

Quasi-uniform and Syntopogenous Structures on Categories



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the Degree of Doctor of Philosophy in the Department of Mathematics
and Applied Mathematics, University of the Western Cape.

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Dedication

I dedicate this thesis with much appreciation and love to my mom Kaliza mwa Muhabura, to my late father Iragi Mashema and to all those mathematicians, dead or alive, whose publications have inspired me and all those who will be inspired by this work to carry out their own research works.



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Key words

$(\mathcal{E}, \mathcal{M})$ -factorization system

Categorical closure operator

Categorical interior operator

Categorical neighbourhood operator

Categorical topogenous structure

Quasi-uniform structure

Syntopogenous structure

Initial morphism

Completeness

Continuous functors



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Abstract

In a category \mathcal{C} with a proper $(\mathcal{E}, \mathcal{M})$ -factorization system for morphisms, we further investigate categorical topogenous structures and demonstrate their prominent role played in providing a unified approach to the theory of closure, interior and neighbourhood operators. We then introduce and study an abstract notion of Cászár's syntopogenous structure which provides a convenient setting to investigate a quasi-uniformity on a category. We demonstrate that a quasi-uniformity is a family of categorical closure operators. In particular, it is shown that every idempotent closure operator is a base for a quasi-uniformity. This leads us to prove that for any idempotent closure operator c (interior i) on \mathcal{C} there is at least a transitive quasi-uniformity \mathcal{U} on \mathcal{C} compatible with c (i). Various notions of completeness of objects and precompactness with respect to the quasi-uniformity defined in a natural way are studied.

The great relationship between quasi-uniformities and closure operators in a category inspires the investigation of categorical quasi-uniform structures induced by functors. We introduce the continuity of a \mathcal{C} -morphism with respect to two syntopogenous structures (in particular with respect to two quasi-uniformities) and utilize it to investigate the quasi-uniformities induced by pointed and copointed endofunctors. Amongst other things, it is shown that every quasi-uniformity on a reflective subcategory of \mathcal{C} can be lifted to a coarsest quasi-uniformity on \mathcal{C} for which every reflection morphism is continuous. The notion of continuity of functors between categories endowed with fixed quasi-uniform structures is also introduced and used to describe the quasi-uniform structures induced by an \mathcal{M} -fibration and a functor having a right adjoint.

Declaration

I declare that *Quasi-uniform and Syntopogenous Structures on Categories* is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Minani Iragi



Signed:

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It is great pleasure that I wish to thank immensely my supervisor Professor David Holgate for his exceptional guidance, unwavering encouragements and for having allowed me to work independently during my pleasant time as his graduate student. Above all he has made my dream of pursuing a career in Mathematics come true.

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I thank my family for their continuous love and support which have enabled me to complete this thesis. Finally, I thank the Lord for covering me with his grace during the course of my studies at the University of the Western cape.

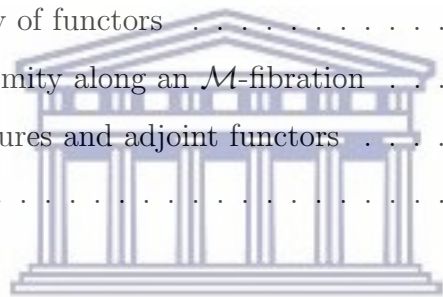
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Introduction

Among the various asymmetric topological structures, one finds the notion of quasi-uniform structure. First introduced by Nachbin ([Nac48]) under the name of semi-uniform structure, the term quasi-uniform structure was proposed by A.Császár in [Csá63] when he introduced a general concept of syntopogenous structure which aimed to provide a single setting study of topological, proximity and (quasi) uniform structures. Quasi-uniform structures have been a subject of intensive research (see e.g.[FL82] with references therein and the survey papers [Kün95, Kün01, Kün02]). Császár ([Csá63]) and Pervin ([Per62]) proved that every topological space has a compatible quasi-uniformity, a result which only holds for uniform spaces if the topological space is completely regular. Thus the study of quasi-uniform spaces provides in some sense an alternative approach to the study of topological spaces. Categorical methods have played an important role in the study of this great relationship between quasi-uniform and topological spaces (see e.g [Brü99, Kün92]). In these papers, the authors essentially studied functors from the category of T_o -topological spaces and continuous maps to the category of T_o -quasi-uniform spaces which endow the T_o -topological spaces with compatible quasi-uniformities and make continuous maps become quasi-uniformly continuous, the so-called functorial quasi-uniformities first pointed out by Brümmer in [Brü69]. Other categorical study of this relationship between topological and quasi-uniform spaces includes [DK00] and [DK18].

The study of topological structures on abstract categories was initiated by D. Dikranjan and Giuli [DG87] who introduced the notion of categorical closure operator. The development of the categorical closure operator led to a beautiful theory (see e.g.[DT95, Cas03]) which constitutes up-to-date an important part of categorical topology. This way of thinking eventually motivated other authors to take a similar approach and introduce the categorical interior ([Vor00]) and neighbourhood ([HŠ11]) operators. The recently in-

roduced notion of topogenous structures on categories ([HIR16]) has provided a unified approach to the categorical closure, interior and neighbourhood operators and has shed a light on the study of a concept of quasi-uniformity on an abstract category.

Our thesis aims to study a quasi-uniformity ([FL82]) on an arbitrary category using an abstract notion of syntopogenous structure ([Csá63]). Classical notions and results on quasi-uniform spaces are expressed in a more general categorical setting. This leads to new results that are applied to specific examples in Topology and Algebra. Departing from a category \mathcal{C} with a proper $(\mathcal{E}, \mathcal{M})$ -factorization system for morphisms, we first focus on providing further development of the categorical topogenous structures. A number of new results that complete our previous study in ([HIR16, Ira16]) and partly lay a basis for the development of the thesis are proved. We then proceed by introducing the notions of quasi-uniformity and syntopogenous structure on a category. Although the categorical syntopogenous structure appears as an appropriate family of order relations on the subobject lattice, $\text{sub}X$, for any object X of the category, a categorical quasi-uniformity is thought of as a suitably axiomatized family of endomaps on $\text{sub}X$ for any object X of the category. The definitions obtained include the fact that every morphism in a category must be continuous with respect to the structure. It is shown that there is a subconglomerate of the conglomerate of all syntopogenous structures which is isomorphic to the conglomerate of all quasi-uniform structures. This leads to the observation that a quasi-uniformity is a family of closure operators. In particular, every idempotent closure operator is a quasi-uniformity. Moreover, we prove that given an idempotent closure operator c (interior i) on \mathcal{C} , there is at least a transitive quasi-uniformity compatible with c (i).

Diverse notions of completeness and precompactness of objects of \mathcal{C} relative to the quasi-uniformity obtained are studied. Our attention will then be turned to the study of continuity of a \mathcal{C} -morphism with respect to two syntopogenous structures on \mathcal{C} which enables us to describe the quasi-uniformity induced by a pointed (resp. copointed) endofunctor. Thinking of categories supplied with quasi-uniformities as large “spaces”, we generalize the continuity of \mathcal{C} -morphisms (with respect to a quasi-uniformity) to functors. We prove that for an \mathcal{M} -fibration or a functor having a right adjoint, one can concretely describe the coarsest quasi-uniformity for which the functor is continuous. Our thesis is organised as below.

The **Chapter 1** is devoted to factorization structures for morphisms as well as the notion of subobject, images and pre-images of subobjects. We also recall a number of definitions and results on closure, interior and neighbourhood operators that will be used throughout our thesis.

In **Chapter 2**, we define, analogous to the \square -strict morphism already studied in [HIR16], the \square -co-strict morphism and show that it generalizes both the c -open and i -closed morphisms. The notions of \square -initial and \square -final morphisms, introduced in [Ira16], are shown to capture their counterparts in the settings of closure, interior and neighbourhood operators. The pullback behaviour of the four types of morphisms is also studied. Our \square -initial morphism leads to the definition of a hereditary topogenous order which enables us to study hereditary closure and interior operators in one setting.

Special topogenous orders that correspond to the additive (respectively grounded) interior and closure operators are identified. We then turn our attention to the lifting of a topogenous order along an \mathcal{M} -fibration. This not only contains the lifting of a closure operator ([DT95]) as a particular case but also provides a way of lifting an interior operator along an \mathcal{M} -fibration. The continuity of a morphism with respect to two topogenous orders is introduced and used to investigate the topogenous order induced by a pointed (resp. copointed) endofunctor.

Chapter 3 introduces the theory of categorical quasi-uniform and syntopogenous structures. We demonstrate the equivalence between quasi-uniform and co-perfect syntopogenous structures, which together with Proposition 2.1.5, leads to the description of a quasi-uniformity as a family of categorical closure operators. Since, every interpolative topogenous order is shown to be a syntopogenous structure and the class of these orders is known (see Corollary 2.1.6(i)) to be essentially equivalent to the conglomerate of all idempotent closure operators, it will be proved that every idempotent closure operator is a quasi-uniformity on \mathcal{C} . This allows to prove a one to one correspondence between idempotent closure operators and the so-called saturated quasi-uniform structures, and obtain a categorical generalization of the Császár-Pervin ([Csá63, Per62]) quasi-uniformity that we characterize as the coarsest transitive quasi-uniform structure compatible with a given idempotent interior operator on \mathcal{C} . The initial morphism with respect to a syntopogenous structure is defined and shown to capture its counterparts in the settings of quasi-uniformity and (idempotent) closure operator. We also study the Hausdorff separa-

tion axiom relative to a syntopogenous structure (in particular relative to a quasi-uniform structure).

Chapter 4 studies complete objects of a category with respect to a syntopogenous (in particular a quasi-uniform) structure. For a quasi-uniform structure, distinct notions of Cauchy filters are defined. Consequently, variant notions of completeness of objects in the category are studied. Categorical proofs of classical theorems of completeness are provided.

In **Chapter 5**, we investigate the continuity of a \mathcal{C} -morphism with respect to two syntopogenous structures (in particular with respect to two quasi-uniformities). It is shown that for a syntopogenous structure \mathcal{S} on \mathcal{C} and an \mathcal{E} -pointed endofunctor (F, η) , there is coarsest syntopogenous structure $\mathcal{S}^{F, \eta}$ on \mathcal{C} for which every $\eta_X : X \rightarrow FX$ is $(\mathcal{S}^{F, \eta}, \mathcal{S})$ -continuous. Since a categorical quasi-uniformity is equivalent to a co-perfect syntopogenous structure and simple co-perfect syntopogenous structures are equivalent to idempotent closure operators, $\mathcal{S}^{F, \eta}$ allows us to construct the quasi-uniform structure and the closure operator induced by a pointed endofunctor. In particular, we demonstrate that every quasi-uniformity on a reflective subcategory of \mathcal{C} can be lifted to a coarsest quasi-uniformity $\mathcal{U}^{F, \eta}$ on \mathcal{C} for which every reflection morphism is $(\mathcal{U}^{F, \eta}, \mathcal{U})$ -continuous. When applied to spaces, $\mathcal{U}^{F, \eta}$ turns out to describe initial structures induced by reflection maps. Dually for \mathcal{M} -copointed endofunctor and syntopogenous structure \mathcal{S} on \mathcal{C} , there is a finest syntopogenous structure $\mathcal{S}^{G, \varepsilon}$ on \mathcal{C} for which every $\varepsilon_X : GX \rightarrow X$ is $(\mathcal{S}, \mathcal{S}^{G, \varepsilon})$ -continuous. If $F : \mathcal{A} \rightarrow \mathcal{C}$ is a functor and \mathcal{U} and \mathcal{V} are quasi-uniformities on \mathcal{A} and \mathcal{C} respectively, we define the $(\mathcal{U}, \mathcal{V})$ -continuity of F and show that if F is an \mathcal{M} -fibration or has a right adjoint, then there is a coarsest quasi-uniformity \mathcal{V}^F on \mathcal{A} for which F is $(\mathcal{V}^F, \mathcal{V})$ -continuous. Investigating the lattice of all quasi-uniform structures on \mathcal{C} , we demonstrate that for a functor $F : \mathcal{A} \rightarrow \mathcal{C}$ with a right adjoint G , there is a Galois connection between the conglomerate of all quasi-uniformities on \mathcal{C} and the conglomerate of all those on \mathcal{A} .

Some of the main results of this thesis have been discussed in

- (1) D. Holgate, M. Iragi, *Quasi-uniform and syntopogenous structures on categories*, Topology and its Applications, 263:16-25, 2019.
- (2) D. Holgate and M. Iragi. *Quasi-uniform structures and functors*.

Quaestiones Mathematicae (under review), 2019.

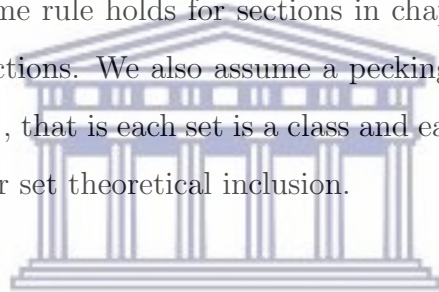
(3) D. Holgate and M. Iragi. *More on categorical topogenous structures*

(In preparation).

(4) D. Holgate and M. Iragi. *Quasi-uniform structures determined by closure*

and interior operators (In preparation).

The reader of this thesis is assumed to have a basic knowledge of general topology, category theory with little more presupposed from algebra, order and lattices ([Fuc73, DP02, Eng89]) and of course familiarity with the topological structures considered in the thesis especially quasi-uniform and syntopogenous spaces. However, we have recalled a number of basics that can help the reader to go through the work without much difficulty. The structure of our thesis is simple, chapters are numbered according to their order of appearance in the text. The same rule holds for sections in chapters and for propositions, lemmas, and definitions in sections. We also assume a pecking order of sets, classes and conglomerates (as in [AHS06]), that is each set is a class and each class is a conglomerate. The symbol \subseteq will be used for set theoretical inclusion.



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Chapter 1

Preliminaries

This chapter is mainly an overview of terminologies and elementary results that will be used throughout the thesis, the aim being to make the work as self-contained as possible. We depart from a fixed category \mathcal{C} . Then discuss factorization structures for morphisms of \mathcal{C} that enable us to efficiently deal with images and inverse images of subobjects. We end the chapter by stating a few definitions on closure, interior and neighbourhood operators that we will frequently use in the subsequent chapters. For the meaning of categorical concepts and notations used without definition in this work, we refer the reader to ([AHS06, HST14]). However, we note some variations from the notations of these two books: $f \in \mathcal{C}$ (resp. $X \in \mathcal{C}$) shall be used when f is a morphism (resp. X is an object) of \mathcal{C} .

1.1 Factorization structures for morphisms

Factorization systems play an important role in this work, in fact from section 2 of this chapter our basic working environment will always be a category \mathcal{C} endowed with an $(\mathcal{E}, \mathcal{M})$ -factorization system for morphisms. Here, we recall its definition and a few results that we shall need throughout. For more details on the topic, the interested reader is referred to ([AHS06], chapter 14).

Definition 1.1.1. [HST14] *A pair of distinguished classes $(\mathcal{E}, \mathcal{M})$ of morphisms of \mathcal{C} is factorization system provided:*

- (1) \mathcal{E} and \mathcal{M} are closed under composition with isomorphisms from the left and the right

respectively i.e if $e \in \mathcal{E}$, $g \in \text{Iso}(\mathcal{C})$ and $g \circ e$ makes sense, then $g \circ e \in \mathcal{E}$, and if $m \in \mathcal{M}$, $g \in \text{Iso}(\mathcal{C})$ and $m \circ g$ makes sense, then $m \circ g \in \mathcal{M}$.

(2) Every morphism $f \in \mathcal{C}$ factors as an \mathcal{E} -morphism and an \mathcal{M} -morphism
i.e $f = m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$,

(3) \mathcal{C} has the unique $(\mathcal{E}, \mathcal{M})$ -diagonalization property: for every commutative diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{e} & \cdot \\ \downarrow u & \swarrow w & \downarrow v \\ \cdot & \xrightarrow{m} & \cdot \end{array}$$

with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there is a uniquely determined morphism w with $w \circ e = u$ and $m \circ w = v$. In this case we say that every \mathcal{E} -morphism e is orthogonal to every \mathcal{M} -morphism m and write $e \perp m$.

The system is called proper if $\mathcal{E} \subseteq \text{Epi}(\mathcal{C})$ and $\mathcal{M} \subseteq \text{Mono}(\mathcal{C})$.

For the rest of this work, by a factorization structure, we will always mean a proper one.

Some usefull stability properties of the classes \mathcal{M} and \mathcal{E} are considered in the next proposition.

Proposition 1.1.2. [AHS06] Let $(\mathcal{E}, \mathcal{M})$ be a factorization system in \mathcal{C} .

- (1) $\mathcal{E} \cap \mathcal{M} = \text{Iso}(\mathcal{C})$;
- (2) If $g \circ f \in \mathcal{M}$, then $f \in \mathcal{M}$;
- (3) If $g \circ f \in \mathcal{E}$, then $g \in \mathcal{E}$;
- (4) \mathcal{E} and \mathcal{M} are closed under composition;
- (5) \mathcal{M} is stable under pullbacks;
- (6) \mathcal{M} is stable under intersections.

Proof. (1) If $f \in \text{Iso}(\mathcal{C})$ and $f = m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{E}$, then $e, m \in \text{Iso}(\mathcal{C})$ by the diagonalization property and hence $f \in \mathcal{M} \cap \mathcal{E}$. On the other hand if $f \in \mathcal{M} \cap \mathcal{E}$, the diagonalization property implies the existance of w which makes

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ \downarrow id & \swarrow w & \downarrow id \\ \cdot & \xrightarrow{f} & \cdot \end{array}$$

commute. Thus $f \in Iso(\mathcal{C})$. (2) and (3) follows from properness of $(\mathcal{E}, \mathcal{M})$.

(4) Let $n, m \in \mathcal{M}$ and $n \circ m = m' \circ e'$ with $m' \in \mathcal{M}$ and $e' \in \mathcal{E}$. Then there is w which makes

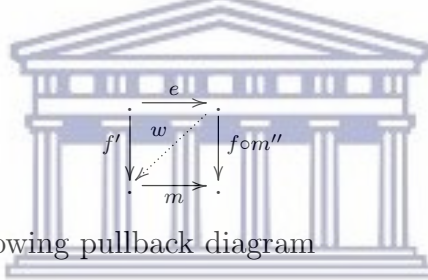
$$\begin{array}{ccc} \cdot & \xrightarrow{e'} & \cdot \\ \downarrow m & \swarrow w & \downarrow m' \\ \cdot & \xrightarrow{n} & \cdot \end{array}$$

commute. By Proposition 1.1.2(2), $e' \in \mathcal{E} \cap \mathcal{M} = Iso(\mathcal{C})$ and so $n \circ m \in \mathcal{M}$. A similar argument holds for \mathcal{E} .

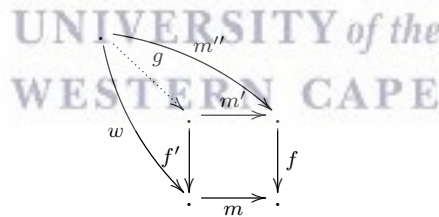
(5) [AHS06] Consider the pullback diagram

$$\begin{array}{ccc} \cdot & \xrightarrow{m'} & \cdot \\ \downarrow f' & & \downarrow f \\ \cdot & \xrightarrow{m} & \cdot \end{array}$$

with $m \in \mathcal{M}$ and let $m' = m'' \circ e$ with $m'' \in \mathcal{M}$ and $e \in \mathcal{E}$. Then by the diagonalization property, there is w such that



commutes. This gives the following pullback diagram



So $m' \circ g = m''$ and $f' \circ g = w$, which implies that $m' \circ (g \circ e) = m'$ and $f' \circ (g \circ e) = f'$ and thus $g \circ e = id$. Now e is an epimorphism and a section, $e \in Iso(\mathcal{C})$ and $m' \in \mathcal{M}$.

An analogous reasoning to the previous proves (6). □

Our next proposition is a consequence of the unique diagonalization property.

