}(X, Y) = \text{Hom}_{\mathbf{Eclo}}(X, Y)$.

It is interesting to note that we have an interrelation between closure spaces and relational structures. That a closure operator induces a relation on a closure space. Similarly, a relational structure enjoys some closure properties. Recall (Section 4.1 and Section 4.5) that one may construct models by taking quotients on a relational structure. Furthermore, in the case where the relational structure satisfies a topological property (T_0 in our case), we may regard the model as an object in the category **Eql**. The relation between **Eql** and **Eclo** present an avenue to understand constructions that arise out of taking quotients, for instance models in $\mathcal{R}$.

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Chapter 6

Category with a cylinder object

In this chapter, we discuss the notion of a cylinder object in a category. In particular, a non-Hausdorff version of a cylinder object. The main references are K.A Hardie *et al.* [23], K.H. Kamps and T. Porter [28] and P.J Witbooi [51].

6.1 The cylinder object

Recall in Definition 3.1.1 that a homotopy between functions $f, g : X \to Y$ in **Top** is a map of the form:

$$H : X \times I \to Y \text{ such that } H(x,0) = f(x), H(x,1) = g(x) \ \ \forall \ x \in X.$$

The underlying concept is that the homotopy theory in **Top** is induced by constructing the cylinder $X \times I$ on a topological space X and then finding an extension for the restriction of the functions to the two ends of $X \times I$. Further, defining a collapsing map taking the points (x,t) to x for all $x \in X$, $t \in I$.

Definition 6.1.1. (cf. Kamps [28]) A *cylinder functor* $\mathcal{I}$, on a category $\mathcal{C}$ is

a functor

$$\_ \times I : \mathcal{C} \to \mathcal{C}$$

together with three natural transformations.

$$i_0 : Id_{\mathcal{C}} \quad \to \quad \_ \times I$$
$$i_1 : Id_{\mathcal{C}} \quad \to \quad \_ \times I$$
$$\sigma : \_ \times I \quad \to \quad Id_{\mathcal{C}}$$

such that $\sigma \circ i_0 = \sigma \circ i_1 = Id_{\mathcal{C}}$.

In **Top**, taking the unit interval $I = [0, 1]$, we have that the space $X \times [0, 1]$ is a cylinder on a topological space X. The cylinder functor in this case is

$$X \times [0, 1] : \textbf{Top} \to \textbf{Top}$$

where the natural transformations are given by the restrictions of the functions to the two ends of the cylinder

$$i_0(x) = (x, 0),$$
$$i_1(x) = (x, 1),$$
and the collapsing map $\sigma(x, t) = x.$

We define the notion of a non-Hausdorff double mapping cylinder.

In the subcategory **Fpos** of $\mathcal{R}$, this construction can have the following interpretation. Let A be a finite poset, with order $\leq$ on it. We first replace the unit interval by some poset

$$\mathbb{I} = 0 \leftarrow \frac{1}{2} \to 1$$

The cylinder in this context is

$$A \times \mathbb{I} : \mathcal{R} \to \mathcal{R}$$

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where the natural transformations are given by

$$*i_0 : a \mapsto 0$$
$$*i_1 : a \mapsto 1$$
$$*\sigma : (a, t) \mapsto a$$

with some additional relations $*\sigma(a, t) \leq *i_0(a)$ and $*\sigma(a, t) \leq *i_1(a)$ for all $a \in A$ and $t \in \mathbb{I}$. Here, the points 0 and 1 are understood to be the two ends of the non-Hausdorff cylinder.

Definition 6.1.2. (cf. P. Witbooi [23]) Let $f_1 : A \to X$ and $f_2 : A \to Y$ be a pair of poset maps, then the *non-Hausdorff double mapping cylinder* of f_1 and f_2 denoted $\mathbb{M}(f_1, f_2)$ is the poset obtained from the disjoint union $X + A + Y$ of finite posets by imposing some additional relations $a \leq f_1(a), a \leq f_2(a) \ \forall \ a \in A$.

Remark 6.1.3. Note that the non-Hausdorff double mapping cylinder $\mathbb{M}(f_1, f_2)$ is a quotient space obtained from the disjoint union $X + A + Y$.

The poset $\mathbb{I}$ can be related to the usual unit interval $[0, 1]$ in $\mathbb{R}$ in the following way: Witbooi in [49] (proposition 3.2) defines a map $h : [0, 1] \to \mathbb{I}$ where

$$0 \mapsto 0, \ (0, 1) \mapsto \frac{1}{2}, \ 1 \mapsto 1$$

which can be looked at as a homotopy from $[0, 1]$ to $\mathbb{I}$. It was proved in [33] that for each cotriad in **Top**, the map h induces a weak equivalence from the ordinary double mapping cylinder to the non-Hausdorff double mapping cylinder. We shall return to these concepts in Chapter 7 where following [25], [24], [23] and [49], we give some examples of non-Hausdorff mapping cylinders.