Proposition 1.1.3. [AHS06]

- (1) The $(\mathcal{E}, \mathcal{M})$ -factorizations of a morphism of \mathcal{C} are unique, up to isomorphism.
- (2) In an $(\mathcal{E}, \mathcal{M})$ -factorization system, the classes \mathcal{E} and \mathcal{M} determines each other i.e.

$$\mathcal{E} = \{e \in \mathcal{C} \mid \forall m \in \mathcal{M} \mid e \perp m\}$$

$$\mathcal{M} = \{m \in \mathcal{C} \mid \forall e \in \mathcal{E} \mid e \perp m\}.$$

Proof. (1) Let $f = m \circ e = m' \circ e'$ with $e', e \in \mathcal{E}$ and $m, m' \in \mathcal{M}$. Then by the diagonalization, there is w for which

$$\begin{array}{ccc} \cdot & \xrightarrow{e} & \cdot \\ \cdot & \swarrow w & \cdot \\ \cdot & \xrightarrow{m'} & \cdot \end{array} \quad \begin{array}{c} \downarrow e' \\ \cdot \\ \cdot \end{array} \quad \begin{array}{c} \downarrow m \\ \cdot \\ \cdot \end{array}$$

commutes. By Properness of $(\mathcal{E}, \mathcal{M})$, $w \in \mathcal{E} \cap \mathcal{M} = Iso(\mathcal{C})$. Thus $m \cong m'$ and $e \cong e'$.

(2) If $f \in \mathcal{E}$, then for all $m \in \mathcal{M}$, $f \perp m$ by diagonalization property. On the other hand if $f \perp m$ for all $m \in \mathcal{M}$ and $f = m' \circ e$ with $e \in \mathcal{E}$ and $m' \in \mathcal{M}$, there is w such that

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ \cdot & \swarrow w & \cdot \\ \cdot & \xrightarrow{m'} & \cdot \end{array} \quad \begin{array}{c} \downarrow e \\ \cdot \\ \cdot \end{array} \quad \begin{array}{c} \downarrow id \\ \cdot \\ \cdot \end{array}$$

commutes i.e $m' \circ w = id$ and $w \circ f = e$. Thus $m' \in Iso(\mathcal{C})$ and we obtain $f = m' \circ e \in \mathcal{E}$.

□

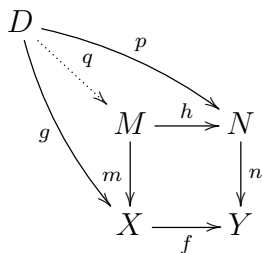
1.2 \mathcal{M} -subobjects, Images and Inverse images

Throughout this section we assume that the category \mathcal{C} is endowed with $(\mathcal{E}, \mathcal{M})$ -factorization system for morphisms. In accordance with [DT95], the class $subX$ of all \mathcal{M} -morphisms with codomain X , for every object X in \mathcal{C} , will be called the subobjects of X . Subobjects represent an appropriate categorical treatment of the notion of sub-structures. $subX$ is preordered as follows : if $m \leq n$ in $subX$ if and only if there exists j such that $n \circ j = m$

$$\begin{array}{ccc} M & \xrightarrow{j} & N \\ & \searrow m & \swarrow n \\ & & X \end{array}$$

The morphisms n and m are isomorphic ($m \cong n$) if it holds that $m \leq n$ and $n \leq m$. Clearly \cong is an equivalence relation. The collection of equivalence classes $\{[m] \mid m \in subX\}$ can be preordered as $[m] \subseteq [n] \Leftrightarrow m \leq n$. Thus, instead of working with these equivalence classes, we use their representatives. We think of isomorphic subobjects as being the same and for the rest of the thesis, we shall simply write $n = m$ for $m \cong n$.

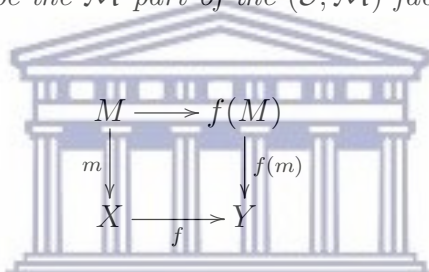
Definition 1.2.1. [DT95] We will say that \mathcal{C} has \mathcal{M} -pullbacks, if for every morphism $f : X \rightarrow Y$ and every $n \in \text{sub}Y$ a pullback diagram



exists in \mathcal{C} with $m \in \text{sub}X$.

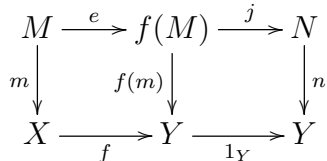
The morphism m is uniquely determined up to isomorphism, it is called the inverse image of n under f and denoted by $f^{-1}(n) : f^{-1}(N) \rightarrow X$.

Definition 1.2.2. [CGT04] For a morphism $f : X \rightarrow Y$ in \mathcal{C} and $m : M \rightarrow X$, one defines $f(m) : (M) \rightarrow Y$ to be the \mathcal{M} -part of the $(\mathcal{E}, \mathcal{M})$ -factorization of the composite $f \circ m$.

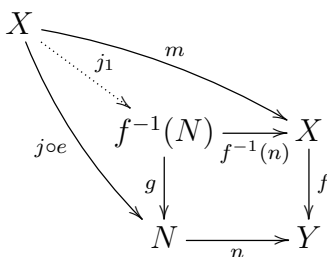


Proposition 1.2.3. [DT95] For every morphism $f : X \rightarrow Y$ in \mathcal{C} , $f(-)$ and $f^{-1}(-)$ are adjoint to each other with $f(-)$ being the left adjoint.

Proof. We need to show that, $f(m) \leq n \Leftrightarrow m \leq f^{-1}(n)$ for all $m \in \text{sub}X$ and $n \in \text{sub}Y$. Assume that $f(m) \leq n$, then there is $j : f(M) \rightarrow N$ such that the diagram below commutes.



This implies that $f \circ m = n \circ j \circ e$ and we have the commutative diagram below



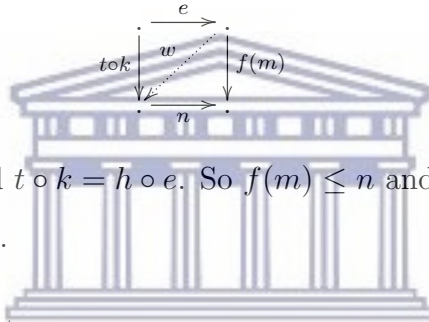
The arrow j_1 exists by the pullback property of the diagram. So $m = f^{-1}(n) \circ j_1$ and $j \circ e = g \circ j_1$. Hence $m \leq f^{-1}(n)$.

On the other hand if $m \leq f^{-1}(n)$, then there is $k : M \rightarrow f^{-1}(N)$ such that $m = f^{-1}(n) \circ k$. Now consider the diagram below

$$\begin{array}{ccccc}
 M & \xrightarrow{k} & f^{-1}(N) & \xrightarrow{t} & N \\
 & \searrow m & \downarrow f^{-1}(n) & & \downarrow n \\
 & & X & \xrightarrow{f} & Y
 \end{array}$$

We get that $f(m) \circ e = f \circ m = f \circ f^{-1}(n) \circ k = n \circ t \circ k$.

By the diagonalization property, there is w which makes



commute i.e $f(m) = n \circ h$ and $t \circ k = h \circ e$. So $f(m) \leq n$ and thus $m \leq f^{-1}(n) \Leftrightarrow f(m) \leq n$. □

It follows from adjointness that:

- (1) $f(-)$ and $f^{-1}(-)$ are monotone maps
- (2) $m \leq f^{-1}(f(m))$ and $f(f^{-1}(n)) \leq n$;
- (3) $f(\bigvee_{i \in I} m_i) \cong \bigvee_{i \in I} f(m_i)$;
- (4) $f^{-1}(\bigwedge_{i \in I} n_i) \cong \bigwedge_{i \in I} f^{-1}(n_i)$.

Lemma 1.2.4. *Let*

$$\begin{array}{ccc}
 & \xrightarrow{f'} & \\
 p' \downarrow & & \downarrow p \\
 & \xrightarrow{f} &
 \end{array}$$

be a commutative diagram. Then for any suitable subobjects n and m ,

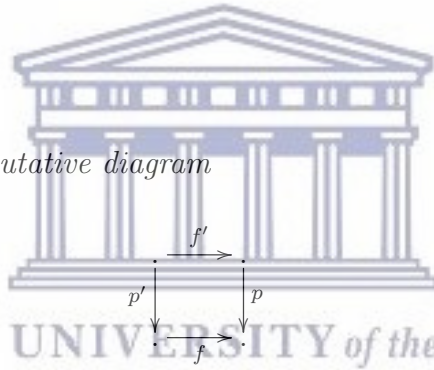
- (1) [DT95] $p'(f'^{-1}(n)) \leq f^{-1}(p(n))$.
- (2) $f'(p'^{-1}(m)) \leq p^{-1}(f(m))$.

Proof.

$$\begin{aligned}
 (1) \quad f'(f'^{-1}(n)) \leq n &\Rightarrow p(f'(f'^{-1}(n))) \leq p(n) \\
 &\Rightarrow f(p'(f'^{-1}(n))) \leq p(n) && \text{commutativity of the diagram} \\
 &\Leftrightarrow p'(f'^{-1}(n)) \leq f^{-1}(p(n)) && \text{adjointness.}
 \end{aligned}$$

Likewise,

$$\begin{aligned}
 (3) \quad p'(p'^{-1}(m)) \leq m &\Rightarrow f(p'(p'^{-1}(m))) \leq f(m) \\
 &\Rightarrow p(f'(p'^{-1}(m))) \leq f(m) && \text{commutativity of the diagram} \\
 &\Rightarrow f'(p'^{-1}(m)) \leq p^{-1}(f(m)) && \text{adjointness.}
 \end{aligned}$$



□

Definition 1.2.5. *The commutative diagram*

is said to satisfy Beck-Chevalley's Property (BCP) if $f'(p'^{-1}(m)) = p^{-1}(f(m))$. Equivalently if $p'(f'^{-1}(n)) = f^{-1}(p(n))$ for appropriate subobjects n and m .

If \mathcal{C} has \mathcal{M} -pullbacks, then the preordered class $\text{sub}X$ has binary meets for all $X \in \mathcal{C}$. In fact, for $m : M \rightarrow X$ and $n : N \rightarrow X$ subobjects of X the binary meet is given by the diagonal of the following pullback diagram

$$\begin{array}{ccc}
 M \wedge N & \longrightarrow & N \\
 \downarrow & & \downarrow n \\
 M & \xrightarrow{m} & X
 \end{array}$$

This means that $m \wedge n = m \circ m^{-1}(n) = n \circ n^{-1}(m)$.

We are interested in the existence of arbitrary meets in $\text{sub}X$ as we need $\text{sub}X$ to be a complete lattice for each $X \in \mathcal{C}$.

Definition 1.2.6. [DT95] We shall say that \mathcal{C} has \mathcal{M} -intersections if for every family $(m_i)_{i \in I}$ in $\text{sub}X$ (I may be infinite class or empty), if a multiple pullback diagram

$$\begin{array}{ccc} M & \xrightarrow{\gamma_i} & M_i \\ & \searrow m & \swarrow m_i \\ & & X \end{array}$$

exists in \mathcal{C} with $m \in \text{sub}X$.

This also implies the existence of the join \bigvee of subobjects and in particular the least subobject $0_X : 0_X \rightarrow X$ exists for every $X \in \mathcal{C}$.

Definition 1.2.7. We shall say that \mathcal{C} is \mathcal{M} -complete if it has \mathcal{M} -pullbacks and \mathcal{M} -intersections.

It is now clear from Definition 1.2.6 that if \mathcal{C} has \mathcal{M} -intersections, then the preordered class $\text{sub}X$ is a complete lattice for every object of \mathcal{C} . The largest element of $\text{sub}X$ always exist, it is the identity morphism $1_X : X \rightarrow X$ on X .

The next proposition provides sufficient conditions for the image and inverse image of subobjects to be partially inverse to each other.

Proposition 1.2.8. [DT95] Let $f : X \rightarrow Y$ be a morphism in \mathcal{C} .

- (1) If $f \in \mathcal{M}$, then $f^{-1}(f(m)) = m$ for all $m \in \text{sub}X$.
- (2) If $f \in \mathcal{E}$ and \mathcal{E} is stable under pullback then $f(f^{-1}(n)) = n$ for all $n \in \text{sub}Y$
- (3) $f \in \mathcal{E}$ if and only if $f(1_X) = 1_Y$

Proof. (1) Consider the diagram

$$\begin{array}{ccc} M & \xrightarrow{e} & f(M) \\ m \downarrow & & \downarrow f(m) \\ X & \xrightarrow{f} & Y \end{array}$$

Since $f \in \mathcal{M}$ by taking $e = 1_M$, the diagram becomes a pullback. This implies by Definition 1.2.1 that m is the inverse image of $f(m)$ under f . Thus, $f^{-1}(f(m)) = m$.

(2) Consider the diagram

$$\begin{array}{ccc} f^{-1}(N) & \xrightarrow{f'} & f(X) \\ f^{-1}(n) \downarrow & & \downarrow n \\ X & \xrightarrow{f} & Y \end{array}$$

with $f \in \mathcal{E}$ and \mathcal{E} is stable under pullback. Then $f' \in \mathcal{E}$. This implies by Definition 1.2.2 that n is the image of $f^{-1}(n)$ under f . Hence $f(f^{-1}(n)) = n$.

(3) Consider the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{e} & f(X) \\ f \downarrow & & \downarrow f(1_X) \\ Y & \xrightarrow{1_Y} & Y \end{array}$$

Since $f \in \mathcal{E}$ and $f(1_X) \in \mathcal{M}$, by the diagonalization property of $(\mathcal{E}, \mathcal{M})$ factorizations, there is a morphism $t : Y \rightarrow f(X)$ such that $f(1_X) \circ t = 1_Y$, that is $1_Y \leq f(1_X)$.

Conversely if $1_Y = f(1_X)$, then the commutativity of the above diagram gives $f = f(1_X) \circ e = 1_Y \circ e = e$. Hence, $f \in \mathcal{E}$ □

Proposition 1.2.9. [DT95] *Let $f : X \rightarrow Y$ be a morphism in \mathcal{C} . For any morphism $g : Y \rightarrow Z$ in \mathcal{C} , one has that $(g \circ f)(-) = g(f(-))$ and $(g \circ f)^{-1}(-) = f^{-1}(g^{-1}(-))$.*

Proof. One uses Definition 1.2.2 and Proposition 1.1.3(1) to prove $(g \circ f)(-) = g(f(-))$ while $(g \circ f)^{-1}(-) = f^{-1}(g^{-1}(-))$ follows from Definition 1.2.1 and the uniqueness of pullbacks. □

Definition 1.2.10. *A \mathcal{C} -morphism $f : X \rightarrow Y$ reflects o if $f^{-1}(o_Y) = o_X$ (equivalently $f(m) = o_Y \Leftrightarrow m = o_X$).*

Besides the image pre-image adjunction studied in Proposition 1.2.3, we shall often find it important to assume that for any \mathcal{C} -morphism $f : X \rightarrow Y$, the inverse image f^{-1} commutes with the joins of subobjects so that it has a right adjoint f_* given by $f_*(m) = \bigvee \{n \in \text{sub}Y \mid f^{-1}(n) \leq m\}$. Thus $f^{-1}(n) \leq m \Leftrightarrow n \leq f_*(m)$, $f^{-1}(f_*(m)) \leq m$ (with $f^{-1}(f_*(m)) = m$ if $f \in \mathcal{M}$) and $n \leq f_*(f^{-1}(n))$ (with $f_*(f^{-1}(n)) = n$ if $f \in \mathcal{E}$ and \mathcal{E} stable under pullback).

Lemma 1.2.11. *Assume that for any morphism $f \in \mathcal{C}$, the inverse image f^{-1} commutes with the joins of subobjects and*

$$\begin{array}{ccc} \cdot & \xrightarrow{f'} & \cdot \\ p' \downarrow & & \downarrow p \\ \cdot & \xrightarrow{f} & \cdot \end{array}$$

be a commutative diagram. Then for suitable subobjects n and m ,

$$(1) \ p^{-1}(f_*(n)) \leq f'_*(p'^{-1}(n));$$

$$(2) \ f^{-1}(p_*(m)) \leq p'_*(f'^{-1}(m)).$$

Proof. Let l be an appropriate subobject. Since

$$(1) \ f^{-1}(l) \leq n \Rightarrow p'^{-1}(f^{-1}(l)) \leq p'^{-1}(n) \Rightarrow f'^{-1}(p^{-1}(l)) \leq p'^{-1}(n);$$

we have that $\{p^{-1}(l) \mid f^{-1}(l) \leq n\} \subseteq \{l \mid f'^{-1}(l) \leq p'^{-1}(n)\} \Rightarrow$

$$p^{-1}(f_*(n)) = \bigvee \{p^{-1}(l) \mid f^{-1}(l) \leq n\} \leq \bigvee \{l \mid f'^{-1}(l) \leq p'^{-1}(n)\} = f'_*(p'^{-1}(n)).$$

Similarly for some suitable t ,

$$(2) \ p^{-1}(t) \leq m \Rightarrow f'^{-1}(p^{-1}(t)) \leq f'^{-1}(m) \Rightarrow p'^{-1}(f^{-1}(t)) \leq f^{-1}(m)$$

gives that $\{f^{-1}(m) \mid p^{-1}(t) \leq m\} \subseteq \{f^{-1}(p'^{-1}(l) \leq n\} \Rightarrow$

$$f^{-1}(p_*(n)) = \bigvee \{f^{-1}(m) \mid p^{-1}(t) \leq m\} \leq \bigvee \{f^{-1}(p'^{-1}(l) \leq n\} = p'_*(f'^{-1}(n)).$$

□

Corollary 1.2.12. *If for any morphism $f \in \mathcal{C}$, the inverse image f^{-1} commutes with the joins of subobjects, then the diagram in the definition above satisfies Beck-Chevalley Property (BCP) if $f^{-1}(p_*(n)) = p'_*(f'^{-1}(n))$. Equivalently $p^{-1}(f_*(n)) = f'_*(p'^{-1}(n))$ for appropriate subobjects m and n .*

Lemma 1.2.13. *If f^{-1} commutes with the join of subobjects for any $f \in \mathcal{C}$, then $\text{sub}X$ is a distributive lattice for every $X \in \mathcal{C}$*

Proof. For all $X \in \mathcal{C}$ and $m, n, p \in \text{sub}X$, then $m \wedge (n \vee p) = m \circ m^{-1}(n \vee p) = m(m^{-1}(n) \vee m^{-1}(p)) = m(m^{-1}(n)) \vee m(m^{-1}(p)) = (m \circ m^{-1}(n)) \vee (m \circ m^{-1}(p)) = (m \wedge n) \vee (m \wedge p)$. □

1.3 Closure, Interior and Neighbourhood operators

In the sequel, we shall assume that the category \mathcal{C} is endowed with an $(\mathcal{E}, \mathcal{M})$ -factorization system for morphisms and that it is \mathcal{M} -complete.

Definition 1.3.1. [DG87] *A closure operator c on \mathcal{C} with respect to \mathcal{M} is given by a family of maps*

$\{c_X: \text{sub}X \longrightarrow \text{sub}X \mid X \in \mathcal{C}\}$ *such that:*

(C1) $m \leq c_X(m)$ *for all $m \in \text{sub}X$;*

(C2) $m \leq n \Rightarrow c_X(m) \leq c_X(n)$ for all $m, n \in \text{sub}X$;

(C3) every morphism $f : X \longrightarrow Y$ is c -continuous, that is: $f(c_X(m)) \leq c_Y(f(m))$ for all $m \in \text{sub}X$.

We denote by $CL(\mathcal{C}, \mathcal{M})$ the conglomerate of all closure operators on \mathcal{C} with respect to \mathcal{M} ordered as follows: $c \leq c'$ if $c_X(m) \leq c'_X(m)$ for all $m \in \text{sub}X$ and $X \in \mathcal{C}$.

According to [DT95], a closure operator c on \mathcal{C} is :

- (1) grounded if $c_X(0_X) = 0_X$ for all $X \in \mathcal{C}$,
- (2) additive if $c_X(m \vee n) = c_X(m) \vee c_X(n)$ for all $m, n \in \text{sub}X$ and $X \in \mathcal{C}$,
- (3) idempotent if $c_X(c_X(m)) = c_X(m)$ for all $m \in \text{sub}X$ and $X \in \mathcal{C}$,
- (4) hereditary if $c_M(p) = m^{-1}(c_X(m(p)))$ for all $p \in \text{sub}M$.