Chapter 7

Pairings in a category

7.1   Introduction
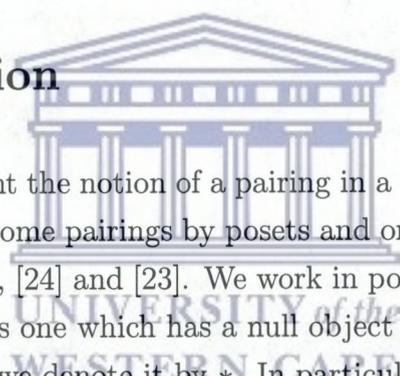
In this chapter, we present the notion of a pairing in a category, following Oda [36]. Further, we model some pairings by posets and order preserving maps as from Hardie *et al.* in [25], [24] and [23]. We work in pointed categories. Recall that a pointed category is one which has a null object (i.e the initial and final objects are isomorphic); we denote it by $*$. In particular, we will consider the objects of the subcategory $\mathbf{LFT_0}$ of $\mathbf{Top}$. By way of the interrelations between $\mathbf{Top}$, $\mathbf{LFT_0}$, $\mathbf{FPos}$ and $\mathcal{R}$ (Section 2.8, Theorem 2.11.3 and Proposition 4.3.2), the models of the pairings are in the category of relational structures and $\mathcal{R}$-morphisms.

Before we discuss the concept of a pairing in a category, we first present in Section 7.2 the notion of homotopy relation in a category. Further, we recall how the product relates to the pushout in a category.

7.2 Preliminaries

Definition 7.2.1. (cf. Oda [38]) A *homotopy relation* in a category $\mathcal{C}$ is an equivalence relation $\simeq$ among morphisms $f, g : X \to Y$ which satisfy the following conditions. If $f \simeq g$ then $h \circ f \simeq h \circ g$ for any morphism $h : Y \to Z$, and that $f \circ w \simeq g \circ w$ for any morphism $w : W \to X$.

This is more general than the *homotopy* discussed in Chapter 3, since in Chapter 3, as we were only restricted to topological spaces.

Definition 7.2.2. Objects X, Y in the category $\mathcal{C}$ are said to be of the same homotopy type when there exist some morphisms $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$.

Definition 7.2.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories equipped with homotopy relations. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *homotopy preserving functor* if whenever f, g are morphisms in the category $\mathcal{C}$ such that $f \simeq g$ then $F(f) \simeq F(g)$.

Given categories, one can define a homotopy relation on their product which is induced in a natural way by the respective homotopy relations on the categories. The relation is in the following result.

Proposition 7.2.4. (Oda [36]) *Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories with homotopy relation. Then their respective homotopy relations induce a homotopy relation on the product category $\mathcal{C} \times \mathcal{D}$.*

Proof. Let $f_1 \simeq f_2$ in $\mathcal{C}$ where $f_1, f_2 : C \to C'$ and

$$g_1 \simeq g_2 \text{ in } \mathcal{D} \text{ where } g_1, g_2 : D \to D'.$$

Then for any other morphisms $h : C' \to Z$, $w : W \to C$, $u : D' \to Y$ and $v : X \to D$, we have that $(f_1, g_1) \simeq (f_2, g_2)$ will imply

$$(h, u) \circ (f_1, g_1) = (h \circ f_1, u \circ g_1) \simeq (h \circ f_2, u \circ g_2) = (h, u) \circ (f_2, g_2).$$

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We therefore have that the homotopy relation $(h \circ f_1, u \circ g_1) \simeq (h \circ f_2, u \circ g_2)$ is induced by the respective homotopy relations in $\mathcal{C}$ and $\mathcal{D}$. $\square$

In general, one can define a pairing in a category as in the case of [38], if the category has a pseudo product and pseudo-coproduct. In this thesis, we work in a category which has product, coproduct and zero object. The pseudo-product is given by the product, the pseudo-coproduct in given by the coproduct [37]. We recall some details on how the object $X \times Y$ relates to $X \vee Y$ the one point union (pushout) in a category. Let the morphism which factors through the null object $*$ be denoted by $* : X \to Y$, thus $* : X \to * \to Y$.

Write $\triangle_X : X \to X \times X$ to mean the *diagonal map*, as in the pullback diagram
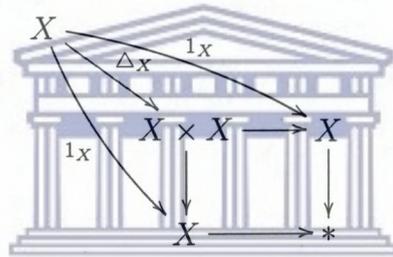
Dually, we write the *folding map* $\nabla_X : X \vee X \to X$ as in the pushout diagram
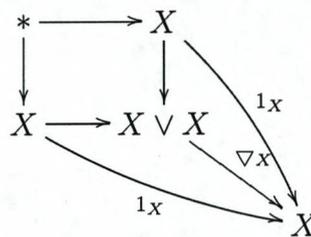
For instance, restricted to categories which are constructs, one may write $\triangle_X(x) = (x, x)$ and the dual $\nabla_X(x, *) = * = \nabla_X(*, x)$ for every $x \in X$.