The ordered conglomerate of all grounded (resp. additive, idempotent) closure operators will be denoted by $gCL(\mathcal{C}, \mathcal{M})$ (resp. $aCL(\mathcal{C}, \mathcal{M})$, $iCL(\mathcal{C}, \mathcal{M})$).

Definition 1.3.2. [Vor00] An interior operator i on \mathcal{C} with respect to \mathcal{M} is given by a family of maps

$\{i_X : \text{sub}X \longrightarrow \text{sub}X \mid X \in \mathcal{C}\}$ such that

- (I1) $i_X(m) \leq m$ for every $m \in \text{sub}X$ and $X \in \mathcal{C}$;
- (I2) $m \leq n \Rightarrow i_X(m) \leq i_X(n)$ for every $m, n \in \text{sub}X$, $X \in \mathcal{C}$;
- (I3) every morphism $f : X \longrightarrow Y$ in \mathcal{C} is i -continuous, $f^{-1}(i_Y(n)) \leq i_X(f^{-1}(n))$ for each $n \in \text{sub}Y$.

The ordered conglomerate of all interior operators on \mathcal{C} with respect to \mathcal{M} is denoted by $INT(\mathcal{C}, \mathcal{M})$. We also note from [HŠ18] that an interior operator i on \mathcal{C} is :

- (1) grounded if $i_X(1_X) = 1_X$ for all $X \in \mathcal{C}$,
- (2) additive if $i_X(m \wedge n) = i_X(m) \wedge i_X(n)$ for all $m, n \in \text{sub}X$ and $X \in \mathcal{C}$,
- (3) idempotent if $i_X(i_X(m)) = i_X(m)$ for all $m \in \text{sub}X$ and $X \in \mathcal{C}$,
- (4) ([AH19]) hereditary if $i_M(p) = m^{-1}(i_X(m_*(p)))$ for all $m : M \longrightarrow X$ and $p \in \text{sub}M$.

The symbols $gINT(\mathcal{C}, \mathcal{M})$, $aINT(\mathcal{C}, \mathcal{M})$ and $iINT(\mathcal{C}, \mathcal{M})$ will denote the ordered conglomerate of all grounded, additive and idempotent interior operators respectively.

Definition 1.3.3. [HŠ11] A neighbourhood operator ν on \mathcal{C} with respect to \mathcal{M} is a family of maps $\{\nu_X : \text{sub}X \longrightarrow P(\text{sub}X) \mid X \in \mathcal{C}\}$ such that

(N1) $n \in \nu_X(m) \Rightarrow m \leq n$ for every $m \in \text{sub}X$ and $X \in \mathcal{C}$;

(N2) $m \leq n \Rightarrow \nu_X(n) \subseteq \nu_X(m)$ for every $m, n \in \text{sub}X$ and $X \in \mathcal{C}$;

(N3) $p \in \nu_X(m)$ and $p \leq q$ then $q \in \nu_X(m)$ for every $m, p, q \in \text{sub}X$ and $X \in \mathcal{C}$;

(N4) every morphism $f : X \longrightarrow Y$ in \mathcal{C} is ν -continuous, $n \in \nu_Y(f(m)) \Rightarrow f^{-1}(n) \in \nu_X(m)$ for every $m \in \text{sub}X$ and $n \in \text{sub}Y$.

The conglomerate of all neighbourhood operators on \mathcal{C} with respect to \mathcal{M} is denoted by $NBH(\mathcal{C}, \mathcal{M})$



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Chapter 2

More on topogenous structures

This chapter aims to continue the investigating categorical topogenous structures and demonstrate their prominent role played in providing a unified approach to the theory of closure, interior and neighbourhood operators. The notions of strict, co-strict, initial and final morphisms with respect to a topogenous order are systematically studied. Besides showing that they allow simultaneous study of four classes of morphisms obtained separately with respect to closure, interior and neighbourhood operators, the initial and final morphisms lead us to the study of topogenous structures induced by pointed and co-pointed endofunctors. Hereditariness, additivity and groundedness for topogenous structures are defined. We also lift a topogenous order along an \mathcal{M} -fibration. This permits to obtain the lifting of interior and neighbourhood operators along an \mathcal{M} -fibration and includes the lifting of closure operators found in the literature. A number of examples presented at the end of the chapter demonstrate our results.

2.1 The Basic Results

This section covers fundamental definitions and results on topogenous structures. Some of them are already known from ([HIR16]) or ([Ira16]) while others appear here for the first time.

Definition 2.1.1. *A topogenous order \sqsubset on \mathcal{C} is a family $\sqsubset = \{\sqsubset_X \mid X \in \mathcal{C}\}$ of relations, each \sqsubset_X on $\text{sub}X$, such that:*

(T1) $m \sqsubset_X n \Rightarrow m \leq n$ for every $m, n \in \text{sub}X$,

(T2) $m \leq n \sqsubset_X p \leq q \Rightarrow m \sqsubset_X q$ for every $m, n, p, q \in \text{sub}X$, and

(T3) every morphism $f : X \rightarrow Y$ in \mathcal{C} is \sqsubset -continuous, $f(m) \sqsubset_Y n \Rightarrow m \sqsubset_X f^{-1}(n)$ for all $n \in \text{sub}Y$, $m \in \text{sub}X$.

Given two topogenous orders \sqsubset and \sqsubset' on \mathcal{C} , $\sqsubset \subseteq \sqsubset' \Leftrightarrow (m \sqsubset_X n \Rightarrow m \sqsubset'_X n)$ for all $m, n \in \text{sub}X$ and $X \in \mathcal{C}$. The resulting ordered conglomerate of all topogenous orders on \mathcal{C} is denoted by $TORD(\mathcal{C}, \mathcal{M})$.

Proposition 2.1.2. $TORD(\mathcal{C}, \mathcal{M})$ and $NBH(\mathcal{C}, \mathcal{M})$ are order isomorphic with the inverse assignments $\sqsubset \rightarrow \nu^\sqsubset$ and $\nu \rightarrow \sqsubset^\nu$ given by

$$\nu_X^\sqsubset(m) = \{n \mid m \sqsubset_X n\} \text{ and } m \sqsubset_X^\nu n \Leftrightarrow n \in \nu_X(m) \text{ for all } X \in \mathcal{C}$$

Proof. (N1) follows from (T1) while (N2) and (N3) follows from (T2). For (N4), let $f : X \rightarrow Y$ be a \mathcal{C} morphism and $p \in \nu_Y^\sqsubset(f(m)) \Rightarrow f(m) \sqsubset_Y p \Rightarrow m \sqsubset_X f^{-1}(p) \Leftrightarrow f^{-1}(p) \in \nu_X^\sqsubset(m)$. Similarly (T1) and (T2) follows from (N1) and (N3) respectively. Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism. Then $f(m) \sqsubset_Y^\nu n \Leftrightarrow n \in \nu_Y(f(m)) \Rightarrow f^{-1}(n) \in \nu_X(m) \Leftrightarrow m \sqsubset_X f^{-1}(n)$. The assignments clearly preserve order and they are inverse to each other. □

Particular topogenous orders will be of importance.

Definition 2.1.3. A topogenous order \sqsubset is said to be

- (1) \bigvee -preserving if $(\forall i \in I : m_i \sqsubset_X n) \Rightarrow \bigvee m_i \sqsubset_X n$,
- (2) \bigwedge -preserving if $(\forall i \in I : m \sqsubset_X n_i) \Rightarrow m \sqsubset_X \bigwedge n_i$, and
- (3) interpolative $m \sqsubset_X n \Rightarrow \exists p \mid m \sqsubset_X p \sqsubset_X n$ for all $X \in \mathcal{C}$.

The ordered conglomerate of all \bigvee -preserving, \bigwedge -preserving and interpolative topogenous orders is denoted by $\bigvee-TORD(\mathcal{C}, \mathcal{M})$, $\bigwedge-TORD(\mathcal{C}, \mathcal{M})$ and $INTORD(\mathcal{C}, \mathcal{M})$ respectively.

Our interest in the above classes is due to the fact that the first two are the equivalent to the conglomerate of interior and closure operators respectively while the last one when considered in $\bigwedge-TORD(\mathcal{C}, \mathcal{M})$ and $\bigvee-TORD(\mathcal{C}, \mathcal{M})$ corresponds to the conglomerate of idempotent closure and interior operators respectively.

Proposition 2.1.4. $\bigvee -TORD(\mathcal{C}, \mathcal{M})$ is order isomorphic to $INT(\mathcal{C}, \mathcal{M})$ with the inverse assignments given by

$$i_X^{\square}(m) = \bigvee \{p \mid p \square_X m\} \text{ and } m \square_X^i n \Leftrightarrow m \leq i_X(n) \text{ for all } X \in \mathcal{C}$$

Proof. We use (T1) and (T2) to see that (I1) and (I2) are respectively satisfied. Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism. Since $\square \in \bigvee -TORD$, $i_Y^{\square}(m) = \bigvee \{p \mid p \square_Y m\}$ and so by (T3), $f^{-1}(i_Y^{\square}(m)) \in \{q \mid q \square_X f^{-1}(m)\} \Rightarrow f^{-1}(i_Y^{\square}(m)) \leq \bigvee \{q \mid q \square_X f^{-1}(m)\}$. Hence $f^{-1}(i_Y^{\square}(m)) \leq i_X^{\square}(f^{-1}(m))$. On the other hand (I1) and (I2) follows from (T1) and (T2) respectively. Let $f : X \rightarrow Y$ be any \mathcal{C} -morphism and $m \square_Y^i n$ with $m, n \in \text{sub}Y$. Then $m \leq i_Y(n) \Rightarrow f^{-1}(m) \leq f^{-1}(i_Y(n)) \leq i_Y(f^{-1}(m)) \Rightarrow f^{-1}(m) \leq i_X(f^{-1}(n)) \Leftrightarrow f^{-1}(m) \square_X^i f^{-1}(n)$. \square

In similar way to the above we obtain the following:

Proposition 2.1.5. $\bigwedge -TORD(\mathcal{C}, \mathcal{M})$ is order isomorphic to $CL(\mathcal{C}, \mathcal{M})$ with the inverse assignments given by

$$c_X^{\square}(m) = \bigwedge \{p \mid m \square_X p\} \text{ and } m \square_X^c n \Leftrightarrow c_X(m) \leq n \text{ for all } X \in \mathcal{C}$$

If we denote by $\bigvee -INTORD(\mathcal{C}, \mathcal{M})$ and $\bigwedge -INTORD(\mathcal{C}, \mathcal{M})$ the conglomerate of all interpolative topogenous orders in $\bigvee -TORD(\mathcal{C}, \mathcal{M})$ and $\bigwedge -TORD(\mathcal{C}, \mathcal{M})$ respectively, then

Corollary 2.1.6. (i) $\bigwedge -INTORD(\mathcal{C}, \mathcal{M}) \cong iCL(\mathcal{C}, \mathcal{M})$.

(ii) $\bigvee -INTORD(\mathcal{C}, \mathcal{M}) \cong iINT(\mathcal{C}, \mathcal{M})$.

Proof. (i) If c is idempotent, then $m \square^c n \Leftrightarrow c_X(c_X(m)) = c_X(m) \leq n \Leftrightarrow m \square^c c_X(m) \square^c n$. Conversely if $\square \in \bigwedge -INTORD$ then $c_X^{\square}(c_X^{\square}(m)) \leq c_X^{\square}(m)$.

(ii) Similar reasoning to the above. \square

Definition 2.1.7. Let $\square, \square' \in TORD(\mathcal{C}, \mathcal{M})$. $\square \circ \square'$ is the topogenous order defined by

$$m \square_X \circ \square'_X n \Leftrightarrow \exists p \in \text{sub}X \mid m \square_X p \square'_X n$$

for all $m, n \in \text{sub}X$ and $X \in \mathcal{C}$ and called the composition of \square and \square' .

It is clear from the above definition that \square is interpolative if $\square \circ \square = \square$.

Proposition 2.1.8. (1) If $\sqsubset, \sqsubset' \in \bigwedge -TORD(\mathcal{C}, \mathcal{M})$, then $m \sqsubset_X \circ \sqsubset'_X n \Leftrightarrow c_{\bar{X}}^{\sqsubset'}(c^{\sqsubset}(m)) \leq n$

(2) If $\sqsubset, \sqsubset' \in \bigvee -TORD(\mathcal{C}, \mathcal{M})$, then $m \sqsubset_X \circ \sqsubset'_X n \Leftrightarrow m \leq i_{\bar{X}}^{\sqsubset}(i_{\bar{X}}^{\sqsubset'}(n))$

Proof. (1) Assume that $\sqsubset, \sqsubset' \in \bigwedge -TORD(\mathcal{C}, \mathcal{M})$ and $m \sqsubset_X \circ \sqsubset'_X n$. Then there is $p \in \text{sub}X \mid m \sqsubset_X p \sqsubset'_X n$ and by Proposition 2.1.5 $c_{\bar{X}}^{\sqsubset}(m) \leq p$ and $c_{\bar{X}}^{\sqsubset'}(p) \leq n$. Thus $c_{\bar{X}}^{\sqsubset'}(c^{\sqsubset}(m)) \leq n$. On the other hand if $c_{\bar{X}}^{\sqsubset'}(c^{\sqsubset}(m)) \leq n$, put $p = c^{\sqsubset}(m)$ to get $m \sqsubset_X c^{\sqsubset}(m) \sqsubset'_X n \Leftrightarrow m \sqsubset_X \circ \sqsubset'_X n$.

(2) If $\sqsubset, \sqsubset' \in \bigvee -TORD(\mathcal{C}, \mathcal{M})$, then by Proposition 2.1.4 $m \sqsubset_X \circ \sqsubset'_X n \Leftrightarrow m \leq i_{\bar{X}}^{\sqsubset}(p)$ and $p \leq i_{\bar{X}}^{\sqsubset'}(n) \Leftrightarrow m \leq i_{\bar{X}}^{\sqsubset}(i_{\bar{X}}^{\sqsubset'}(n))$. \square

2.2 Family of Morphisms

Investigating different ways of expressing the continuity condition of a \mathcal{C} -morphism with respect to categorical closure, interior and neighbourhood operators led to the study of particular classes of morphisms with respect to each of the operators (see e.g [GT00, Raz12, CGT01]). We show that this approach when applied to a topogenous order produces special classes of morphisms that provide a common generalization of those obtained previously with respect to each of the three operators.

Proposition 2.2.1. Assume that for every \mathcal{C} -morphism $f : X \rightarrow Y$, f^{-1} has a right adjoint. Let $\sqsubset \in TORD$. The following are equivalent to the \sqsubset -continuity. For suitable subobjects m, n and p ,

$$(1) m \sqsubset_Y n \Rightarrow f^{-1}(m) \sqsubset_X f^{-1}(n);$$

$$(2) m \sqsubset_Y f_*(n) \Rightarrow f^{-1}(m) \sqsubset_X n;$$

$$(3) f(m) \sqsubset_Y f_*(n) \Rightarrow m \sqsubset_X n.$$

Proof. If (T3) holds, then $m \sqsubset_X n \Rightarrow f(f^{-1}(m)) \leq m \sqsubset_X n \Rightarrow f(f^{-1}(m)) \sqsubset_X n \Rightarrow f^{-1}(m) \sqsubset_X f^{-1}(n)$.

Assume (1) holds, then $m \sqsubset_Y f_*(n) \Rightarrow f^{-1}(m) \sqsubset_X f^{-1}(f_*(n)) \leq n \Rightarrow f^{-1}(m) \sqsubset_X n$.

If (2) holds then $f(m) \sqsubset_Y f_*(n) \Rightarrow m \leq f^{-1}(f(m)) \sqsubset_X n \Rightarrow m \sqsubset_X n$.

If (3) holds, then $f(m) \sqsubset_Y p \Rightarrow f(m) \sqsubset_X p \leq f_*(f^{-1}(p)) \Rightarrow f(m) \sqsubset_Y f_*(f^{-1}(p)) \Rightarrow m \sqsubset_X f^{-1}(p)$. \square

The point in the proposition above that each equivalent description of (T3) fullfils one implication leads to the natural definition of morphisms that satisfy the other implication.

Definition 2.2.2.

Given a topogenous order \sqsubset , we say that a \mathcal{C} -morphism $f : X \rightarrow Y$ is

- (1) ([HIR16]) \sqsubset -strict if $f(m) \sqsubset_Y p \Leftrightarrow m \sqsubset_X f^{-1}(p)$ for all $m \in \text{sub}X$ and $p \in \text{sub}Y$;
- (2) \sqsubset -final if $m \sqsubset_Y n \Leftrightarrow f^{-1}(m) \sqsubset_X f^{-1}(n)$ for all $n, m \in \text{sub}Y$;
- (3) \sqsubset -co-strict if $m \sqsubset_Y f_*(n) \Leftrightarrow f^{-1}(m) \sqsubset_X n$ for all $m \in \text{sub}Y$ and $n \in \text{sub}X$,
- (4) \sqsubset -initial if $f(m) \sqsubset_Y f_*(n) \Leftrightarrow m \sqsubset_X n$ for all $m, n \in \text{sub}X$.

We note that in a category where f^{-1} does not have a right adjoint the definition of \sqsubset -initial and \sqsubset -co-strict morphisms can be written as follows. A morphism $f : X \rightarrow Y$ is :

- (i) \sqsubset -initial if $m \sqsubset_X n \Rightarrow \exists p \in \text{sub}Y \mid f(m) \sqsubset_Y p$ and $f^{-1}(p) \leq n$ for all $m, n \in \text{sub}X$.
- (ii) \sqsubset -co-strict if $f^{-1}(m) \sqsubset_X n \Rightarrow \exists p \in \text{sub}Y \mid m \sqsubset_Y p$ and $f^{-1}(p) \leq n$ for all $m \in \text{sub}Y, n \in \text{sub}X$.

It follows immediately from Proposition 2.1.2 that our classes correspond to those obtained in [Raz12] with respect to a neighbourhood operator as this can be seen in the next proposition.

Proposition 2.2.3. Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism, $m \in \text{sub}X$ and $n \in \text{sub}Y$.

- (1) f is \sqsubset -strict if and only if $\nu_Y^{\sqsubset}(f(m)) = f(\nu_X^{\sqsubset}(m))$.
- (2) f is \sqsubset -co-strict if and only if $\nu_X^{\sqsubset}(f^{-1}(n)) = f^{-1}(\nu_Y^{\sqsubset}(n))$.
- (3) f is \sqsubset -final if and only if $f(\nu_X^{\sqsubset}(f^{-1}(n))) = \nu_Y^{\sqsubset}(n)$.
- (4) f is \sqsubset -initial if and only if $f^{-1}(\nu_Y^{\sqsubset}(f(m))) = \nu_X^{\sqsubset}(m)$.

The behaviour of our morphisms can be summarized in the following proposition.

Proposition 2.2.4. (1) *Each of the classes is closed under composition, contains all isomorphisms of \mathcal{C} .*

(2) *\sqsubset -initial morphisms are left-cancelable, while \sqsubset -co-strict, \sqsubset -strict and \sqsubset -final morphisms are left cancellable with respect to \mathcal{M} .*

(3) *\sqsubset -final morphisms are right cancelable, while \sqsubset -initial, \sqsubset -co-strict and \sqsubset -strict morphisms are right cancellable with respect to \mathcal{E} provided \mathcal{E} is pullback stable.*

Proof. (1) First note that if $f : X \rightarrow Y$ is an isomorphism then $f_*(m) = f(m)$. Now, let $g : Y \rightarrow X$ be the inverse of f , then $f^{-1}(n) \sqsubset_X n \Rightarrow m = g^{-1}(f^{-1}(m)) \sqsubset_Y g^{-1}(m) = f(m) = f_*(m)$. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are \sqsubset -strict, then $m \sqsubset_X (g \circ f)^{-1}(n) = f^{-1}(g^{-1}(n) \Leftrightarrow f(m) \sqsubset_Y g^{-1}(n) \Leftrightarrow (g \circ f)(m) = g(f(m)) \sqsubset_Z n$. A similar argument holds for \sqsubset -initial and \sqsubset -final.

(2) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are \sqsubset -initial, then $f^{-1}(g^{-1}(m)) \sqsubset n \Rightarrow g^{-1}(m) \sqsubseteq_Y f_*(n) \Rightarrow m \sqsubset_Z g_*(f_*(n))$. If $g \circ f$ is \sqsubset -strict and $g : Y \rightarrow Z$ is in \mathcal{M} , then $g^{-1}(g(n)) = n$. Now $m \sqsubset_X f^{-1}(n) = f^{-1}(g^{-1}(g(n))) = (g \circ f)^{-1}(g(n)) \Rightarrow g(f(m)) = (g \circ f)(m) \sqsubset g(n) \Rightarrow f(m) \sqsubset_Y g^{-1}(g(n)) \Rightarrow f(m) \sqsubset_Y n$. A similar argument holds for \sqsubset -co-strict and \sqsubset -final.

(3) If $g \circ f$ is \sqsubset -final, then $g^{-1}(m) \sqsubset_Y g^{-1}(n) \Rightarrow f^{-1}(g^{-1}(m)) \sqsubset_Y f^{-1}(g^{-1}(n)) \Rightarrow m \sqsubset_Z n$. Assume that $g \circ f$ is \sqsubset -strict, \mathcal{E} is stable under pullbacks and $f \in \mathcal{E}$, then $f(f^{-1}(m)) = m$. Hence $m \sqsubset_Y g^{-1}(n) \Rightarrow f^{-1}(m) \sqsubset_X f^{-1}(g^{-1}(n)) = (g \circ f)^{-1}(n) \Rightarrow (g \circ f)(f^{-1}(m)) \sqsubset_Z n \Rightarrow g(m) \sqsubset_Z n$.

A similar reasoning works for \sqsubset -initial, \sqsubset -co-strict. □

The following is an observation concerning the relationship between the types of morphisms.

Proposition 2.2.5. (1) *Every \sqsubset -co-strict morphism in \mathcal{M} is \sqsubset -initial.*

(2) *Every \sqsubset -initial morphism in \mathcal{E} is \sqsubset -co-strict provided \mathcal{E} is pullback stable.*

(3) *Any \sqsubset -strict morphism in \mathcal{M} is \sqsubset -initial.*

(4) *Every \sqsubset -strict in \mathcal{E} is \sqsubset -final provided \mathcal{E} is pullback stable.*

(5) *If $g \circ f = 1$ in \mathcal{C} then f is a \sqsubset -initial morphism and g is a \sqsubset -final morphism in \mathcal{E} .*

(6) *Any \sqsubset -final morphism in \mathcal{M} is \sqsubset -strict.*

(7) Every \sqsubset -co-strict morphism in \mathcal{E} is \sqsubset -final provided \mathcal{E} is pullback stable.

Proof. Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and consider suitable subobjects m and n in each case. (1) If f is \sqsubset -co-strict, then $m \sqsubset_X n \Leftrightarrow f^{-1}(f(m)) \sqsubset_X n \Rightarrow f(m) \sqsubset_X f_*(n)$.

(2) If f is \sqsubset -initial in \mathcal{E} with \mathcal{E} pullback stable, then $f^{-1}(m) \sqsubset_X n \Rightarrow m = f(f^{-1}(m)) \sqsubset_X f_*(n)$.

(3) If f is \sqsubset -strict in \mathcal{M} , then $m \sqsubset_X n \Leftrightarrow m \sqsubset_X f^{-1}(f_*(n)) \Rightarrow f^{-1}(m) \sqsubset_X f_*(n)$.

(4) If f is \sqsubset -strict in \mathcal{E} with \mathcal{E} pullback stable, then $f^{-1}(m) \sqsubset_Y f^{-1}(n) \Leftrightarrow m = f(f^{-1}(m)) \sqsubset_Y f(f^{-1}(n)) = n$.

(5) Follows from Proposition 2.2.4.

(6) If f is \sqsubset -final in \mathcal{M} , then $f^{-1}(m) \sqsubset_X n \Leftrightarrow f^{-1}(m) \sqsubset_X f^{-1}(f(n)) \Rightarrow m \sqsubset_Y f(n)$.

(7) If f is \sqsubset -co-strict in \mathcal{E} , then $f^{-1}(m) \sqsubset_X f^{-1}(n) \Leftrightarrow m \sqsubset_Y f_*(f^{-1}(m)) = n$.

□

Propositions 2.2.4 and 2.2.5 were already obtained in [Ira16] without use of the fact that $f^{-1}[-]$ has both left and right adjoint. This condition plays an important role in the study of the relationship between these classes and those obtained for the interior operators.

Definition 2.2.6. ([Ira16]) A subobject m of an object $X \in \mathcal{C}$ is \sqsubset -strict if $m \sqsubset_X m$.

Proposition 2.2.7. Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism.

(1) If f is \sqsubset -final then a subobject n of Y is \sqsubset -strict iff $f^{-1}(n)$ is \sqsubset -strict in X .

(2) If $\sqsubset \in \text{INTORD}$ and f is \sqsubset -initial then for every \sqsubset -strict subobject m of X , there is $p \in \text{sub}Y$ such that $m = f^{-1}(p)$.

Proof. (1) is clear. For (2), assume $\sqsubset \in \text{INTORD}$, f is \sqsubset -initial and $m \in \text{sub}X$ is \sqsubset -strict. Then $m \sqsubset_X m \Rightarrow f(m) \sqsubset_X f_*(m) \Rightarrow \exists p \in \text{sub}Y$ such that $f(m) \sqsubset_X p \sqsubset_X f_*(m) \Rightarrow m \sqsubset_X f^{-1}(p) \sqsubset_X f^{-1}(f_*(m)) \leq m \Rightarrow m \leq f^{-1}(p) \leq m \Rightarrow m = f^{-1}(p)$. □

We are interested in the pullback behaviour of the morphisms. We show that each of the classes ascends along \sqsubset -initial morphisms and descends along \sqsubset -final morphisms

Proposition 2.2.8. Let

$$\begin{array}{ccc} P & \xrightarrow{f'} & Q \\ p' \downarrow & & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

be a pullback diagram satisfying Beck Chevalley Property (BCP). Then the following statements are true.

(i) If p' is \sqsubset -initial, then f' is \sqsubset -initial (resp. \sqsubset -strict, \sqsubset -co-strict, \sqsubset -final) provided f is \sqsubset -initial (resp. \sqsubset -strict, \sqsubset -co-strict, \sqsubset -final).

(i) If p is \sqsubset -final, then f is \sqsubset -final (resp. \sqsubset -strict, \sqsubset -co-strict, \sqsubset -initial) provided f' is \sqsubset -final (resp. \sqsubset -strict, \sqsubset -co-strict, \sqsubset -initial).

Proof. (i) Let p' be \sqsubset -initial. If f is \sqsubset -initial, then f' is \sqsubset by Proposition 2.2.4.

Assume f is \sqsubset -strict, then

$$\begin{aligned}
m \sqsubset_P n &\Rightarrow p'(m) \sqsubset_X p'_*(n) && \text{initiality of } p' \\
&\Rightarrow f(p'(m)) \sqsubset_Y f(p'_*(n)) && \text{strictness of } f \\
&\Rightarrow p(f'(m)) \sqsubset_Y f(p'_*(n)) && \text{commutativity of the diagram} \\
&\Rightarrow f'(m) \sqsubset_Q p^{-1}(f(p'_*(n))) && \sqsubset\text{-continuity of } p \\
&\Rightarrow f'(m) \sqsubset_Q f'(p'^{-1}(p'_*(n))) \leq f'(n) && \text{BCP} \\
&\Rightarrow f'(m) \sqsubset_Q f'(n)
\end{aligned}$$

If f is \sqsubset -co-strict, then

$$\begin{aligned}
f'^{-1}(m) \sqsubset_P n &\Rightarrow p'(f'^{-1}(m)) \sqsubset_X p'_*(n) && \text{initiality of } p' \\
&\Rightarrow f^{-1}(p(m)) \sqsubset_X p'_*(n) && \text{BCP} \\
&\Rightarrow p(m) \sqsubset_Y f_*(p'_*(n)) && \text{co-strictness of } f \\
&\Rightarrow m \sqsubset_Q p^{-1}(f_*(p'_*(n))) && \sqsubset\text{-continuity of } p \\
&\Rightarrow m \sqsubset_Q f'_*(p'^{-1}(p'_*(n))) \leq f'_*(n) && \text{BCP} \\
&\Rightarrow m \sqsubset_Q f'_*(n)
\end{aligned}$$

Suppose f is \sqsubset -final, then

$$\begin{aligned}
f'^{-1}(m) \sqsubset_P f'^{-1}(n) &\Rightarrow p'(f'^{-1}(m)) \sqsubset_X p'_*(f'^{-1}(n)) && \text{initiality of } p' \\
&\Rightarrow f^{-1}(p(m)) \sqsubset_X f^{-1}(p_*(n)) && \text{BCP} \\
&\Rightarrow p(m) \sqsubset_Y p_*(n) && \text{finality of } f \\
&\Rightarrow m \sqsubset_Y n && \text{continuity of } p
\end{aligned}$$

(ii) Let p be final. If f' is \sqsubset -final then f is \sqsubset -final by Proposition 3.2.24.

Assume f' is \sqsubset -strict, then

$$\begin{aligned}
m \sqsubset_X n &\Rightarrow p'^{-1}(m) \sqsubset_P p'^{-1}(n) && \sqsubset\text{-continuity of } p' \\
&\Rightarrow f'(p'^{-1}(m)) \sqsubset_Q f'(p'^{-1}(n)) && \sqsubset\text{-strictness of } f' \\
&\Rightarrow p^{-1}(f(m)) \sqsubset_Q p^{-1}(f(n)) && \text{BCP} \\
&\Rightarrow f(m) \sqsubset_Y f(n) && \text{finality of } p
\end{aligned}$$

Suppose f' is \sqsubseteq -co-strict, then

$$\begin{aligned}
f^{-1}(m) \sqsubseteq_X n &\Rightarrow p'^{-1}(f^{-1}(m)) \sqsubseteq_P p'^{-1}(n) && \sqsubseteq\text{-continuity of } p' \\
&\Rightarrow f'^{-1}(p^{-1}(m)) \sqsubseteq_P p'^{-1}(n) && \text{BCP} \\
&\Rightarrow p^{-1}(m) \sqsubseteq_Q f'_*(p^{-1}(n)) && \text{co-strictness of } f' \\
&\Rightarrow p^{-1}(m) \sqsubseteq_Q p^{-1}(f_*(n)) && \text{BCP} \\
&\Rightarrow m \sqsubseteq_Y f_*(n) && \text{finality of } p
\end{aligned}$$

If f' is \sqsubset -initial, then

$$\begin{aligned}
m \sqsubset_X n &\Rightarrow p'^{-1}(m) \sqsubset_P p'^{-1}(n) && \sqsubseteq\text{-continuity of } p' \\
&\Rightarrow f'(p'^{-1}(m)) \sqsubset_Q f'_*(p'^{-1}(n)) && \text{initiality of } f' \\
&\Rightarrow p^{-1}(f(m)) \sqsubset_Q p^{-1}(f_*(n)) && \text{BCP} \\
&\Rightarrow f(m) \sqsubset_Y f_*(n) && \text{finality of } p
\end{aligned}$$

□

Definition 2.2.9.

([DT95, GT00, CGT01]) Let $f : X \rightarrow Y$ be a \mathcal{C} and $c \in CL(\mathcal{C}, \mathcal{M})$. Then f is

- (1) c -closed if $f(c_X(m)) = c_Y(f(m))$ for all $m \in \text{sub}X$.
- (2) c -open if $f^{-1}(c_Y(n)) = c_X(f^{-1}(n))$ for all $n \in \text{sub}Y$.
- (3) c -initial if $c_X(m) = f^{-1}(c_Y(f(m)))$ for all $m \in \text{sub}X$.
- (4) c -final if $c_Y(n) = f(c_X(f^{-1}(n)))$ for all $n \in \text{sub}Y$.

Our next proposition shows that when $\sqsubset \in \bigwedge -\text{TORD}(\mathcal{C}, \mathcal{M})$, then the \sqsubset -strict morphism (resp. \sqsubset -co-strict, \sqsubset -initial, \sqsubset -final) correspond the c^\sqsubset -closed (resp. c^\sqsubset -open, c^\sqsubset -initial and c -final) morphisms.

Proposition 2.2.10. Let $\sqsubset \in \bigwedge -\text{TORD}(\mathcal{C}, \mathcal{M})$, and let for any morphism $f \in \mathcal{C}$, the inverse image f^{-1} commutes with the join of subobjects. Then $f : X \rightarrow Y$ is \sqsubset -initial (resp. \sqsubset -co-strict, \sqsubset -final, \sqsubset -strict) if and only if it is c^\sqsubset -initial (resp c^\sqsubset -open, c^\sqsubset -final, c^\sqsubset -closed).

Proof. (1) Let f be a c^\sqsubset -initial and $\sqsubset \in \bigwedge -\text{TORD}(\mathcal{C}, \mathcal{M})$. Then $f(m) \sqsubset_Y f_*(n) \Leftrightarrow c_Y^\sqsubset(f(m)) \leq f_*(n) \Leftrightarrow f^{-1}(c_Y^\sqsubset(f(m))) \leq n \Leftrightarrow c_X^\sqsubset(m) \leq n \Leftrightarrow m \sqsubset_X n$. Conversely if f is \sqsubset -initial, then $f^{-1}(c_Y^\sqsubset(f(m))) \leq n \Leftrightarrow c_Y^\sqsubset(f(m)) \leq f_*(n) \Leftrightarrow f(m) \sqsubset_Y f_*(n) \Leftrightarrow m \sqsubset_X n \Leftrightarrow c_X^\sqsubset(m) \leq n$.

(2) Assume f is c^\sqsubset -open, then $m \sqsubset_Y f_*(n) \Leftrightarrow c_Y^\sqsubset(m) \leq f_*(n) \Leftrightarrow f^{-1}(c_Y^\sqsubset(m)) \leq n \Leftrightarrow c_X^\sqsubset(f^{-1}(m)) \leq n \Leftrightarrow f^{-1}(m) \sqsubset_X n$. Conversely if f \sqsubset -co-strict then $c^\sqsubset(f^{-1}(m)) \leq n \Leftrightarrow f^{-1}(m) \sqsubset_X n \Leftrightarrow m \sqsubset_Y f_*(n) \Leftrightarrow c_Y^\sqsubset(m) \leq f_*(n) \Leftrightarrow f^{-1}(c_Y^\sqsubset(m)) \leq n$.

(3) If f is c^\sqsubset -final, then $f^{-1}(m) \sqsubset_X f^{-1}(n) \Leftrightarrow c^\sqsubset(f^{-1}(m)) \leq f^{-1}(n) \Leftrightarrow f(c^\sqsubset(f^{-1}(m))) \leq n \Leftrightarrow c_Y^\sqsubset(m) \leq n \Leftrightarrow m \sqsubset_Y n$. On the other hand if f is \sqsubset -final, then $f(c^\sqsubset(f^{-1}(m))) \leq n \Leftrightarrow c_Y^\sqsubset(m) \leq n \Leftrightarrow m \sqsubset_Y n \Leftrightarrow c_X^\sqsubset(f^{-1}(m)) \leq f^{-1}(n) \Leftrightarrow f^{-1}(m) \sqsubset_X f^{-1}(n) \Leftrightarrow m \sqsubset_Y n \Leftrightarrow c_Y^\sqsubset(m) \leq n$.

(4) ([HIR16]) If f is \sqsubset -strict, then $f(c_X^\sqsubset(m)) \leq n \Leftrightarrow c_X^\sqsubset(m) \leq f^{-1}(n) \Leftrightarrow m \sqsubset_X f^{-1}(n) \Leftrightarrow f(m) \sqsubset_Y n \Leftrightarrow c_Y^\sqsubset(f(m)) \leq n$. Conversely if $f(c_X^\sqsubset(m)) = c_Y^\sqsubset(f(m))$ then, $m \sqsubset_X f^{-1}(n) \Leftrightarrow c_X^\sqsubset(m) \leq f^{-1}(n) \Leftrightarrow f(c_X^\sqsubset(m)) \leq n \Leftrightarrow c_Y^\sqsubset(f(m)) \leq n \Leftrightarrow f(m) \sqsubset_Y n$.

□

Open morphism with respect to an interior operator was studied in ([Cas15]). Assuming the pre-image commutes with the join of subobjects, i -initial and i -final morphisms were introduced in [Raz12]. Recently in ([AH19]), the i -closed morphism has been defined and a systematic study of the four classes of morphisms with respect to an interior operator is provided. In the next proposition we prove that if $\sqsubset \in \check{\vee}\text{-TORD}(\mathcal{C}, \mathcal{M})$, then the \sqsubset -strict morphism (resp. \sqsubset -co-strict, \sqsubset -initial, \sqsubset -final) correspond to the i -open (resp. i -closed, i -initial and i -final) morphisms. As introduced in ([AH19]),

Definition 2.2.11. *Assume that for every \mathcal{C} -morphism $f : X \rightarrow Y$, f^{-1} has a right adjoint f_* and $i \in \text{INT}(\mathcal{C}, \mathcal{M})$. Then f is*

- (1) i -closed if $f_*(i_X(m)) = i_Y(f_*(m))$ for all $m \in \text{sub}X$.
- (2) i -open if $i_X(f^{-1}(n)) = f^{-1}(i_Y(n))$ for all $n \in \text{sub}Y$.
- (3) i -initial if $i_X(m) = f^{-1}(i_Y(f_*(m)))$ for all $m \in \text{sub}X$.
- (4) i -final if $i_Y(n) = f_*(i_X(f^{-1}(n)))$ for all $n \in \text{sub}Y$.

Proposition 2.2.12. *Let $\sqsubset \in \check{\vee}\text{-TORD}(\mathcal{C}, \mathcal{M})$, and let for any morphism $f \in \mathcal{C}$, the inverse image f^{-1} commutes with the join of subobjects. Then $f : X \rightarrow Y$ is \sqsubset -initial (resp. \sqsubset -co-strict, \sqsubset -final, \sqsubset -strict) if and only if it is i -initial (resp. i -closed, i -final, i -open).*

Proof. (1) If f is i -initial and $\sqsubset \in \check{\vee}\text{-TORD}(\mathcal{C}, \mathcal{M})$, then $f(m) \sqsubset_Y f_*(n) \Leftrightarrow f(m) \leq i_Y^\sqsubset(f_*(n)) \Leftrightarrow m \leq f^{-1}(i_Y^\sqsubset(f_*(n))) \Leftrightarrow m \leq i_X^\sqsubset(n) \Leftrightarrow m \sqsubset_X n$. Conversely if f is \sqsubset -initial then $m \leq f^{-1}(i_Y(f_*(n))) \Leftrightarrow f(m) \leq i_Y(f_*(n)) \Leftrightarrow f(m) \sqsubset_Y f_*(n) \Leftrightarrow m \sqsubset_X n \Leftrightarrow m \leq i_X(n)$.

(2) Let f be i -closed, then $f^{-1}(m) \sqsubset_X n \Leftrightarrow f^{-1}(m) \leq i_X(n) \Leftrightarrow m \leq f_*(i_X^\sqsubset(n)) \Leftrightarrow m \leq i_Y^\sqsubset(f_*(n)) \Leftrightarrow m \sqsubset_Y f_*(n)$. Conversely if f is \sqsubset -co-strict, $m \leq i_Y^\sqsubset(f_*(n)) \Leftrightarrow m \sqsubset_Y f_*(n) \Leftrightarrow f^{-1}(m) \sqsubset_X n \Leftrightarrow f^{-1}(m) \leq i_X^\sqsubset(n) \Leftrightarrow m \leq f_*(i_X^\sqsubset(n))$.

(3) If f is i -final, then $f^{-1}(m) \sqsubset_X f^{-1}(n) \Leftrightarrow f^{-1}(m) \leq i_X^\sqsubset(f^{-1}(n)) \Leftrightarrow m \leq f_*(i_X^\sqsubset(f^{-1}(n))) \Leftrightarrow m \leq i_Y^\sqsubset(n) \Leftrightarrow m \sqsubset_Y n$. Conversely if f is \sqsubset -final then $m \leq f_*(i_X(f^{-1}(n))) \Leftrightarrow f^{-1}(m) \leq i_X(f^{-1}(n)) \Leftrightarrow f^{-1}(m) \sqsubset_X f^{-1}(n) \Leftrightarrow m \sqsubset_Y n \Leftrightarrow m \leq i_Y(n)$.