Consider the following for the product:

$$X \underset{i_1}{\overset{p_1}{\rightleftarrows}} X \times Y \underset{i_2}{\overset{p_2}{\rightleftarrows}} Y$$

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We have $i_1(x) = (x, *)$ and $i_2(y) = (*, y)$ are inclusions, $p_1(x, y) = x$ and $p_2(x, y) = y$ are projections for any $x \in X$ and $x \in Y$. These maps satisfy the following:

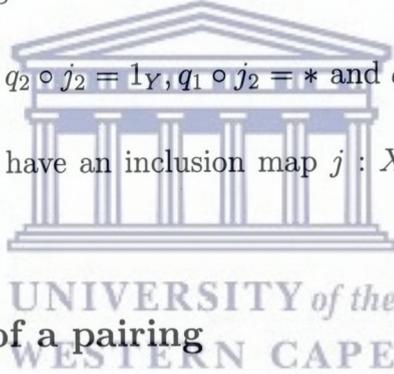$$p_1 \circ i_1 = 1_X, p_2 \circ i_2 = 1_Y, p_1 \circ i_2 = * \text{ and } p_2 \circ i_1 = *.$$

Similarly, for the one point union,

$$X \underset{j_1}{\overset{q_1}{\rightleftarrows}} X \vee Y \underset{j_2}{\overset{q_2}{\rightleftarrows}} Y$$

where $j_1(x) = (x, *)$ and $j_2(y) = (*, y)$ are inclusions, $q_1(x, *) = x, q_1(*, y) = *$ and $q_2(x, *) = *$, $q_2(*, y) = y$ are projections for any $x \in X$ and $y \in Y$. These maps satisfy the following:

$$q_1 \circ j_1 = 1_X, q_2 \circ j_2 = 1_Y, q_1 \circ j_2 = * \text{ and } q_2 \circ j_1 = *.$$

It is easy to see that we have an inclusion map $j : X \vee Y \to X \times Y$ where $j = (i_1 \circ q_1, i_2 \circ q_2)$.

7.2.1 Definition of a pairing

Definition 7.2.5. If X, Y and Z are objects in a pointed category, then a morphism $\mu : X \times Y \to Z$ is a *pairing* with axes $f : X \to Z$ and $g : Y \to Z$ if it satisfies

$$\mu \mid X \vee Y \simeq \nabla_Z \circ (f \vee g) \to Z,$$

that is, the diagram

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\ \mu\ } & Z \\
{\scriptstyle j}\big\uparrow & & \big\uparrow{\scriptstyle \nabla_Z} \\
X \vee Y & \xrightarrow[f \vee g]{} & Z \vee Z
\end{array}
$$

is homotopy relation commutative.

In the case where we have a multiplication $m : X \times X \to X$ on a topological

space X such that m has a two sided homotopy unit, that is the diagram

$$
\begin{array}{ccc}
X \times X & \xrightarrow{\ m\ } & X \\
{\scriptstyle j}\big\uparrow & & \big\uparrow{\scriptstyle \nabla x} \\
X \vee X & \xrightarrow[\ 1_X \vee 1_X\ ]{} & X \vee X
\end{array}
$$

is homotopy relation commutative, then X is said to be a Hopf space. In what follows, we present some models of pairings.

7.3 The multiplication $\mu : S_8 \times S_8 \to S_4$

At this point, we show as presented in [24] by way of an example that the connected T_0 spaces S_4 and its barycentric subdivision S_8 admit a non-trivial pairing with axes f where $f : S_8 \to S_4$ is a weak homotopy equivalence. The model of S^1 is as in Section 4.5 where we represented S^1 by its finite four point model S_4 obtained from the quotient space $\mathbb{F}_4/_{0=4}$. For convenience, we take a slight modification of the model in Section 4.5 as follows: we write $i = \bar{3}, -i = \bar{1}, 1 = \bar{2}, -1 = \bar{4}$. This is done so that we get a multiplication reminiscent of the complex number multiplication. Hence we have
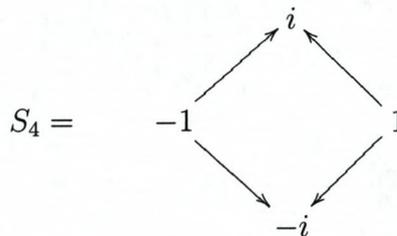
$$
S_4 = \qquad
\begin{array}{ccc}
 & i & \\
\nearrow & & \nwarrow \\
-1 & & 1 \\
\searrow & & \swarrow \\
 & -i &
\end{array}
$$

Figure 7.1:

and its barycentric subdivision $(S_4)' = S_8$ is in the diagram below.

$$S_8 = \qquad \begin{array}{ccc} -1+i \xleftarrow{\quad i \quad} 1+i \\ \end{array} \qquad = \qquad$$

In the diagram of S_8, the square on the extreme right has some dots in the place of $-1+i, 1+i, -1-i$ and $1-i$ for simplicity. The (complex number) multiplication of $S_4 \times S_4 \to S_4$ is represented by the grid in Figure 7.2. In the
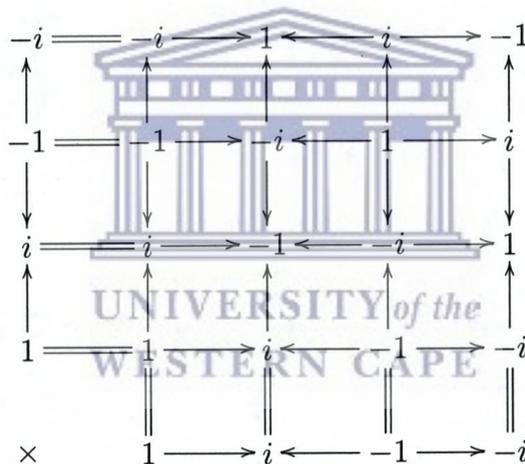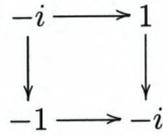

Figure 7.2:

grid, the column on the left as well as the row on the bottom each represents a copy of S_4. The double lines between the points on each of the S_4 and the respective points on the second column and the second row from down imply equality. In particular, an arrow in any direction is permissible.