(4) ([HIR16]) Let f be \sqsubset -strict, $m \leq f^{-1}(i_Y(n)) \Leftrightarrow f(m) \leq i_Y(n) \Leftrightarrow f(m) \sqsubset_Y n \Leftrightarrow m \sqsubset_X f^{-1}(n) \Leftrightarrow m \leq i_X(f^{-1}(n))$. On the other hand if $f^{-1}(i_Y^\sqsubset(m)) = i_X^\sqsubset(f^{-1}(m))$ then, $m \sqsubset_X f^{-1}(n) \Leftrightarrow m \leq i_X^\sqsubset(f^{-1}(n)) \Leftrightarrow m \leq f^{-1}(i_Y^\sqsubset(n)) \Leftrightarrow f(m) \leq i_Y^\sqsubset(n) \Leftrightarrow$

$$f(m) \sqsubset_Y n.$$

□

Apart from the four classes of morphisms studied in above, a weaker notion of \sqsubset -final morphism will be usefull.

Definition 2.2.13. Let $\sqsubset \in \text{TORD}$. A \mathcal{C} -morphism $f : X \rightarrow Y$ is said to be weakly \sqsubset -final if for any $m, n \in \text{sub}Y$ such that $m \leq n$, $m \sqsubset_Y n \Leftrightarrow f^{-1}(m) \sqsubset_X f^{-1}(n)$.

We note that if $f \in \mathcal{E}$, then f is weakly \sqsubset -final if and only if it is \sqsubset -final.

Proposition 2.2.14. Let $\sqsubset \in \text{TORD}$ and $f : X \rightarrow Y$ be a \mathcal{C} -morphism.

- (1) If $\sqsubset \in \wedge\text{-TORD}$, then f is weakly \sqsubset -final if and only if $c_Y^{\sqsubset}(m) = m \vee f(c_X^{\sqsubset}(f^{-1}(m)))$.
- (2) If $\sqsubset \in \vee\text{-TORD}$, then f is weakly \sqsubset -final if and only if $i_Y^{\sqsubset}(m) = m \wedge f_*(i_X^{\sqsubset}(f^{-1}(m)))$.

Proof. (1) Let $\sqsubset \in \wedge\text{-TORD}$ and f be weakly \sqsubset -final and $m \in \text{sub}Y$. For any $n \in \text{sub}Y$ such that $m \leq n$, $c_Y^{\sqsubset}(m) \leq n \Leftrightarrow m \sqsubset_Y n \Leftrightarrow f^{-1}(m) \sqsubset_X f^{-1}(n) \Leftrightarrow f(c_Y^{\sqsubset}(f^{-1}(m))) \leq n \Leftrightarrow m \vee f(c_X^{\sqsubset}(f^{-1}(m))) \leq n$.

On the other hand if $c_Y^{\sqsubset}(p) = p \vee f(c_X^{\sqsubset}(f^{-1}(p)))$ for any $p \in \text{sub}Y$, then for all $m, n \in \text{sub}Y$ such that $m \leq n$, $m \sqsubset_Y n \Leftrightarrow c_Y^{\sqsubset}(m) \leq n \Leftrightarrow m \vee f(c_X^{\sqsubset}(f^{-1}(m))) \leq n \Leftrightarrow f(c_Y^{\sqsubset}(f^{-1}(m))) \leq n \Leftrightarrow c_X^{\sqsubset}(f^{-1}(m)) \leq f^{-1}(n) \Leftrightarrow f^{-1}(m) \sqsubset_X f^{-1}(n)$.

- (2) Assume that $\sqsubset \in \vee\text{-TORD}$, f is weakly \sqsubset -final and $m \in \text{sub}Y$. Then for any $n \in \text{sub}Y$ such that $m \leq n$, $m \leq i_Y^{\sqsubset}(n) \Leftrightarrow m \sqsubset_Y n \Leftrightarrow f^{-1}(m) \sqsubset_X f^{-1}(n) \Leftrightarrow f^{-1}(m) \leq i_X^{\sqsubset}(f^{-1}(n)) \Leftrightarrow m \leq f_*(i_X^{\sqsubset}(f^{-1}(n))) \Leftrightarrow m \leq n \wedge f_*(i_X^{\sqsubset}(f^{-1}(n)))$.

Conversely if $i_Y^{\sqsubset}(p) = p \wedge f_*(i_X^{\sqsubset}(f^{-1}(p)))$ for any $p \in \text{sub}Y$, then for all $m, n \in \text{sub}Y$ such that $m \leq n$, $m \sqsubset_Y n \Leftrightarrow m \leq i_Y^{\sqsubset}(n) \Leftrightarrow m \leq n \wedge f_*(i_X^{\sqsubset}(f^{-1}(n))) \Leftrightarrow m \leq f_*(i_X^{\sqsubset}(f^{-1}(n))) \Leftrightarrow f^{-1}(m) \leq i_X^{\sqsubset}(f^{-1}(n)) \Leftrightarrow f^{-1}(m) \sqsubset_X f^{-1}(n)$.

□

2.3 Some Properties of Topogenous orders

Having observed that a topogenous order provides a unified approach to closure and interior operators, it is natural to think of properties of topogenous orders that would

specialize into known ones for these two operators. This is the point we wish to make in this section.

Definition 2.3.1. A topogenous order \sqsubset is hereditary if $n \sqsubset_M p \Leftrightarrow m(n) \sqsubset_X m_*(p)$ for any $M : M \rightarrow X, p, n \in \text{sub}M$ and $X \in \mathcal{C}$.

It is seen from Definition 2.2.2(4) that our definition for hereditariness is equivalent to the fact that every morphism in \mathcal{M} is \sqsubset -initial.

We shall now show that the definition above corresponds to the hereditary closure operator if $\sqsubset \in \bigwedge -\text{TORD}(\mathcal{C}, \mathcal{M})$ and to the hereditary interior operator if $\sqsubset \in \bigvee -\text{TORD}(\mathcal{C}, \mathcal{M})$.

Proposition 2.3.2. Let $\sqsubset \in \bigwedge -\text{TORD}(\mathcal{C}, \mathcal{M})$. Then \sqsubset is hereditary if and only if $n^{-1}(c_X^\sqsubset(n(p))) = c_N^\sqsubset(p)$ for any $p \in \text{sub}N$.

Proof. If \sqsubset is hereditary, then $c_N^\sqsubset(p) \leq j \Leftrightarrow p \sqsubset_N j \Leftrightarrow n(p) \sqsubset_X n_*(j) \Leftrightarrow c_X^\sqsubset(n(p)) \leq n_*(j) \Leftrightarrow n^{-1}(c_X^\sqsubset(n(p))) \leq j$. Conversely if $n^{-1}(c_X^\sqsubset(n(p))) = c_N^\sqsubset(p)$ then $p \sqsubset_N j \Leftrightarrow c_N^\sqsubset(p) \leq j \Leftrightarrow n^{-1}(c_X^\sqsubset(n(p))) \leq j \Leftrightarrow c_X^\sqsubset(n(p)) \leq n_*(j) \Leftrightarrow n(p) \sqsubset_X n_*(j)$. \square

Proposition 2.3.3. Let $\sqsubset \in \bigvee -\text{TORD}(\mathcal{C}, \mathcal{M})$. Then \sqsubset is hereditary if and only if $n^{-1}(i_X^\sqsubset(n_*(j))) = c_N^\sqsubset(j)$ for any $n \in \text{sub}X$ and $j \in \text{sub}N$.

Proof. If $n^{-1}(i_X^\sqsubset(n_*(j))) = c_N^\sqsubset(j)$, then $p \sqsubset_N j \Leftrightarrow p \leq i_N^\sqsubset(j) \Leftrightarrow p \leq n^{-1}(i_X^\sqsubset(n_*(j))) \Leftrightarrow n(p) \leq i_X^\sqsubset(n_*(j)) \Leftrightarrow n(p) \sqsubset_X n_*(j)$. Conversely if \sqsubset is hereditary, $p \leq i_N^\sqsubset(j) \Leftrightarrow p \sqsubset_N j \Leftrightarrow n(p) \sqsubset_X n_*(j) \Leftrightarrow n(p) \leq i_X^\sqsubset(n_*(j)) \Leftrightarrow p \leq n^{-1}(i_X^\sqsubset(n_*(j)))$. \square

Since being a hereditary topogenous order means that every morphism in \mathcal{M} is \sqsubset -initial, the above two propositions can be obtained by specializing Propositions 2.2.10(1) and 2.2.12(2) to morphisms in \mathcal{M} .

Proposition 2.3.4. Consider the following pullback diagram.

$$\begin{array}{ccc} f^{-1}(N) & \xrightarrow{f'} & N \\ n' \downarrow & & \downarrow n \\ X & \xrightarrow{f} & Y \end{array}$$

and let \sqsubset be a hereditary topogenous order. Then the restriction f' is \sqsubset -final (resp. \sqsubset -strict, \sqsubset -initial, \sqsubset -co-strict) provided that f is \sqsubset -final (resp. \sqsubset -strict, \sqsubset -initial, \sqsubset -co-strict).

Proof. Similar to the one of Proposition 2.2.8(i) \square

Proposition 2.3.5. *Assume that every morphism in \mathcal{E} is \sqsubset -final. Let p be an \mathcal{E} -morphism in the following pullback diagram.*

$$\begin{array}{ccc} & \xrightarrow{f'} & \\ p' \downarrow & \square & \downarrow p \\ & \xrightarrow{f} & \end{array}$$

Then f is \sqsubset -final (resp. \sqsubset -strict, \sqsubset -initial, \sqsubset -co-strict) provided that f' is \sqsubset -final (resp. \sqsubset -strict, \sqsubset -initial, \sqsubset -co-strict).

Proof. Similar to the one of Proposition 2.2.8(ii) □

We next prove that there are particular topogenous orders that correspond to the additive and grounded closure and interior operators.

Let us now consider the following classes of topogenous orders.

Definition 2.3.6.

- (1) $\bigwedge -aTORD(\mathcal{C}, \mathcal{M})$: the class of all topogenous orders in $\bigwedge -TORD(\mathcal{C}, \mathcal{M})$ satisfying (T4): $m \sqsubset_X n$ and $p \sqsubset_X q \Rightarrow m \vee n \sqsubset_X p \vee q$, for all $m, n \in subX$.
- (2) $\bigvee -aTORD(\mathcal{C}, \mathcal{M})$: the class of all topogenous orders in $\bigvee -TORD(\mathcal{C}, \mathcal{M})$ satisfying (T5): $m \sqsubset_X n$ and $p \sqsubset_X q \Rightarrow m \wedge n \sqsubset_X p \wedge q$,
- (3) $\bigwedge -gTORD(\mathcal{C}, \mathcal{M})$: the class of all topogenous orders in $\bigwedge -TORD(\mathcal{C}, \mathcal{M})$ satisfying (T6): $0_X \sqsubset_X 0_X$,
- (4) $\bigvee -gTORD(\mathcal{C}, \mathcal{M})$: the class of all topogenous orders in $\bigvee -TORD(\mathcal{C}, \mathcal{M})$ satisfying (T7): $1_X \sqsubset_X 1_X$.
- (5) $\bigwedge -a'TORD(\mathcal{C}, \mathcal{M})$: the class of all topogenous orders in $\bigwedge -TORD(\mathcal{C}, \mathcal{M})$ satisfying (T4'): $\forall i \in I, m_i \sqsubset_X n_i \Rightarrow \bigvee_{i \in I} m_i \sqsubset_X \bigvee_{i \in I} n_i$ for $m_i, n_i \in subX$.

Proposition 2.3.7. *The following statements hold true.*

- (1) $\bigwedge -aTORD(\mathcal{C}, \mathcal{M}) \cong aCL(\mathcal{C}, \mathcal{M})$.
- (2) $\bigvee -aTORD(\mathcal{C}, \mathcal{M}) \cong aINT(\mathcal{C}, \mathcal{M})$.
- (3) $\bigwedge -gTORD(\mathcal{C}, \mathcal{M}) \cong gCL(\mathcal{C}, \mathcal{M})$.
- (4) $\bigvee -gTORD(\mathcal{C}, \mathcal{M}) \cong gINT(\mathcal{C}, \mathcal{M})$.

(5) $\bigwedge -a'TORD(\mathcal{C}, \mathcal{M}) \cong a'CL(\mathcal{C}, \mathcal{M})$.

Proof. (1) Assume \sqsubset satisfies (T4). Obviously, $c_X^\sqsubset(m) \vee c_X^\sqsubset(p) \leq c_X^\sqsubset(m \vee p)$. Let $a \leq c_X^\sqsubset(m \vee p)$. Then $a \leq \bigwedge \{q \mid m \vee p \sqsubset q\}$. Since $m \sqsubset c^\sqsubset(m)$ and $p \sqsubset c^\sqsubset(p)$, by (T4) $m \vee p \sqsubset c^\sqsubset(m) \vee c^\sqsubset(p)$. Thus $a \leq c(m) \vee c(p)$. Conversely, if c is additive and $m \sqsubset^c n$, $p \sqsubset^c q$. Then $c_X(m) \leq n$ and $c_X(p) \leq q$. This implies that $c_X(m \vee n) = c_X(m) \vee c_X(n) \leq p \vee q$. Thus $m \vee n \sqsubset^c p \vee q$.

(2) Assume that (T5) holds. Clearly $i_X^\sqsubset(m \wedge p) \leq i^\sqsubset(m) \wedge i^\sqsubset(p)$. Let $a \leq i^\sqsubset(m) \wedge i^\sqsubset(p)$ then $a \leq i^\sqsubset(m) = \bigvee \{q \mid p \sqsubset q\}$. This means that there are n and q such that $a \leq n, a \leq q$ with $m \sqsubset n$ and $n \sqsubset q$. By assumption, $m \wedge p \sqsubset n \wedge q$ and $a \leq n \wedge q = l$. Thus $a \leq \bigvee \{l \mid m \wedge p \sqsubset l\} = i^\sqsubset(m \wedge p)$ and $i^\sqsubset(m) \wedge i^\sqsubset(p) \leq i^\sqsubset(m \wedge p)$. Conversely, if i is additive and $m \sqsubseteq^i n$, $p \sqsubseteq^i q$ then $m \leq i_X(m)$ and $p \leq i_X(q) \Rightarrow m \wedge p \leq i_X(n) \wedge i_X(q) = i_X(m \wedge q)$. Thus $m \wedge p \sqsubset^i n \wedge q$.

(3) and (4) are clear.

(5) Assume that \sqsubset satisfies (T4'). Clearly, $\bigvee_{i \in I} c_X^\sqsubset(m_i) \leq c_X^\sqsubset(\bigvee_{i \in I} m_i)$. Let $n \leq c_X^\sqsubset(\bigvee_{i \in I} m_i) = \bigwedge \{p \mid \bigvee_{i \in I} m_i \sqsubset_X p\}$. Since $m_i \sqsubset_X c^\sqsubset(m_i)$ for each $i \in I$, by (T4'), $\bigvee_{i \in I} m_i \sqsubset_i \bigvee_{i \in I} c_X^\sqsubset(m_i)$. Thus $n \leq \bigvee_{i \in I} c_X^\sqsubset(m_i)$.

On the other hand if c is fully additive and $m \sqsubset_X^c n_i$ for all $i \in I$, then $c_X(m_i) \leq n_i \Rightarrow \bigvee_{i \in I} c_X(m_i) = c_X(\bigvee_{i \in I} m_i) \leq \bigvee_{i \in I} n_i \Leftrightarrow \bigvee_{i \in I} m_i \sqsubset_X \bigvee_{i \in I} n_i$.

□

Proposition 2.3.8.

Let $\sqsubset, \sqsubset' \in TORD(\mathcal{C}, \mathcal{M})$. If \sqsubset and \sqsubset' satisfy (T4), (T5). Then so does $\sqsubset \circ \sqsubset'$.

Proof. Assume that \sqsubset and \sqsubset' satisfy (T4). If $m \sqsubset_X \circ \sqsubset'_X n$ and $m' \sqsubset_X \circ \sqsubset'_X n'$, then $m \sqsubset_X p \sqsubset'_X n$ and $m' \sqsubset_X p' \sqsubset'_X n'$ for some $p, p' \in \text{sub}X$. Thus $m \vee m' \sqsubset_X p \vee p' \sqsubset'_X n \vee n'$, that is $m \vee m' \sqsubset_X \circ \sqsubset'_X n \vee n'$. A similar argument holds for the case of (T5). □

Proposition 2.3.9. Let $\{\sqsubset_X^i \mid i \in I\} \subseteq TORD(\mathcal{C}, \mathcal{M})$ for all $X \in \mathcal{C}$ and consider the topogenous orders $\sqsubset_X^* = \bigcup \{\sqsubset_X^i \mid i \in I\}$ and $\sqsubset_X^\diamond = \bigcap \{\sqsubset_X^i \mid i \in I\}$ for all $X \in \mathcal{C}$. Then

(1) \sqsubset_X^* is hereditary if and only if there is $i \in I$ such that \sqsubset_X^i is hereditary.

(2) \sqsubset_X^\diamond is hereditary if and only if \sqsubset_X^i is hereditary for each $i \in I$.

- (3) \sqsubset_X^* satisfies (T4) to (T7) if and only if there is $i \in I$ such that \sqsubset_X^i satisfies (T4) to (T7).
- (4) \sqsubset_X^\diamond satisfies (T4) to (T7) if and only if \sqsubset_X^i satisfies for each $i \in I$ (T4) to (T7).

2.4 Lifting a Topogenous order along an \mathcal{M} -fibration

Considering categories supplied with fixed closure operators with respect to classes of subobjects, D. Dikranjan and W. Tholen [DT95] generalized the notion of c -continuity of morphisms to functors and defined the least and largest closure operators for which the functor is continuous. A concrete description of this largest closure operator was obtained in the case of an \mathcal{M} -fibration. In this section we wish to define a topogenous order induced by an \mathcal{M} -fibration which, includes D. Dikranjan and W. Tholen's closure as a particular case and allows also to lift an interior operator along this functor.

Let us start by recalling from [DT95] that for an \mathcal{M} -fibration $F : \mathcal{A} \rightarrow \mathcal{C}$, $(\mathcal{E}_F, \mathcal{M}_F)$ where $\mathcal{E}_F = F^{-1}\mathcal{E} = \{e \in \mathcal{A} \mid Fe \in \mathcal{E}\}$ and $\mathcal{M}_F = F^{-1}\mathcal{M} \cap \text{Ini}F$, with $\text{Ini}F$ the class of F -initial morphisms, is a factorization system in \mathcal{A} and \mathcal{M} -subobject properties in \mathcal{C} are inherited by \mathcal{M}_F -subobjects in \mathcal{A} . In particular,

- (1) \mathcal{A} has \mathcal{M}_F -pullbacks if \mathcal{C} has \mathcal{M} -pullbacks.
- (2) \mathcal{A} is \mathcal{M}_F -complete if \mathcal{C} is \mathcal{M} -complete.
- (3) the \mathcal{M}_F -images and \mathcal{M}_F -inverse images are obtained by initially lifting \mathcal{M} -images and \mathcal{M} -inverse images. Consequently $Ff^{-1}(n) = (Ff)^{-1}(Fn)$ and $(Ff)(Fm) = Ff(m)$ for any $f \in \mathcal{A}$ and suitable subobjects n and m .

Lemma 2.4.1. [DT95] *Let $F : \mathcal{A} \rightarrow \mathcal{C}$ be a faithful \mathcal{M} -fibration.*

- (1) *For any $X \in \mathcal{A}$, $\text{sub}X$ and $\text{sub}FX$ are order equivalent with the inverse assignments, $\gamma_X : \text{sub}X \rightarrow \text{sub}FX$ and $\delta_X : \text{sub}FX \rightarrow \text{sub}X$, given by $\gamma_X(m) = Fm$ and $\delta_X(n) = p$ with $Fp = n$ and $p \in \text{Ini}F$.*
- (2) *For any $f : X \rightarrow Y \in \mathcal{A}$ and suitable subobjects n, m, n' and m' .*
 - (1) $\gamma_Y(f(m)) = (Ff)(\gamma_X(m))$.
 - (2) $f(\delta_X(n)) = \delta_Y(Ff)(n)$.

$$(3) f^{-1}(\delta_Y(m')) = \delta_X((Ff)^{-1}(m')).$$

$$(4) \gamma_X(f^{-1}(n')) = (Ff)^{-1}(\gamma_Y(n')).$$

Proof. (1) is clear and (2) follows from the fact that F preserves images and inverse images of subobjects. \square

Proposition 2.4.2. *Let $F : \mathcal{A} \rightarrow \mathcal{C}$ be a faithful \mathcal{M} -fibration and \sqsubset be a topogenous order on \mathcal{C} with respect to \mathcal{M} . Defines \sqsubset^F by $m \sqsubset_X^F n \Leftrightarrow Fm \sqsubset_{FX} \gamma_X(n)$.*

(1) \sqsubset^F is a topogenous order on \mathcal{A} with respect to \mathcal{M}_F .