Lemma 7.3.1. *The complex number multiplication $\times : S_4 \times S_4 \to S_4$ fails to be order preserving.*

Proof. On the grid (fig. 7.2) which represents the multiplication $\times : S_4 \times S_4 \to$

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S_4, consider the small square near the upper left corner of the grid,

$$-i \longrightarrow 1$$
$$\downarrow \qquad \downarrow$$
$$-1 \longrightarrow -i$$

We see for instance that $-i \to 1$ and $-i \to -1$. The directions of the arrows are not in line with the arrows between the respective points on S_4 in Figure 7.1. This clearly shows that $\times : S_4 \times S_4 \to S_4$ is not continuous. $\qquad \square$

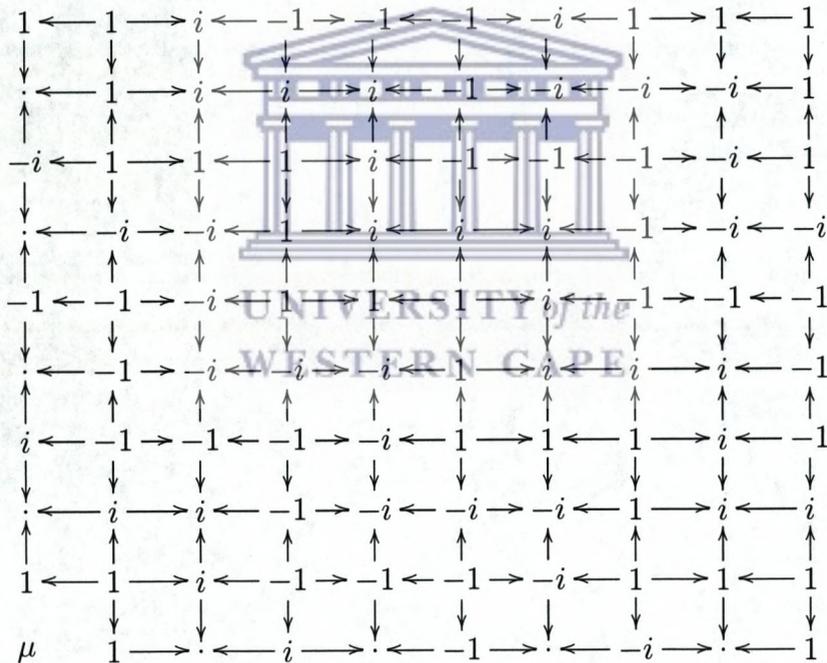We now consider a different map $\mu : S_8 \times S_8 \to S_4$. We define the map μ as follows:

Figure 7.3:

The multiplication can be explained as follows: The grid represents $\mathbb{F}_8 \times \mathbb{F}_8$ where $\mathbb{F}_8$ is the 8 point fence. When we identify 0 and 8 in $\mathbb{F}_8$, we obtain S_8. So we can view this grid as representing $S_8 \times S_8$. Now in place of (x, y) in $S_8 \times S_8$ we put $\mu(x, y)$ where of course $\mu(x, y) \in S_4$.

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Checking for associativity, we see for example that

$$\mu(\mu(1,-1),i) = 1 \neq \mu(1,\mu(-1,i)) = \mu(1,-1) = -1,$$

hence this property fails. The element 1 fails to be the strict identity for this multiplication. For example, consider $\mu(1,i) = -i \neq i$. Nevertheless, since $\mu(1,x) = \mu(x,1)$ for all $x \in S_8$, 1 can be regarded as a right and left homotopy identity. The function μ so defined is base point preserving taking the point 1 onto $1 \in S_4$. The following result appears in [24].

Lemma 7.3.2. *The function $\mu : S_8 \times S_8 \to S_4$ as described is order preserving.*

Proof. By inspecting the grid in Figure 7.3 which represents $\mu : S_8 \times S_8 \to S_4$, we observe that the arrows in the cartesian product are in line with the partial order on S_4. $\square$

Proposition 7.3.3. *The function $\mu : S_8 \times S_8 \to S_4$ defines a non trivial pairing with axes f and f, where $f : S_8 \to S_4$ is some weak homotopy equivalence.*

Proof. The left column and the bottom row in the diagram for $S_8 \times S_8$ each defines a map $S_8 \to S_4$. Note that it winds the 8-point circle once around S_4. Hence by [24] (Theorems 0.1 and 0.2), we have that the map $f : S_8 \to S_4$ is a weak homotopy equivalence. Since the map $S_8 \times S_8 \to S_4$ is continuous (Lemma 7.3.2), we have that the diagram

$$
\begin{array}{ccc}
S_8 \times S_8 & \xrightarrow{\ \mu\ } & S_4 \\
{\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \nabla S_4} \\
S_8 \vee S_8 & \xrightarrow[f \vee f]{} & S_4 \vee S_4.
\end{array}
$$

is homotopy relation commutative. This completes the proof. $\square$

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7.4 Model of Hopf map

In Section 7.3, we discussed the existence of an order preserving map $\mu :$ $S_8 \times S_8 \to S_4$. In this section, we take a step further. Using the notion of the non-Hausdorff join and suspension, we use the map μ to model the Hopf map. The Hopf map $h : S^3 \to S^2$ was the first example of a null homotopic map from a higher dimensional sphere to one with a lower dimension [14]. The construction we give closely follows that of [24]. We first give some definitions.