(2) \sqsubset^F is interpolative and satisfies (T4) provided \sqsubset has the same properties.

Proof. (1) (T1) $m \sqsubset_X^F n \Leftrightarrow Fm \sqsubset_{FX} \gamma_X(n) \Rightarrow Fm \leq \gamma_X(n) \Rightarrow m = \delta_X(Fm) \leq \delta_X(\gamma_X(n)) = n$. (T2) is clear.

For (T3), let $f : X \rightarrow Y$ be an \mathcal{A} -morphism and $f(m) \sqsubset_Y^F n$. Then $Ff(m) \sqsubset_{FY} \gamma_Y(n) \Rightarrow (Ff)(Fm) \sqsubset_{FY} \gamma_Y(n) \Rightarrow Fm \sqsubset_{FX} (Ff)^{-1}(\gamma_Y(n)) = \gamma_X(f^{-1}(n)) \Leftrightarrow m \sqsubset_X^F f^{-1}(n)$.

(2) If \sqsubset^F is interpolative, then $m \sqsubset_X^F n \Leftrightarrow Fm \sqsubset_{FX} \gamma_X(n) \Rightarrow \exists p \in \text{sub}FX \mid Fm \sqsubset_{FX} p \sqsubset_{FX} \gamma_X(n) \Rightarrow Fm \sqsubset_{FX} \gamma_X(\delta_X(p))$ and $F(\delta_X(p)) \sqsubset_{FX} \gamma_X(n)$, since $F\delta_X(p) = p$ and $\gamma_X(\delta_X(p)) = p$. Thus $m \sqsubset_X^F \delta_X(p) \sqsubset_X^F n$.

If \sqsubset^F satisfies (T4), then $m \sqsubset_X^F n$ and $m' \sqsubset_X^F n'$. This implies that $Fm \sqsubset_{FX} \gamma_X(n)$ and $Fm' \sqsubset_{FX} \gamma_X(n)$. Thus $Fm \wedge Fm' \sqsubset_{FX} \gamma_X(n) \wedge \gamma_X(n') \Rightarrow F(m \wedge m') \sqsubset_{FX} \gamma_X(n \wedge n') \Leftrightarrow m \wedge m' \sqsubset_X^{FX} n \wedge n'$. \square

In the light of Propositions 2.1.4 and 2.1.5, we can prove the following.

Proposition 2.4.3. (1) *If $\sqsubset \in \bigwedge$ -TORD, then $m \sqsubset_X^F n \Leftrightarrow \delta_X(c_{FX}^{\sqsubset}(Fm)) \leq n$.*

(2) *If $\sqsubset \in \bigvee$ -TORD, then $m \sqsubset_X^F n \Leftrightarrow m \leq \delta_X(i_{FX}^{\sqsubset}(\gamma_X(n)))$.*

Proof. (1) If $\sqsubset \in \bigwedge$ -TORD, then $m \sqsubset_X^F n \Leftrightarrow Fm \sqsubset_{FX} \gamma_X(n) \Leftrightarrow c_{FX}^{\sqsubset}(Fm) \leq \gamma_X \Leftrightarrow \delta_X(c_{FX}^{\sqsubset}(Fm)) \leq \delta_X(\gamma_X(n)) \Leftrightarrow \delta_X(c_{FX}^{\sqsubset}(Fm)) \leq n$.

(2) If $\sqsubset \in \bigvee$ -TORD, then $m \sqsubset_X^F n \Leftrightarrow Fm \sqsubset_{FX} \gamma_X(n) \Leftrightarrow Fm \leq i_{FX}^{\sqsubset}(\gamma_X(n)) \Leftrightarrow \delta_X(Fm) \leq \delta_X(i_{FX}^{\sqsubset}(\gamma_X(n))) \Leftrightarrow m \leq \delta_X(i_{FX}^{\sqsubset}(\gamma_X(n)))$. \square

2.5 Topogenous orders induced by (co)pointed endofunctors

We define the continuity of a \mathcal{C} -morphism with respect to two topogenous orders on \mathcal{C} and use it to construct new topogenous orders from old. It is shown that for a pointed endofunctor (F, η) of \mathcal{C} and a topogenous order \sqsubset on \mathcal{C} , there is a coarsest topogenous order $\sqsubset^{F, \eta}$ on \mathcal{C} for which every $\eta_X : X \rightarrow FX$ is $(\sqsubset^{F, \eta}, \sqsubset)$ -continuous and dually for a copointed endofunctor of \mathcal{C} , there is a finest topogenous order $\sqsubset^{G, \varepsilon}$ on \mathcal{C} for which every $\varepsilon_X : GX \rightarrow X$ is $(\sqsubset, \sqsubset^{G, \varepsilon})$ -continuous. In particular, for meet preserving topogenous order, $\sqsubset^{F, \eta}$ and $\sqsubset^{G, \varepsilon}$ correspond to the closure operators obtained by Dikranjan and W. Tholen in ([DT95]) while join preserving $\sqsubset^{F, \eta}$ and $\sqsubset^{G, \varepsilon}$ allow us to construct the interior operators induced by F and G respectively.

Definition 2.5.1. Let $\sqsubset, \sqsubset' \in \text{TOR}D(\mathcal{C}, \mathcal{M})$. A \mathcal{C} -morphism $f : X \rightarrow Y$ is (\sqsubset, \sqsubset') -continuous if $f(m) \sqsubset'_Y n \Rightarrow m \sqsubset_X f^{-1}(n)$ or equivalently $p \sqsubset'_Y n \Rightarrow f^{-1}(p) \sqsubset_X f^{-1}(n)$ for all $n, p \in \text{sub}Y$ and $m \in \text{sub}X$.

It is clear from the definition that every \mathcal{C} -morphism $f : X \rightarrow Y$ is (\sqsubset, \sqsubset) -continuous and (\sqsubset', \sqsubset') -continuous, it is (\sqsubset, \sqsubset') -continuous if $\sqsubset' \subseteq \sqsubset$.

Proposition 2.5.2. Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism.

(1) If $\sqsubset, \sqsubset' \in \bigwedge\text{-TOR}D(\mathcal{C}, \mathcal{M})$, then $f : X \rightarrow Y$ is (\sqsubset, \sqsubset') -continuous if and only if $f(c_X^{\sqsubset}(m)) \leq c_Y^{\sqsubset'}(f(m))$.

(2) If $\sqsubset, \sqsubset' \in \bigvee\text{-TOR}D(\mathcal{C}, \mathcal{M})$, then $f : X \rightarrow Y$ is (\sqsubset, \sqsubset') -continuous if and only if $f^{-1}(i_Y^{\sqsubset'}(n)) \leq i_X^{\sqsubset}(f^{-1}(n))$.

Proof. (1) If $\sqsubset, \sqsubset' \in \bigwedge\text{-TOR}D(\mathcal{C}, \mathcal{M})$ and $f : X \rightarrow Y$ is (\sqsubset, \sqsubset') -continuous,

$\{f^{-1}(n) \mid f(m) \sqsubset'_Y n\} \subseteq \{p \mid m \sqsubset_X p\} \Rightarrow c_X^{\sqsubset}(m) = \bigwedge\{p \mid m \sqsubset_X p\} \leq \bigwedge\{f^{-1}(n) \mid f(m) \sqsubset'_Y n\} = f^{-1}(c_Y^{\sqsubset'}(f(m)))$. On the other hand, if $f(c_X^{\sqsubset}(m)) \leq c_Y^{\sqsubset'}(f(m))$, then $f(m) \sqsubset'_Y n \Leftrightarrow c_Y^{\sqsubset'}(f(m)) \leq n \Rightarrow f(c_X^{\sqsubset}(m)) \leq n \Leftrightarrow c_X^{\sqsubset}(m) \leq f^{-1}(n) \Leftrightarrow m \sqsubset_X f^{-1}(n)$.

(2) If $\sqsubset, \sqsubset' \in \bigvee\text{-TOR}D(\mathcal{C}, \mathcal{M})$ and $f : X \rightarrow Y$ is (\sqsubset, \sqsubset') -continuous, then $f^{-1}(i_Y^{\sqsubset'}(n)) \in \{q \mid q \sqsubset_X f^{-1}(n)\} \Rightarrow f^{-1}(i_Y^{\sqsubset'}(n)) \leq \bigvee\{q \mid q \sqsubset_X f^{-1}(n)\} = i_X^{\sqsubset}(f^{-1}(n))$. Conversely if $f^{-1}(i_Y^{\sqsubset'}(n)) \leq i_X^{\sqsubset}(f^{-1}(n))$, then $p \leq f^{-1}(i_Y^{\sqsubset'}(n)) \Leftrightarrow f(p) \sqsubset'_Y n \Rightarrow p \sqsubset_X f^{-1}(n) \Leftrightarrow p \leq i_X^{\sqsubset}(f^{-1}(n))$. \square

Morphisms in \mathcal{C} satisfying condition (1) (resp. (2)) in Proposition 2.5.2 will be referred to as $(c^\square, c^{\square'})$ -continuous (resp. $(i^\square, i^{\square'})$ -continuous.

Definition 2.5.3. A pointed endofunctor of \mathcal{C} is a pair (F, η) consisting of a functor $F : \mathcal{C} \rightarrow \mathcal{C}$ and a natural transformation $\eta : 1_{\mathcal{C}} \rightarrow F$.

For any \mathcal{C} -morphism $f : X \rightarrow Y$, (F, η) induces the commutative diagram below.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & FX \\ f \downarrow & & \downarrow Ff \\ Y & \xrightarrow{\eta_Y} & FY \end{array}$$

The dual notion is the copointed endofunctor, that is a pair (G, ε) consisting of a functor $G : \mathcal{C} \rightarrow \mathcal{C}$ and a natural transformation $\varepsilon : G \rightarrow 1_{\mathcal{C}}$. (G, ε) induces the commutative diagram

$$\begin{array}{ccc} GX & \xrightarrow{\varepsilon_X} & X \\ Gf \downarrow & & \downarrow f \\ GY & \xrightarrow{\varepsilon_Y} & Y \end{array}$$

for any $f : X \rightarrow Y$ in \mathcal{C} . If $\eta_X \in \mathcal{E}$ for any $X \in \mathcal{C}$, then (F, η) is said to be \mathcal{E} -pointed. Dually if $\varepsilon_X \in \mathcal{M}$ for any $X \in \mathcal{C}$, then (G, ε) is \mathcal{M} -copointed.

For a pointed endofunctor (F, η) of \mathcal{C} and a topogenous order \square on \mathcal{C} , we wish to construct the coarsest topogenous order \square' on \mathcal{C} for which every morphism in $\mathcal{F} = \{\eta_X : X \in \mathcal{C}\}$ is (\square', \square) -continuous and the dual case. This method was developed for categorical closure operators in ([DT95]) and it is used in chapter 5 in the case of categorical syntopogenous structures (in particular quasi-uniform structures).

Theorem 2.5.4. Let (F, η) be an \mathcal{E} -pointed endofunctor of \mathcal{C} and \square be a topogenous order on \mathcal{C} . Then for all $m, n \in \text{sub}X$, $m \square_X^{F, \eta} n \Leftrightarrow \eta_X(m) \square_{FX} p$ and $\eta_X^{-1}(p) \leq n$ is a topogenous order on \mathcal{C} . It is the least topogenous order for which every $\eta_X : X \rightarrow FX$ is $(\square_X^{F, \eta}, \square)$ -continuous. Moreover, $\square_X^{F, \eta}$ is interpolative provided \square interpolates.

Proof. (1) (T1) is easily seen to be satisfied.

$$\begin{aligned}
\text{For (T2), } m' \leq m \sqsubset_X^{F,\eta} n \leq n' &\Leftrightarrow \exists p \in \text{sub}FX \mid \eta_X(m) \sqsubset_{FX} p \text{ and } r_X^{-1}(p) \leq n. \\
&\Rightarrow \eta_X(m') \leq \eta_X(m) \sqsubset_{FX} p \text{ and } \eta_X^{-1}(p) \leq n \leq n' \\
&\Rightarrow \eta_X(m') \sqsubset_{FX} p \text{ and } \eta_X^{-1}(p) \leq n' \\
&\Leftrightarrow m \sqsubset_X^{F,\eta} n
\end{aligned}$$

To check (T3), let $X \rightarrow Y$ be a \mathcal{C} morphism and $f(m) \sqsubset_Y^{F,\eta} n$. Then there is $p \in \text{sub}FY$ such that $\eta_Y(m) \sqsubset_{FY} p$ and $\eta_Y^{-1}(p) \leq n$. By Definition 2.5.3, $Ff \circ \eta_X = \eta_Y \circ f$. So

$$\begin{aligned}
(Ff)(\eta_X(m)) \sqsubset_{FY} p \text{ and } \eta_Y^{-1}(p) \leq n &\Rightarrow (Ff)(\eta_X(m)) \sqsubset_{FX} p \text{ and } f^{-1}(\eta_Y^{-1}(p)) \leq f^{-1}(n) \\
&\Rightarrow \eta_X(m) \sqsubset_X (Ff)^{-1}(p) \text{ and } \eta_X^{-1}((Ff)^{-1}(p)) \leq f^{-1}(n) \\
&\Rightarrow \eta_X(m) \sqsubset_{FX} l \text{ and } \eta_X^{-1}(l) \leq f^{-1}(n) \text{ (with } l = g^{-1}(p)) \\
&\Leftrightarrow m \sqsubset_X^{F,\eta} f^{-1}(n)
\end{aligned}$$

Since $\eta_X(m) \sqsubset_{FX} n \Rightarrow \eta_X(m) \sqsubset_{FX} n \leq \eta_X(\eta_X^{-1}(n)) \Rightarrow \eta_X(m) \sqsubset_{FX} \eta_X(\eta_X^{-1}(n)) \Leftrightarrow m \sqsubset_X^{F,\eta} \eta_X^{-1}(n)$, F is $(\sqsubset^{F,\eta}, \sqsubset)$ -continuous. If \sqsubset' is another topogenous order on \mathcal{C} such that F is (\sqsubset', \sqsubset) -continuous, then $m \sqsubset_X^{F,\eta} n \Leftrightarrow \eta_X(m) \sqsubset_{FX} p$ and $\eta_X^{-1}(p) \leq n \Rightarrow m \sqsubset_X \eta_X^{-1}(p) \leq n \Rightarrow m \sqsubset'_X n$.

Lastly, if \sqsubset is interpolative and $\eta_X \in \mathcal{E}$ for all $X \in \mathcal{C}$, then $m \sqsubset_X^{F,\eta} n \Leftrightarrow \eta_X(m) \sqsubset_{FX} p$ and $\eta_X^{-1}(p) \leq n$ for some $p \in \text{sub}FX$. This implies that there is $l \in \text{sub}FX$ such that $\eta_X(m) \sqsubset_{FX} l \sqsubset_{FX} p$. Thus $\eta_X(m) \sqsubset_{FX} \eta_X(\eta_X^{-1}(l)) \sqsubset_{FX} p$, that is $m \sqsubset_X^{F,\eta} \eta_X^{-1}(l) \sqsubset_X^{F,\eta} n$. \square

Definition 2.5.5. [AHS06] A full subcategory \mathcal{A} of \mathcal{C} is reflective if for every $X \in \mathcal{C}$, there is an object $FX \in \mathcal{A}$ and a \mathcal{C} -morphism $\eta_X : X \rightarrow FX$ (called the reflection morphism) with the property that for any \mathcal{C} -morphism $f : X \rightarrow Y$ with $Y \in \mathcal{A}$, there is a unique \mathcal{A} -morphism $g : FX \rightarrow Y$ such that $f = g \circ \eta_X$.

If $\eta_X \in \mathcal{E}$ for any $X \in \mathcal{C}$, then \mathcal{A} is \mathcal{E} -reflective. Viewing a reflector as pointed endofunctor, one obtains the following proposition that will turn out to be useful in the examples.

Proposition 2.5.6. Let \mathcal{A} be an \mathcal{E} -reflective subcategory of \mathcal{C} and \sqsubset be a topogenous order on \mathcal{A} . Then for all $X \in \mathcal{C}$ and $m, n \text{ sub}X$, $m \sqsubset^{\mathcal{A}} n \Leftrightarrow \eta_X(m) \sqsubset_{FX} p$ and $\eta_X^{-1}(p) \leq n$ is a

topogenous order on \mathcal{C} . It is the least one for which every reflection morphism $\eta_X : X \rightarrow FX$ is $(\sqsubset^{F,\eta}, \sqsubset)$ -continuous. Moreover, $\sqsubset^{F,\eta}$ is interpolative provided \sqsubset interpolates.

Proposition 2.5.7. *Let (F, η) be a pointed endofunctor of \mathcal{C} and $\sqsubset \in \bigwedge -TORD$. Then $c^{\sqsubset^{F,\eta}}(m) = \eta_X^{-1}(c_{FX}^{\sqsubset}(\eta_X(m)))$ is the largest closure operator on \mathcal{C} for which every $\eta_X : X \rightarrow FX$ is $(c^{\sqsubset^{F,\eta}}, c^{\sqsubset})$ -continuous. Moreover, if \sqsubset is interpolative, then $c^{\sqsubset^{F,\eta}}$ is idempotent.*

Proof. (C1) follows from the fact that, $\eta_X(m) \leq \bigwedge \{p \mid \eta_X(m) \sqsubset_{FX} p\} \Leftrightarrow m \leq \eta_X^{-1}(\bigwedge \{p \mid \eta_X(m) \sqsubset_{FX} p\}) = \eta_X^{-1}(c_{FX}^{\sqsubset}(\eta_X(m)))$. (C2) is clear. For (C3), Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and $m \in \text{sub}X$.

$$\begin{aligned}
\text{Then } f(c^{\sqsubset^{F,\eta}}(m)) &= f(\eta_X^{-1}(c_{FX}^{\sqsubset}(\eta_X(m)))) \\
&\leq \eta_Y^{-1}((Ff)(c_{FX}^{\sqsubset}(\eta_X(m)))) && \text{Lemma 1.2.4} \\
&\leq \eta_Y^{-1}(c_{FY}^{\sqsubset}((Ff)(\eta_X(m)))) && c^{\sqsubset}\text{-continuity} \\
&\leq \eta_Y^{-1}(c_{FY}^{\sqsubset}(\eta_Y(f(m)))) && \text{Definition 2.5.3} \\
&\leq c^{\sqsubset^{F,\eta}}(f(m))
\end{aligned}$$

Since $\eta_X(c^{\sqsubset^{F,\eta}}(m)) \leq c_{FX}^{\sqsubset}(\eta_X(m))$, F is $(c^{\sqsubset^{F,\eta}}, c^{\sqsubset})$ -continuous. If c' is another closure operator such that F is (c', c^{\sqsubset}) -continuous, then $\eta_X(c'(m)) \leq c^{\sqsubset}(\eta_X(m)) \Leftrightarrow c'(m) \leq \eta_X^{-1}(c_X^{\sqsubset}(\eta_X(m))) = c^{\sqsubset^{F,\eta}}(m)$. For \sqsubset is interpolative,

$$\begin{aligned}
c^{\sqsubset^{F,\eta}}(c^{\sqsubset^{F,\eta}}(m)) &= c^{\sqsubset^{F,\eta}}(\eta_X^{-1}(c_{FX}^{\sqsubset}(\eta_X(m)))) \\
&= \eta_X^{-1}(c_{FX}^{\sqsubset}(\eta_X(\eta_X^{-1}(c_{FX}^{\sqsubset}(\eta_X(m))))) \\
&\leq \eta_X^{-1}(c_{FX}^{\sqsubset}(c_{FX}^{\sqsubset}(\eta_X(m)))) \\
&= \eta_X^{-1}(c_{FX}^{\sqsubset}(\eta_X(m))), \quad \sqsubset \in \bigwedge -INTORD \\
&= c^{\sqsubset^{F,\eta}}(m).
\end{aligned}$$

□

Proposition 2.5.8. *Let \mathcal{A} be a reflective subcategory of \mathcal{C} and $\sqsubset \in \bigwedge -TORD$. Then $c^{\mathcal{A}}(m) = \eta_X^{-1}(c_{FX}^{\sqsubset}(\eta_X(m)))$ is the largest closure operator on \mathcal{C} for which every reflection morphism $\eta_X : X \rightarrow FX$ is $(c^{\mathcal{A}}, c^{\sqsubset})$ -continuous. Moreover, if \sqsubset is interpolative, then $c^{\mathcal{A}}$ is idempotent.*

The closure operator in Proposition 2.5.8 was studied on the category of topological spaces and continuous maps by L. Stramaccia ([Str88]), on topological categories by D. Dikranjan ([Dik92]) and later on an arbitrary category by Dikranjan and Tholen ([DT95]). It is a special case of the pullback closure studied by D. Holgate in [Hol96, Hol95].