Definition 7.4.1. Let X be a poset. The *non-Hausdorff Cone* $\mathbb{C}X = (X, \hat{x})$ of X is the poset equipped with an additional point $\hat{x}$ as upper bound.

Definition 7.4.2. The *non-Hausdorff suspension*, $\mathbb{S}X = (X, \hat{n}, \hat{s})$ is the union of two copies of $\mathbb{C}X$ such that their intersection is X.

These constructions are analogous to the cone and suspension as defined in Section 3.2 for Hausdorff spaces. We define the non-Hausdorff join of X and Y to be the poset
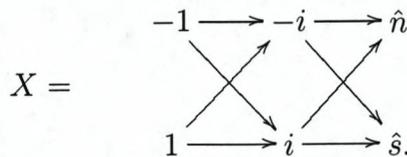
$$X \circledast Y = \mathbb{C}X \times Y \cup X \times \mathbb{C}Y.$$

Let X, Y and Z be finite posets and $\mu : X \times Y \to Z$ a pairing with axes $f : X \to Z$ and $g : Y \to Z$.

Using the non-Hausdorff join, we construct a map $\Gamma(\mu) : X \circledast Y \to \mathbb{S}Z$ as follows (cf. [24]):

$$\Gamma(\mu)(x, y) = \mu(x, y), \Gamma(\hat{x}, y) = \hat{n}, \Gamma(x, \hat{y}) = \hat{s} \ (x \in X, y \in Y).$$

Note that the sphere S^2 is weakly equivalent to the poset X displayed below:

$$X =$$

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We take $\hat{n}$ and $\hat{s}$ to denote the north and south poles on S_4. $\Gamma(\mu)$ is a map of posets in the category **Pos**. Thus we can apply the functor $\mathcal{K}$ described in 2.11.3 on $\Gamma(\mu)$. The homotopy class $|\mathcal{K}\Gamma(\mu)|$ is equivalent to the class of the Hopf construction $|\mathcal{K}\mu|$. We give a sketch to represent $\Gamma(\mu)$ where $\mu : S_8 \times S_8 \to S_4$. First observe that

$$
\begin{aligned}
S_8 \circledast S_8 &= \mathbb{C}S_8 \times S_8 \cup S_8 \times \mathbb{C}S_8 \\
&= S_8 \times S_8 \cup S_8 \times (\hat{y}) \cup (\hat{x}) \times S_8.
\end{aligned}
$$

This is a finite model of the 3-sphere. $S_8 \times S_8$ has 64 points, $S_8 \times (\hat{y})$ and $(\hat{x}) \times S_8$ have 8 points each. Thus the model has 80 points. The associated poset is given in Figure 7.4. The diagram is to be interpreted as follows: We consider
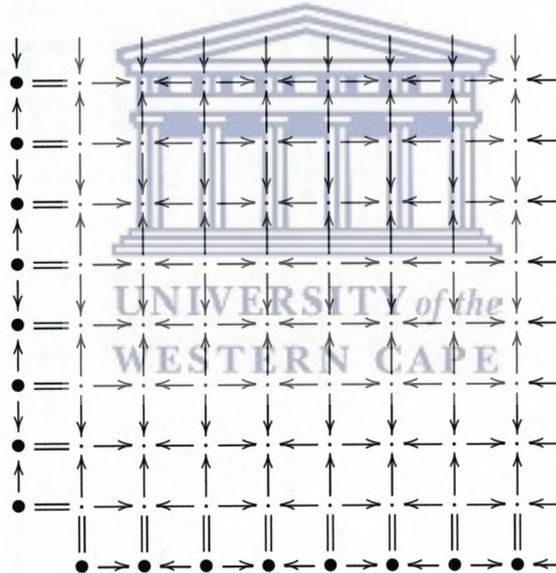

Figure 7.4:

the figure to be a $k \times l$ grid denoted by $T(k, l)$ where $1 \le k \le 9, 1 \le l \le 9$. In this case, k represents the rows and l the columns. The subposet $(\hat{x}) \times S_8$ i.e., $T(9, l)$ is represented by the larger bullets at the bottom of the diagram. The bullets on the left hand extreme of the grid i.e., $T(k, 1)$ represents the subposet $S_8 \times (\hat{y})$. The inside of the diagram i.e., $T(k, l)_{1 \le k \le 8, \, 2 \le l \le 8}$ constitutes an 8×8 grid of smaller dots which represents the product $\mu : S_8 \times S_8 \to S_4$ as discussed

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in Section 7.3. The arrows on the extreme right edge have their sources, the small dots on the left hand side of the 8×8 grid thus completing the circles. In a similar manner, the downward pointing arrows on the top edge of the 8×8 grid $T(k, l)_{1 \leq k \leq 8, \; 2 \leq l \leq 8}$ have their sources the small dots on the bottom of the grid. The subposets $S_8 \times (\hat{y})$ and $(\hat{x}) \times S_8$ each has arrows between their points whose intention is to complete the circle. The double lines between the bullets and the grid represent equality between representative points in the column (row) to the bullet in the column (row). Hence an arrow in either direction is permissible.

7.5   The construction $\Gamma(\mu)$

In [24], where the map is constructed, the function $\Gamma(\mu)$ is described as follows. Each point of the grid is sent to a respective point as described in Section 7.3. The "larger" bullets on the left edge are all identified with the point $(\hat{s})$ and the bullets on the lower edge are all identified with the point $(\hat{n})$. Hence, we have that the map $\Gamma(\mu) : S_8 \circledast S_8 \to \mathbb{S}S_4$ is induced by the diagram

$$
\begin{array}{ccccc}
S_8 & \xleftarrow{\;\pi\;} & S_8 \times S_8 & \xrightarrow{\;\pi\;} & S_8 \\
\downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow \\
\hat{n} & \xleftarrow{\;n\;} & S_4 & \xrightarrow{\;s\;} & \hat{s}
\end{array}
$$

where μ is a pairing, π_1 and π_2 are projection maps. The map $\Gamma(\mu)$ is a Hopf construction which is a Model of the Hopf map $S^3 \to S^2$.

The join $S_8 \circledast S_8 = \mathbb{C}S_8 \times S_8 \cup S_8 \times \mathbb{C}S_8$ as in Section 7.4 is a special case of the non-Hausdorff double mapping cylinder. We have that $S_8 \circledast S_8 = \mathbb{M}(\pi, \pi)$ where the projections are given by $\pi : S_8 \times S_8 \to S_8$. In relation to the model of the Hopf map $\Gamma(\mu) : S_8 \circledast S_8 \to \mathbb{S}S_4$, we have that the suspension $\mathbb{S}S_4$ is equal to $\mathbb{M}(n, s)$, where the maps $n : \mathbb{S}S_4 \to \hat{n}$ and $s : \mathbb{S}S_4 \to \hat{s}$. We have that the non-Hausdorff suspension $\mathbb{S}S_4$ is a special case of the non-Hausdorff double mapping cylinder. Hence the model of the Hopf map can be looked

at as a map $\Gamma(\mu) : \mathbb{M}(\pi, \pi) \to \mathbb{M}(n, s)$ of the non-Hausdorff double mapping cylinders.

7.6 Model of generalised Whitehead product

The Whitehead product on the homotopy groups of a space is constructed as follows. Let $[f] \in \pi_n(X)$, $[g] \in \pi_m(X)$. When it is clear from the context, it is convenient to write f in the place of $[f]$. Thus, we may represent $[f]$ and $[g]$ by

$$f : S^n \to X,$$
$$g : S^m \to X \quad \text{respectively.}$$

We then denote by

$$w = [f, g] \in \pi_{n+m-1}(X)$$

the map derived as follows: The product $S^n \times S^m$ can be obtained by attaching an $n+m$ cell to the one point union $S^n \vee S^m$ by the attaching map $\alpha : S^{n+m-1} \to S^n \vee S^m$. Thus composing the wedge sum

$$(f, g) = \beta : S^n \vee S^m \to X$$

with an attaching map α we obtain the map

$$\omega = [f, g] = \alpha \circ \beta : S^{n+m-1} \to S^n \vee S^m \to X.$$

The term $[f, g]$ is the Whitehead bracket. This resulting map has a homotopy class which does not depend on the choice of the representative maps. For example, in the fundamental group $\pi_1(X, *)$, we have that

$$\omega = [f, g] = [fgf]^{-1} \circ [g]^{-1} = [fgf^{-1}g^{-1}],$$

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and this is the usual commutator in a group. Let

$$A \circledast B = A \times \mathbb{C}B \cup \mathbb{C}A \times B \subseteq \mathbb{C}A \times \mathbb{C}B$$

denote the join of the spaces A and B as defined in section 7.4 and $A * B$ the join as in Definition 3.2.3. Then there is a natural map [5]

$$\nu : A * B \rightarrow A \circledast B, \text{ which may be defined as follows:}$$

$$\nu(a, b, t) = \begin{cases} (a, (b, 1-2t)) \ (0 \le t \le \frac{1}{2}) \\ (a, 2t-1, b) \ (\frac{1}{2} \le t \le 1) \end{cases}$$

Remark 7.6.1. In the case where A and B are polyhedra, then the map ν is a homeomorphism [13].

Following [25] definition 2.1, we give a definition of the generalized Whitehead product. In short, we shall write GWP to mean the generalized Whitehead product.