Proposition 2.5.9. *Let (F, η) be a pointed endofunctor of \mathcal{C} and $\sqsubseteq \in \mathcal{V} - \text{TOR}D$. Then $i_X^{\sqsubseteq, F, \eta}(m) = \eta_X^{-1}(i_{FX}^{\sqsubseteq}(\eta_X)_*(m))$ is the least interior operator on \mathcal{C} for which every $\eta_X : X \rightarrow FX$ is $(i^{\sqsubseteq}, i^{\sqsubseteq, F, \eta})$ -continuous. $i^{F, \eta}$ is idempotent provided \sqsubseteq is interpolative and each $\eta_X \in \mathcal{E}$.*

Proof. Since $i_{FX}^{\sqsubseteq}(\eta_X)_*(m) = \bigvee \{n \mid n \sqsubseteq_{FX} (\eta_X)_*(m)\} \leq (\eta_X)_*(m)$, $i_X^{\sqsubseteq, F, \eta}(m) \leq m$. (I2) is clearly satisfied. For (I3), let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and $m \in \text{sub}Y$,

$$\begin{aligned}
f^{-1}(i_Y^{\sqsubseteq, F, \eta}(m)) &= f^{-1}(\eta_X^{-1}(i_{FX}^{\sqsubseteq}(\eta_X)_*(m))) \\
&= \eta_X^{-1}(Ff)^{-1}(i_{FY}^{\sqsubseteq}(\eta_Y)_*(m)) && \text{Definition 2.5.3} \\
&\leq \eta_X^{-1}(i_{FX}^{\sqsubseteq}((Ff)^{-1}(\eta_Y)_*(m))) && c^{\sqsubseteq}\text{-continuity} \\
&\leq \eta_X^{-1}(i_{FX}^{\sqsubseteq}(\eta_X)_*(f^{-1}(m))) && \text{Lemma 1.2.11} \\
&= i_X^{\sqsubseteq, F, \eta}(f^{-1}(m))
\end{aligned}$$

Since $\eta_X^{-1}(i_{FX}^{\sqsubseteq}(n)) \leq \eta_X^{-1}(i_{FX}^{\sqsubseteq}(\eta_X)_*(\eta_X(n))) = i_X^{\sqsubseteq, F, \eta}(\eta_X^{-1}(n))$, η_X is $(i^{\sqsubseteq, F, \eta}, i^{\sqsubseteq})$ -continuous for any $X \in \mathcal{C}$ and $n \in \text{sub}X$. If i' is another interior operator \mathcal{C} such that η_X is (i', i^{\sqsubseteq}) -continuous, then $i_X^{\sqsubseteq, F, \eta}(m) = \eta_X^{-1}(i_{FX}^{\sqsubseteq}((\eta_X)_*(m))) \leq i'_X(\eta_X^{-1}(\eta_X)_*(m)) \leq i'_X(m)$.

If $\sqsubseteq \in \mathcal{V} - \text{TOR}D$ and $\eta_X \in \mathcal{E}$ for all $X \in \mathcal{C}$, then $i^{\sqsubseteq, F, \eta}(i^{\sqsubseteq, F, \eta}(m)) = i^{\sqsubseteq, F, \eta}(\eta_X^{-1}(i_{FX}^{\sqsubseteq}(\eta_X)_*(m))) = \eta_X^{-1}(i_{FX}^{\sqsubseteq}(\eta_X)_*(\eta_X^{-1}(i_{FX}^{\sqsubseteq}(\eta_X)_*(m)))) = \eta_X^{-1}(i_{FX}^{\sqsubseteq}(i_{FX}^{\sqsubseteq}(\eta_X)_*(m))) = \eta_X^{-1}(i_{FX}^{\sqsubseteq}(\eta_X)_*(m))$.

□

While the topogenous order induced by a pointed endofunctor was obtained with the help of \sqsubseteq -initial morphism, the notion of \sqsubseteq -weakly final morphism is used in the next theorem to obtain the topogenous order induced by a co-pointed endofunctor.

Theorem 2.5.10. *Let (G, ε) be a co-pointed endofunctor of \mathcal{C} and \sqsubseteq a topogenous order on \mathcal{C} , then for all $m \in \text{sub}X$ and $n \geq m$, $m \sqsubseteq^{G, \varepsilon} n \Leftrightarrow \varepsilon_X^{-1}(n) \sqsubseteq_{GX} \varepsilon_X^{-1}(m)$ is a topogenous order on \mathcal{C} . It is the largest one for which every $\varepsilon_X : GX \rightarrow X$ is $(\sqsubseteq, \sqsubseteq^{G, \varepsilon})$ -continuous.*

Proof. (T1) and (T2) are easily seen to be satisfied. For (T3), let $f : X \rightarrow Y$ be a \mathcal{C} -morphism. Then for all $m \in \text{sub}X$ and $n \in \text{sub}Y$ such that $f(m) \leq n$, $f(m) \sqsubseteq^{G, \varepsilon}$

$n \Leftrightarrow \varepsilon_Y^{-1}(f(m)) \sqsubset_{GY} \varepsilon_Y^{-1}(n) \Rightarrow (Gf)(\varepsilon_X^{-1}(m)) \sqsubset_{GY} \varepsilon_Y^{-1}(n) \Rightarrow \varepsilon_X^{-1}(m) \sqsubset_{GX} \varepsilon_X^{-1}(f^{-1}(n)) \Rightarrow m \sqsubset_X^{G,\varepsilon} f^{-1}(n)$. Now, $\varepsilon_X : GX \rightarrow X$ is trivially $(\sqsubset, \sqsubset^{G,\varepsilon})$ -continuous and if \sqsubset' is another topogenous order on \mathcal{C} such that ε_X is (\sqsubset, \sqsubset') -continuous, then $m \sqsubset'_X n \Rightarrow \varepsilon_X(\varepsilon_X^{-1}(m)) \sqsubset'_X n \Rightarrow \varepsilon_X^{-1}(m) \sqsubset_X \varepsilon_X^{-1}(n) \Leftrightarrow m \sqsubseteq_X^{G,\varepsilon} n$ for all $n \geq m$. \square

Definition 2.5.11. A full subcategory \mathcal{A} of \mathcal{C} is coreflective if for every $X \in \mathcal{C}$, there is an object $GX \in \mathcal{A}$ and a morphism $\varepsilon_X : GX \rightarrow X$ (called coreflection morphism) with the property that for any \mathcal{C} -morphism $f : Y \rightarrow X$ (with $Y \in \mathcal{A}$), there is a unique morphism \mathcal{A} -morphism $f' : Y \rightarrow GX$ such that $\varepsilon_X \circ f' = f$.

Viewing a coreflector as a co-pointed endofunctor, we get the next proposition.

Proposition 2.5.12. Let \mathcal{A} be a coreflective subcategory of \mathcal{C} and \sqsubset a topogenous order on \mathcal{A} , then for all $m \in \text{sub}X$ and $n \geq m$, $m \sqsubset^{\mathcal{A}} n \Leftrightarrow \varepsilon_X^{-1}(n) \sqsubset_{GX} \varepsilon_X^{-1}(m)$ is a topogenous order on \mathcal{C} . It is the largest one for which every coreflection morphism $\varepsilon_X : GX \rightarrow X$ is $(\sqsubset, \sqsubset^{\mathcal{A}})$ -continuous.

Proposition 2.5.13. Let (G, ε) be a co-pointed endofunctor of \mathcal{C} and $\sqsubset \in \bigwedge -\text{TORD}$, then for all $m \in \text{sub}X$, $c^{\sqsubset^{G,\varepsilon}}(m) = m \vee \varepsilon_X(c^{\sqsubset}(\varepsilon_X^{-1}(m)))$ is a closure operator on \mathcal{C} . It is the least closure operator for which every $\varepsilon_X : GX \rightarrow X$ is $(c, c^{\sqsubset^{G,\varepsilon}})$ -continuous.

Proof. (C1) and (C2) are easily seen to be satisfied. To check (C3), let $f : X \rightarrow Y$ be a \mathcal{C} -morphism, and $m \in \text{sub}X$,

$$\begin{aligned}
f(c^{\sqsubset^{G,\varepsilon}}(m)) &= f(m \vee \varepsilon_X(c^{\sqsubset}(\varepsilon_X^{-1}(m)))) \\
&= f(m) \vee f(\varepsilon_X(c^{\sqsubset}(\varepsilon_X^{-1}(m)))) \\
&= f(m) \vee \varepsilon_Y(Gf)(c^{\sqsubset}(\varepsilon_X^{-1}(m))) && \text{Definition 2.5.3} \\
&\leq f(m) \vee \varepsilon_Y(c^{\sqsubset}(Gf)(\varepsilon_X^{-1}(m))) && c^{\sqsubset}\text{-continuity} \\
&\leq f(m) \vee \varepsilon_Y(c^{\sqsubset}(\varepsilon_Y^{-1}(f(m)))) && \text{Corollary 1.2.4} \\
&= c^{\sqsubset^{G,\varepsilon}} f(m)
\end{aligned}$$

For any $X \in \mathcal{C}$, ε_X is $(c^{\sqsubset}, c^{\sqsubset^{G,\varepsilon}})$ -continuous since $\varepsilon_X(c^{\sqsubset}(\varepsilon_X^{-1}(m))) \leq c^{\sqsubset^{G,\varepsilon}}(m) \Leftrightarrow c^{\sqsubset}(\varepsilon_X^{-1}(m)) \leq \eta_X^{-1}(c^{\sqsubset^{G,\varepsilon}}(m))$. If c' is another topogenous order on \mathcal{C} such that ε_X is (c, c') -continuous, then $c^{\sqsubset}(\varepsilon_X^{-1}(m)) \leq \varepsilon_X^{-1}(c'(m)) \Leftrightarrow \varepsilon_X(c^{\sqsubset}(\varepsilon_X^{-1}(m))) \leq c'(m) \Rightarrow m \vee \varepsilon_X(c^{\sqsubset}(m)) \leq c'(m) \Rightarrow c^{\sqsubset^{G,\varepsilon}}(m) \leq c'(m)$. \square

Proposition 2.5.14. *Let \mathcal{A} be a coreflective subcategory of \mathcal{C} and $\square \in \wedge -\text{TORD}$, then for all $m \in \text{sub}X$, $c^{\mathcal{A}}(m) = m \vee \varepsilon_X(c^{\square}(\varepsilon_X^{-1}(m)))$ is a closure operator on \mathcal{C} . It is the least closure operator for which every coreflection morphism $\varepsilon_X : GX \rightarrow X$ is $(c, c^{\mathcal{A}})$ -continuous.*

Proposition 2.5.15. *Assume that for any morphism $f \in \mathcal{C}$, the inverse image f^{-1} commutes with the joins of subobjects. Let (G, ε) be a copointed endofunctor of \mathcal{C} and $\square \in \vee -\text{TORD}$, then for all $m \in \text{sub}X$, $i^{\square(G, \varepsilon)}(m) = m \wedge (\varepsilon_X)_*(i^{\square}(\varepsilon_X^{-1}(m)))$ is a interior operator on \mathcal{C} . It is the largest interior operator for which every $\varepsilon_X : GX \rightarrow X$ is $(i, i^{G, \varepsilon})$ -continuous.*

Proof. (I1) and (I2) are clearly seen to be satisfied.

For (I3), let $f : X \rightarrow Y$ be a \mathcal{C} -morphism, then

$$\begin{aligned}
f^{-1}(i_Y^{\square(G, \varepsilon)}(m)) &= f^{-1}(m \wedge (\varepsilon_Y)_*(i_Y^{\square}(\varepsilon_Y^{-1}(m)))) \\
&= f^{-1}(m) \wedge f^{-1}((\varepsilon_Y)_*(i_Y^{\square}(\varepsilon_Y^{-1}(m)))) \\
&\leq f^{-1}(m) \wedge (\varepsilon_X)_*(Gf)^{-1}(i_Y^{\square}(\varepsilon_Y^{-1}(m))) && \text{Lemma 1.2.11} \\
&\leq f^{-1}(m) \wedge (\varepsilon_X)_*(i_{GX}^{\square}(Gf)^{-1}(\varepsilon_Y^{-1}(m))) && c^{\square}\text{-continuity} \\
&\leq f^{-1}(m) \wedge (\varepsilon_X)_*(i_{GX}^{\square}(\varepsilon_X^{-1}(f^{-1}(m)))) && \text{Definition 2.5.3} \\
&= i_X^{\square(G, \varepsilon)}(f^{-1}(m))
\end{aligned}$$

□

Since $\varepsilon_X^{-1}(i_Y^{\square(G, \varepsilon)}(m)) = \varepsilon_X^{-1}(m) \wedge \varepsilon_X^{-1}((\varepsilon_X)_*(i_{GX}^{\square}(\varepsilon_X^{-1}(m)))) \leq i_{GX}^{\square}(\varepsilon_X^{-1}(m))$, ε_X is $(i^{\square}, i^{\square(G, \varepsilon)})$ -continuous.

If i' is another interior operator on \mathcal{C} such that ε_X is (i^{\square}, i') -continuous, then $\varepsilon_X^{-1}(i'(m)) \leq i^{\square}(\varepsilon_X^{-1}(m)) \Leftrightarrow i'(m) \leq (\varepsilon_X)_*(i_{GX}^{\square}(\varepsilon_X^{-1}(m))) \Rightarrow i'(m) \leq m \wedge (\varepsilon_X)_*(i_{GX}^{\square}(\varepsilon_X^{-1}(m)))$.

2.6 Examples

1. In the category **Top** of topological spaces and continuous maps with its (surjections, emdembeddings)-factorization structure, consider the following two topogenous orders:

$$A \sqsubset_X B \Leftrightarrow \overline{A} \subseteq B \text{ and } A \sqsubset'_X B \Leftrightarrow A \subseteq B^o \text{ for all } X \in \mathbf{Top} \text{ and } A, B \subseteq X.$$

Propositions 2.2.10 and 2.2.12 provide equivalent ways of characterizing open and closed continuous maps as well as a continuous map whose domain carries the coarsest topology for which the map is continuous. Equivalent ways of describing hereditary quotient maps are provided. It is clear that $f_*(A) = Y \setminus f(X \setminus A)$ for any $A \subseteq X$.

Proposition 2.6.1. *The following are equivalent for a continuous map $f : X \rightarrow Y$.*

- (1) f is open ;
- (2) $\overline{f^{-1}(A)} \subseteq B \Rightarrow \overline{A} \subseteq Y \setminus f(X \setminus B)$;
- (3) $B \subseteq (f^{-1}(A))^{\circ} \Rightarrow f(B) \subseteq A^{\circ}$.

Proof. (1) \Rightarrow (2) then $\overline{f^{-1}(A)} \subseteq B \Leftrightarrow f^{-1}(\overline{A}) \subseteq B \Leftrightarrow X \setminus B \subseteq X \setminus f^{-1}(\overline{A}) \Leftrightarrow X \setminus B \subseteq f^{-1}(Y \setminus \overline{A}) \Leftrightarrow f(X \setminus B) \subseteq Y \setminus \overline{A} \Leftrightarrow \overline{A} \subseteq Y \setminus f(X \setminus B)$.

(2) \Rightarrow (3) $B \subseteq (f^{-1}(A))^{\circ} \Leftrightarrow X \setminus \overline{f^{-1}(A)} \subseteq X \setminus B \Leftrightarrow \overline{f^{-1}(Y \setminus A)} \subseteq X \setminus B \Leftrightarrow \overline{Y \setminus A} \subseteq Y \setminus f(B) \Leftrightarrow f(B) \subseteq Y \setminus \overline{Y \setminus A} \Leftrightarrow f(B) \subseteq A^{\circ}$.

(3) \Rightarrow (1) $B \subseteq (f^{-1}(A))^{\circ} \Leftrightarrow f(B) \subseteq A^{\circ} \Leftrightarrow B \subseteq f^{-1}(A^{\circ})$. □

A similar reasoning proves the following.

Proposition 2.6.2. *The following are equivalent for a continuous map $f : X \rightarrow Y$.*

- (1) f is closed;
- (2) $\overline{B} \subseteq f^{-1}(A) \Leftrightarrow \overline{f(B)} \subseteq A$;
- (3) $f^{-1}(A) \subseteq B^{\circ} \Leftrightarrow A \subseteq [Y \setminus f(X \setminus B)]^{\circ}$.

For a continuous map $f : X \rightarrow Y$, X carries the initial topology induced by f if, $A \subseteq X$ is open iff there is an open $B \subseteq Y$ such that $A = f^{-1}(B)$ or equivalently $\overline{A} = f^{-1}(\overline{f(A)})$ for each $A \subseteq X$ (see e.g [GT00, Eng89]). Such morphism corresponds to the \square -initial.

In fact, if f is \square -initial, then $f^{-1}(\overline{f(A)}) \subseteq B \Leftrightarrow X \setminus B \subseteq X \setminus f^{-1}(\overline{f(A)}) \Leftrightarrow X \setminus B \subseteq f^{-1}(Y \setminus \overline{f(A)}) \Leftrightarrow f(X \setminus B) \subseteq Y \setminus \overline{f(A)} \Leftrightarrow \overline{f(A)} \subseteq Y \setminus f(X \setminus B) \Leftrightarrow \overline{f(A)} \square_Y f_*(B) \Leftrightarrow A \square_X B \Leftrightarrow \overline{A} \subseteq B$. Conversely if X carries the coarsest topology for which f is continuous, $A \square_X B \Leftrightarrow \overline{A} \subseteq B \Leftrightarrow f^{-1}(\overline{f(A)}) \subseteq B \Leftrightarrow X \setminus B \subseteq X \setminus f^{-1}(Y \setminus \overline{f(A)}) \Leftrightarrow f(X \setminus B) \subseteq Y \setminus \overline{f(A)} \Leftrightarrow \overline{f(A)} \subseteq Y \setminus f(X \setminus B) \Leftrightarrow f(A) \square_Y f_*(B)$ for any $A, B \subseteq X$.

An analogous reasoning shows that X carries the coarsest topology for which f is continuous iff

$$f(A) \subseteq [Y \setminus f(X \setminus B)]^{\circ} \Leftrightarrow A \subseteq B^{\circ}$$

According to ([Eng89], Exercise 2F), f is hereditary quotient if it is surjective with the property that every restriction $f' : f^{-1}(A) \rightarrow A$ is quotient for any $A \subseteq Y$ or equivalently f surjective with the property that $f(\overline{f^{-1}(B)}) \subseteq Y$ is closed for every $B \subseteq Y$. We clearly see that f is \sqsubseteq -final iff it is \sqsubset' -final iff it is hereditary quotient.

(T4) (resp. (T5)) are satisfied by \sqsubseteq (resp. \sqsubset'). However, (T4') fails for \sqsubseteq since the Kuratowski closure is not fully additive.

2. In the category **Grp** of groups and group homomorphisms with the (epi, mono)-factorization structure, let \sqsubseteq be the topogenous structure defined by

$$A \sqsubseteq_G B \Leftrightarrow A \leq N \leq B \text{ with } N \triangleleft G \text{ and } A, B \leq G.$$

A group homomorphism is \sqsubseteq -strict iff it preserves normal subgroups.