Definition 7.6.2. [25], Let $\alpha = [f] \in \pi(\mathbb{S}A, X)$, $\beta = [g] \in \pi(\mathbb{S}B, X)$ be homotopy classes. Then the GWP of α and β is defined to be $[\alpha, \beta]$, that is, the class of the composite map:

$$A * B \xrightarrow{\nu} A \circledast B = \mathbb{C}A \times B \cup A \times \mathbb{C}B \xrightarrow{w} \mathbb{S}A \vee \mathbb{S}B \xrightarrow{(f,g)} X$$

where $w \mid (A \times \mathbb{C}B) = \phi_B \pi_2$, $w \mid (\mathbb{C}A \times B) = \phi_A \pi_1$. The maps ϕ_A and ϕ_B are the natural maps $\phi_A : \mathbb{C}A \rightarrow \mathbb{S}A$ and $\phi_B : \mathbb{C}B \rightarrow \mathbb{S}B$ as defined in Section 3.2.

Definition 7.6.2 is essentially consistent with that given by Arkowitz in [5]. Let $f \in \pi(\mathbb{S}A, X)$ and $g \in \pi(\mathbb{S}B, X)$ be a pair of maps in the category **FPos** of finite posets and order preserving functions. We choose their GWP to be the map

$$A \circledast B = \mathbb{C}A \times B \cup A \times \mathbb{C}B \xrightarrow{w} \mathbb{S}A \vee \mathbb{S}B \xrightarrow{(f,g)} X.$$

To construct the Whitehead square of the circle S^2, we choose $\mathbb{S}A = \mathbb{S}B = X = (S_4)^2$ where S_4 is the four point model of S^1. Note that the suspension

of S_4 can be represented graphically as

$$\begin{array}{ccc} & j & \\ i & -1 & \\ 1 & -i & \\ & -j & \end{array}$$
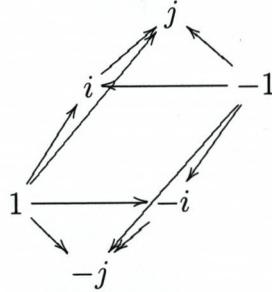
This is a finite model of the two sphere and we shall denote it by $(S_4)^2$. Note that in relation to the model of S^2 in Section 7.4, we represent the antipodal points $\hat{n}$ and $\hat{s}$ by j and $-j$ respectively. The maps f and g become identity maps $S_4 \to S_4$, thus the GWP reduces to

$$S_4 \circledast S_4 = \mathbb{C}S_4 \times S_4 \cup S_4 \times \mathbb{C}S_4 \xrightarrow{w} \mathbb{S}S_4 \vee \mathbb{S}S_4 \xrightarrow{(f,g)} (S_4)^2.$$

Figure 7.5 below displays the resulting GWP function.
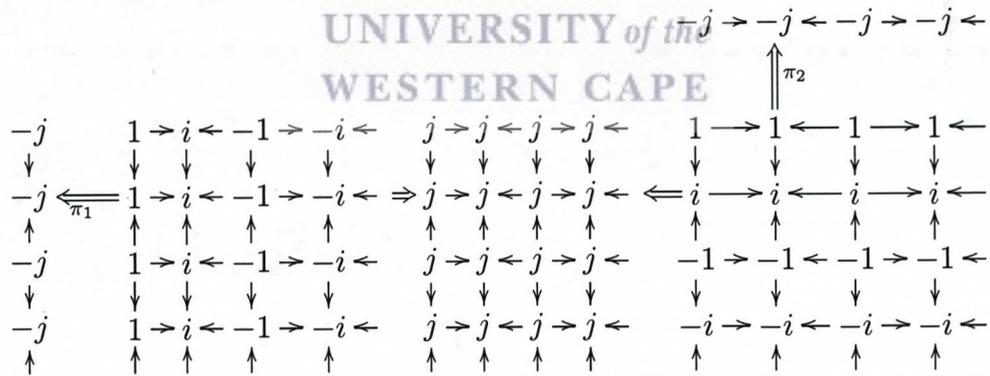
Figure 7.5:

Theorem 7.6.3. (Hardie and Witbooi [25]) *The function described above is a model of the Whitehead square class in $\pi_3(S^2)$.*

Proof. We first check the order preservation of the function. Examining the

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five components of the diagram separated by the $\Rightarrow$ arrows. The extreme left component describes a constant function from the subposet into the codomain $(S_4)^2$. and there is no failure of order preservation. The second component of the diagram from the left is a 16-point subposet, a model of $S^1 \times S^1$. Checking the values indicated and the directions of the arrows displayed, and assuming partial order as in the $\mathbb{S}S_4$ model of S^1, we see that the function is order preserving on this component. On the central portion, each of the entries for each row lies below the entry $-j$ in the row in the extreme left column and also lie below the entry j in the same row and column of the central block. Since the map $\nu : S^1 * S^1 \to S_4 \circledast S_4$ is a homeomorphism, then by Definition 4.4.1, the non-Hausdorff join $S_4 \circledast S_4$ is a model of $S^1 * S^1$. Observe that the map ω is as follows: $\phi_S \pi_1$ modelled by $\omega|_{S \times \mathbb{C}S}$ and $\phi_S \pi_2$ by $\omega|_{\mathbb{C}S \times S}$. This completes the proof. $\square$

In Section 7.7, we discuss (in model form) the relation of the Generalised Whitehead product to the Hopf class.
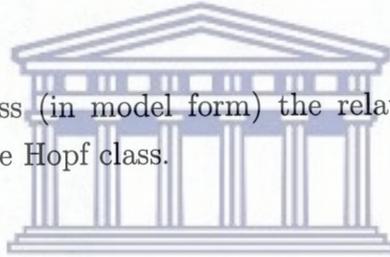
7.7 Relation of the generalised Whitehead product to the Hopf class

Since a pairing is a map in a category, a natural transformation can be applied on it. Oda [39] applies the transformation between pairings to obtain some properties of the Whitehead product and the Γ^*- Hopf construction. In [25], Hardie and Witbooi describe a commutative diagram of the form:

$$
\begin{array}{ccc}
S_8 \circledast S_8 & & \\
\Big\downarrow{\scriptstyle \theta} & \searrow{\scriptstyle \Gamma} & \\
S_4 \circledast S_4 & \xrightarrow[WS]{} & (S_4)^2
\end{array}
$$

Figure 7.6:

In figure 7.6, the map WS is a model of the Whitehead square of the 2-sphere S^2. The Whitehead square being the element $[\iota_2, \iota_2] \in \pi_2(S^2)$. The function θ is described as follows: Similar to the diagram for $S_4 \circledast S_4 \to \mathbb{S}S_4 \vee \mathbb{S}S_4 \to (S_4)^2$, [25] further displays $S_8 \circledast S_8 \to \mathbb{S}S_8 \vee \mathbb{S}S_8 \to (S_8)^2$ as follows:
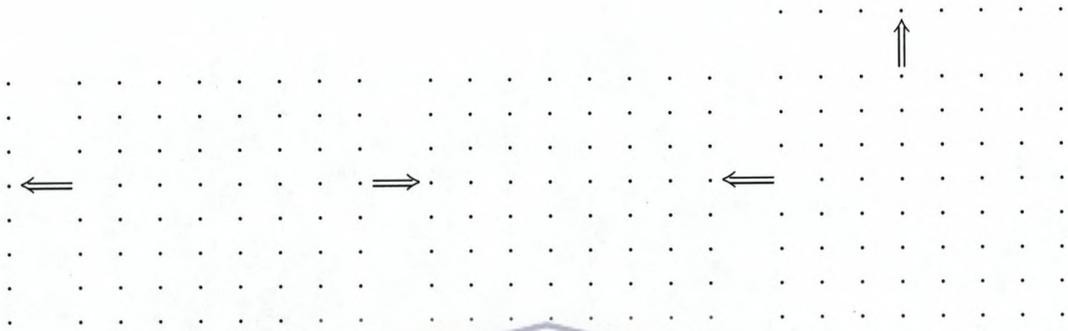

Figure 7.7:

The description of θ uses the notion of a label-preserving function that maps each of the five components of $S_8 \circledast S_8$ onto the corresponding sections of $S_4 \circledast S_4$. The function θ restricted to the central torus is constructed as a 1-fold covering map onto $S_4 \circledast S_4$. Here, the product of pairs of adjacent points are sent to the same point. On the 8-point circles of $S_8 \circledast S_8$, which are the images of the maps π_1 and π_2, each circle is wound exactly once around the corresponding 4-point circle of $S_4 \circledast S_4$. This is done in a similar manner by sending pairs of adjacent points to the same point. As for the torus on the far right (the domain of the projection π_2), when one looks at the top left corner one sees that the corresponding point on $S_8 \times S_8$ corresponds to $\Gamma^{-1}(1)$. This point is sent together with the next three points of the fibre to the top left corner point of the right hand torus of the diagram for the model of WS for $\pi_3(S^2)$. The rest of the points of the fibre of 1 under Γ are wound exactly once around the top row of the far right torus on the model of WS for $\pi_3(S^2)$.

A natural question one might ask is wether it is possible to construct continuous multiplications for the 3 -sphere and the 7-sphere. Further, whether a single barycentric subdivision as we did in Chapter 7 Section 7.3 would suf-

fice to yield a continuous multiplication. In [23], Hardie *et al.* answer this in the affirmative. They describe an order preserving function of the form $v : \text{op}(S^3)' \times (S^3)' \to S^3$ where $\text{op}(S^3)'$ is the poset obtained from reversing the order relations in $(S^3)'$ (the poset (S^3) is the 8-point model of the 3-sphere).

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Chapter 8

Concluding remarks

The thesis was developed from an idea of making a contribution to the program of finding poset models of certain interesting maps that are important in homotopy theory. One such map is a special case of the famous Hopf map $S^3 \to S^2$. Finite posets, in particular the objects (finite T_0 Spaces) of the subcategory $\mathbf{FT_0}$ of $\mathcal{R}$ provide an alternative and rich avenue for such a study. This has much to do with the fact that besides being combinatorial in nature, thus affording the possibility of automation when one wants to carry out some computations (in [23] for instance), they come with an extra topological property. The topological property such as how well a space is endowed with open sets is closely related to its supply of continuous functions. Since continuous functions are of central importance in topology, this property enables us to know whether enough of them are available to make our deliberations fruitful. Though by no means does this thesis provide a comprehensive treatment of pairings, it does however illustrate the availability of an approach for portraying similar representative maps in visual and finite terms. For instance, it will be interesting to use finite posets to model pairings of compact lie groups. One may further devise a general theory that models topological spaces in the category of binary reflexive relational structures by finite poset models.

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