Proposition 2.6.3. *A group homomorphism $f : G \rightarrow H$ is \sqsubseteq -final if and only if it is surjective.*

Proof. Assume f is \sqsubseteq -final. Since $H \triangleleft H$, $f^{-1}(H) \triangleleft G$. Thus $f^{-1}(H) \sqsubseteq_G G \leq f^{-1}(f(G)) \Rightarrow H \sqsubseteq_H f(G) \Rightarrow H \leq f(G) \Rightarrow H = f(G)$, that is f is surjective. Conversely if f is surjective then it preserves normal subgroups and by Proposition 2.2.5(4), f is \sqsubseteq -final. \square

Proposition 2.2.5(3) allows to say that every injective group homomorphism that preserves normal subgroups is \sqsubseteq -initial. It is also clear that \sqsubseteq satisfies (T4'), (T6) and (T7).

3. Let **Prox** be the category of proximity spaces and proximal maps with (surjective, embedding)-factorization. For any $(X, \delta) \in \mathbf{Prox}$ and $A, B \subseteq X$,

$$A \sqsubseteq_{(X, \delta)} B \Leftrightarrow A\bar{\delta}(X \setminus B)$$

is an interpolative topogenous order on **Prox** which satisfies (T5).

4. Consider **TopGrp**, the category of topological groups and continuous group homomorphisms. The forgetfull functor

$$F : \mathbf{TopGrp} \rightarrow \mathbf{Grp}.$$

is a mono-fibration since every subgroup of a topological group is a topological group with the subspace topology (see [DT95]). Thus by Proposition 2.4.2, every topogenous order on **Grp** can be initially lifted to a topogenous order on **TopGrp**.

5. Let **Top** be the category of topological spaces and continuous maps with its (surjections, embeddings)-factorization structure. It is well known that **Top_o**, the category of T_o -topological spaces and continuous maps is a epi-reflective subcategory of **Top**. Define $\mathcal{S}_X = \{\sqsubset_{X_o} \mid X_o \in \mathbf{Top}_o\}$ by $A \sqsubset_{X_o} B \Leftrightarrow \overline{A} \subseteq B$ for any $X_o \subseteq \mathbf{Top}_o$, $A, B \subseteq X_o$. Let (F, η) be the reflector into **Top**. For any $X \in \mathbf{Top}$, $\eta_X : X \rightarrow X/\sim$ takes each $x \in X$ to its equivalence class $[x] = \{y \in X \mid \overline{\{x\}} = \overline{\{y\}}\}$. Thus $\mathcal{S}_X = \{\sqsubset_X^{F, \eta} \mid X \in \mathbf{Top}\}$ with $A \sqsubset_X^{F, \eta} B \Leftrightarrow \eta_X^{-1}(\overline{\eta_X(A)}) \subseteq B$, $A, B \subseteq X$.

6. The category **sTop** of sequentially closed topological spaces (those spaces in which every sequentially closed set is closed) is \mathcal{M} -coreflective in **Top**. Consider \sqsubset on **sTop** defined by $A \sqsubset B \Leftrightarrow A \subseteq C \subseteq B$ for any $X \in \mathbf{sTop}$ and some closed subset C of X . Let (G, ε) be the coreflector into **sTop**. For any $(X, \mathcal{T}) \in \mathbf{Top}$, $\varepsilon_{(X, \mathcal{T})} : (X, \mathcal{T}') \rightarrow (X, \mathcal{T})$, identity map on X , where $\mathcal{T}' = \{A \subseteq X \mid X \setminus A \text{ is sequentially closed}\}$ is an **sTop**-coreflection for any (X, \mathcal{T}) . It is clear that (G, ε)



for some closed subset C of X .

Chapter 3

The syntopogenous structures

In this chapter, we introduce the categorical notions of quasi-uniform and syntopogenous structures that are fundamental to our study. The relationship between the two structures that is established leads to the description of a quasi-uniformity as a family of categorical closure operators. We show that for an idempotent closure operator c (interior i) on \mathcal{C} , there is at least a quasi-uniformity \mathcal{U} on \mathcal{C} compatible with c (i). This allows to find a condition under which a topogenous order is compatible with a transitive quasi-uniformity. We then investigate the initial morphism and Hausdorff separation axiom with respect to a syntopogenous structure (in particular, a quasi-uniformity). This initial morphism will play an important role in the study of completeness of objects. The chapter ends with a list of examples that illustrate the developed theory.

3.1 The definitions

Definition 2.1.1 enables us to axiomatize the notion of syntopogenous structure in a very natural way.

Definition 3.1.1. *A syntopogenous structure on \mathcal{C} with respect to \mathcal{M} is a family $\mathcal{S} = \{\mathcal{S}_X \mid X \in \mathcal{C}\}$ such that each \mathcal{S}_X is a family of relations on $\text{sub}X$ satisfying:*

- (S1) *Each $\sqsubset_X \in \mathcal{S}_X$ satisfies (T1) and (T2).*
- (S2) *\mathcal{S}_X is a directed set with respect to inclusion.*
- (S3) *$\sqsubset_X = \bigcup \mathcal{S}_X$ is an interpolative topogenous order.*

We can extend the ordering of topogenous orders to syntopogenous structures in the following way: $\mathcal{S} \leq \mathcal{S}'$ if for all $X \in \mathcal{C}$ and $\sqsubset_X \in \mathcal{S}_X$, there is $\sqsubset'_X \in \mathcal{S}'_X$ such that $\sqsubset_X \subseteq \sqsubset'_X$. The resulting conglomerate will be denoted by $SYNT$.

(S3) includes the fact that for any \mathcal{C} -morphism $f : X \rightarrow Y$ and $\sqsubset_Y \in \mathcal{S}_Y$, there is $\sqsubset_X \in \mathcal{S}_X$ such $f(m) \sqsubset_Y n \Rightarrow m \sqsubset_X f^{-1}(n)$ for all $m \in \text{sub}X$ and $n \in \text{sub}Y$. This will be referred to as the continuity condition of syntopogenous structure or simply the \mathcal{S} -continuity. Its equivalent descriptions are provided in the next proposition.

Proposition 3.1.2. *Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism such that f^{-1} commutes with the joins of subobjects and $\mathcal{S} \in SYNT$. The following are equivalent to the \mathcal{S} -continuity.*

- (1) *For any $\sqsubset_Y \in \mathcal{S}_Y$, there is $\sqsubset_X \in \mathcal{S}_X \mid m \sqsubset_Y n \Rightarrow f^{-1}(m) \sqsubset_X f^{-1}(n)$ for all $n, m \in \text{sub}Y$,*
- (2) *For any $\sqsubset_Y \in \mathcal{S}_Y$, there is $\sqsubset_X \in \mathcal{S}_X \mid m \sqsubset_Y f_*(n) \Rightarrow f^{-1}(m) \sqsubset_X n$ for all $m \in \text{sub}Y$ and $n \in \text{sub}Y$;*
- (3) *For any $\sqsubset_Y \in \mathcal{S}_Y$, there is $\sqsubset_X \in \mathcal{S}_X \mid f(m) \sqsubset_Y f_*(n) \Rightarrow m \sqsubset_X n$ for all $n, m \in \text{sub}Y$.*

We are interested in particular classes of syntopogenous structures as these will play an important role in what follows.

Definition 3.1.3. *We shall say that $\mathcal{S} \in SYNT(\mathcal{C}, \mathcal{M})$ is:*

- (1) *co-perfect if every $\sqsubset_X \in \mathcal{S}_X$ is \wedge -preserving for all $X \in \mathcal{C}$.*
- (2) *simple if $\mathcal{S}_X = \{\sqsubset_X\}$ for any $X \in \mathcal{C}$.*
- (3) *interpolative if every $\sqsubset_X \in \mathcal{S}_X$ interpolates for all $X \in \mathcal{C}$.*

The ordered conglomerate of all co-perfect (resp. interpolative) syntopogenous structures will be denoted by $CSYNT(\mathcal{C}, \mathcal{M})$ (resp. $INSYNT(\mathcal{C}, \mathcal{M})$). A topogenous order must interpolate to be a syntopogenous structure.

Thus $INTORD(\mathcal{C}, \mathcal{M})$ is just the class of simple syntopogenous structures. In addition, we have the following,

Proposition 3.1.4. *$INTORD(\mathcal{C}, \mathcal{M})$ is coreflective in $SYNT(\mathcal{C}, \mathcal{M})$*

Proof. We just need to observe that the inclusion $\sqsubset_X \mapsto \mathcal{S}_X^{\sqsubset_X} = \{\sqsubset_X \mid X \in \mathcal{C}\}$ has a right adjoint $\mathcal{S} \mapsto \sqsubset_X^{\mathcal{S}} = \bigcup \{\sqsubset_X \in \mathcal{S}_X\}$ for all $X \in \mathcal{C}$. □

Corollary 3.1.5. *$\wedge -INTORD(\mathcal{C}, \mathcal{M})$ is coreflective in $CSYNT(\mathcal{C}, \mathcal{M})$.*

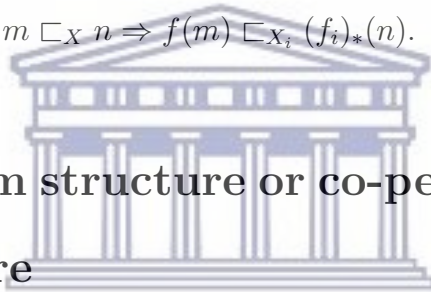
For the rest of this section we assume that for any \mathcal{C} -morphism $f : X \rightarrow Y$, f^{-1} commutes with the joins of subobjects. Proposition 3.1.2 allows us to define an \mathcal{S} -initial morphism.

Definition 3.1.6. *Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and \mathcal{S} be a syntopogenous structure on \mathcal{C} . Then f is \mathcal{S} -initial if for every $\sqsubset_X \in \mathcal{S}_X$ there is $\sqsubset_Y \in \mathcal{S}_Y$ such that $m \sqsubset_X n \Rightarrow f(m) \sqsubset_Y f_*(n)$ for all $m, n \in \text{sub}X$.*

The \mathcal{S} -initial morphism can also be expressed without use of right adjoint f^{-1} . In this case $f : X \rightarrow Y$ is \mathcal{S} -initial if for any $\sqsubset_X \in \mathcal{S}_X$ there is $\sqsubset_Y \in \mathcal{S}_Y$ such that $m \sqsubset_X n$ implies there is $p \in \text{sub}Y$ such that $f(m) \sqsubset_Y p$ and $f^{-1}(p) \leq n$. This definition becomes equivalent to the previous in any category where the preimage commutes with the join of subobjects.

Definition 3.1.7. *A source $(f_i : X \rightarrow X_i)_{i \in I}$ is \mathcal{S} -initial if for any $\sqsubset_X \in \mathcal{S}_X$ there is $i \in I$ and $\sqsubset_{X_i} \in \mathcal{S}_{X_i}$ such that $m \sqsubset_X n \Rightarrow f(m) \sqsubset_{X_i} (f_i)_*(n)$.*

3.2 Quasi-uniform structure or co-perfect syntopogenous structure



Analogous to a uniformity, T. G. Gantner and R. C. Steinlage ([GS72]) proved that a quasi-uniformity on a set X can be described in terms of subsets of $X \times X$ containing the diagonal $\Delta_X = \{(x, x) : x \in X\}$, called the "entourages" or equivalently in terms of covers of X or in terms of quasi-pseudometrics. The observation that this (entourage) quasi-uniformity on X can be equivalently expressed as a family of maps $f : X \rightarrow \mathcal{P}(X)$ which can easily be extended to a family of maps $f : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, motivates the point in our definition that a quasi-uniformity on \mathcal{C} will be given as a family of endomaps on $\text{sub}X$ for each $X \in \mathcal{C}$.

Let $X \in \mathcal{C}$, $\text{sub}X$ being a complete lattice can be seen as a category, monotone maps $f[-], g[-]$ from $\text{sub}X$ to $\text{sub}X$ are the functors and there is a natural transformation $\alpha : f[-] \Rightarrow g[-]$ exactly when $f(m) \leq g(m)$ for all $m \in \text{sub}X$. It turns out that $\mathcal{F}(\text{sub}X)$, the endofunctor category on $\text{sub}X$ is ordered by \leq "pointwise".

Definition 3.2.1. *A quasi-uniformity on \mathcal{C} with respect to \mathcal{M} is a family $\mathcal{U} = \{\mathcal{U}_X \mid X \in \mathcal{C}\}$ with \mathcal{U}_X a full subcategory of $\mathcal{F}(\text{sub}X)$ for each X such that:*

(U1) For any $U \in \mathcal{U}_X$, $1_X \leq U$,

(U2) For any $U \in \mathcal{U}_X$, there is $U' \in \mathcal{U}_X$ such that $U' \circ U' \leq U$.

(U3) For any $U \in \mathcal{U}_X$ and $U \leq U'$, $U' \in \mathcal{U}_X$.

(U4) For any $U, U' \in \mathcal{U}_X$, $U \wedge U' \in \mathcal{U}_X$.

(U5) For any \mathcal{C} -morphism $f : X \rightarrow Y$ and $U \in \mathcal{U}_Y$, there is $U' \in \mathcal{U}_X$ such that $f(-) \circ U' \leq U \circ f(-)$.

We shall denote by $QUNIF(\mathcal{C}, \mathcal{M})$ the conglomerate of all quasi-uniform structures on \mathcal{C} . It is ordered as follows: $\mathcal{U} \leq \mathcal{V}$ if for all $X \in \mathcal{C}$ and $U \in \mathcal{U}_X$, there is $V \in \mathcal{V}_X$ such that $V \leq U$. We shall refer to (U5) as the continuity condition of quasi-uniformities or simply the \mathcal{U} -continuity. Its equivalent expression is provided by the next proposition.

Proposition 3.2.2. *A morphism $f : Y \rightarrow Y$ is \mathcal{U} -continuous if and only if for any $U \in \mathcal{U}_Y$, there is $U' \in \mathcal{U}_X$ such that $U'(f^{-1}(n)) \leq f^{-1}(U(n))$ for any $n \in \text{sub}Y$.*

Proof. If f is \mathcal{U} -continuous and $U \in \mathcal{U}_Y$, then there is $U' \in \mathcal{U}_X$ such that for any $n \in \text{sub}Y$ and $m = f^{-1}(n)$, $f(U'(f^{-1}(m))) \leq U(f(f^{-1}(n))) \leq U(n) \Rightarrow f(U'(f^{-1}(n))) \leq U(n) \Leftrightarrow U'(f^{-1}(n)) \leq f^{-1}(U(n))$. Conversely if the condition in the proposition is satisfied then for any $U \in \mathcal{U}_Y$, there is $U' \in \mathcal{U}_X$ such that for any $m \in \text{sub}X$ and $n = f(m)$, $U'(f^{-1}(f(m))) \leq f^{-1}(U(f(m))) \Rightarrow U'(m) \leq f^{-1}(U(f(m))) \Leftrightarrow f(U'(m)) \leq U(f(m))$. \square

From the image pre-image adjunction we get the following other two equivalent expressions of the \mathcal{U} -continuity of a \mathcal{C} -morphism $f : X \rightarrow Y$.

- (1) For any $U \in \mathcal{U}_Y$, there is $U' \in \mathcal{U}_X$ such that $f(U'(f^{-1}(n))) \leq U(n)$, $n \in \text{sub}Y$.
- (2) For any $U \in \mathcal{U}_Y$, there is $U' \in \mathcal{U}_X$ such that $U'(m) \leq f^{-1}(U(f(m)))$, $m \in \text{sub}X$.

The next proposition immediately follows from (U5) and Proposition 3.2.2.

Proposition 3.2.3.

Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism, $\mathcal{U} \in QUNIF(\mathcal{C}, \mathcal{M})$, $m \in \text{sub}X$ and $n \in \text{sub}Y$.

- (1) If $f(m) \leq n$ then for all $V \in \mathcal{U}_Y$, there is $U \in \mathcal{U}_X$ such that $f(U(m)) \leq V(n)$.
- (2) If $m \leq f^{-1}(n)$ then for any $V \in \mathcal{U}_Y$, there is $U \in \mathcal{U}_X$ such that $U(m) \leq f^{-1}(V(n))$.

We shall often describe a quasi-uniformity by defining a base for it.

Definition 3.2.4. A base for a quasi-uniformity \mathcal{U} on \mathcal{C} is a family $\mathcal{B} = \{\mathcal{B}_X \mid X \in \mathcal{C}\}$ with each \mathcal{B}_X a full subcategory of $\mathcal{F}(\text{sub}X)$ for all $X \in \mathcal{C}$ satisfying all the axioms in Definition 3.2.1 except (U3).

Lemma 3.2.5. Let $\mathcal{U} \in \text{QUNIF}(\mathcal{C}, \mathcal{M})$. Then $\{\mathcal{B}_X \mid X \in \mathcal{C}\}$ is a base for \mathcal{U} if and only if for any $U \in \mathcal{U}_X$, there is $V \in \mathcal{B}_X$ such that $V \leq U$.

Definition 3.2.6. A base for quasi-uniformity on \mathcal{C} is transitive if for all $X \in \mathcal{C}$ and $U \in \mathcal{B}_X$, $U \circ U = U$. A quasi-uniformity with a transitive base is called transitive quasi-uniformity.

The ordered class of all transitive quasi-uniformities on \mathcal{C} will be denoted by $T\text{QUNIF}$.

Proposition 3.2.7. Let \mathcal{U} be a quasi-uniformity on \mathcal{C} .

$\mathcal{U}^* = \{\mathcal{U}_X^* \mid X \in \mathcal{C}\}$ where $\mathcal{U}_X^* = \{U^* \mid U \in \mathcal{U}_X\}$ with $U^*(m) = \bigvee \{n \mid m \leq U(n)\}$ is a base of a quasi-uniformity on \mathcal{C} called the conjugate quasi-uniformity of \mathcal{U} .

Proof. (U1) is easily seen to be satisfied. For (U2), let $U^* \in \mathcal{U}^*$. Then $U \in \mathcal{U}_X \Rightarrow \exists V \in \mathcal{U}_X$ such that $V(V(m)) \leq U(m)$. Thus $V^*(V^*(m)) = \bigvee \{n \mid m \leq V(V(m))\} \leq \bigvee \{n \mid m \leq U(n)\} = U^*(m)$. Since $(U \wedge V)(m) = U(m) \wedge V(m)$, $(U \wedge V)^*(m) = U^*(m) \wedge V^*(m)$ and $U^*(m) \wedge V^*(m) \in \mathcal{U}_X^*$. Thus (U4) is satisfied. Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and $U \in \mathcal{U}_Y$. By (U5), there is $U' \in \mathcal{U}_X$ such that $f(U'(m)) \leq U(f(m))$. Hence $f(U'^*(m)) = f(\bigvee \{n \mid m \leq U'(n)\}) = \bigvee \{f(n) \mid m \leq U'(n)\} \leq \bigvee \{p \mid f(m) \leq U'(p)\} = U^*(f(m))$. \square

Definition 3.2.8. A quasi-uniformity \mathcal{U} on \mathcal{C} is said to be a uniformity provided for every $U \in \mathcal{U}_X$ and $X \in \mathcal{C}$, there is $V \in \mathcal{U}$ such that $m \leq U(n) \Leftrightarrow n \leq V(m)$ for any $m, n \in \text{sub}X$.

Proposition 3.2.9. A quasi-uniformity \mathcal{U} on \mathcal{C} is uniformity if and only if $\mathcal{U}_X^* = \mathcal{U}_X$ for every $X \in \mathcal{C}$.

Proof. Let \mathcal{U} be a uniformity on \mathcal{U} . We must show that for any $X \in \mathcal{C}$, $\mathcal{U}_X \leq \mathcal{U}_X^*$ and $\mathcal{U}_X^* \leq \mathcal{U}_X$. So for any $U \in \mathcal{U}_X$, there is $V \in \mathcal{U}$ such that $n \leq U(m) \Leftrightarrow m \leq V(n)$ so that $n \leq U^*(m) \Rightarrow n \leq V(U^*(m)) \Leftrightarrow U^*(m) \leq V(n) \Rightarrow m \leq V(n) \Leftrightarrow n \leq U(m)$ and for any $U^* \in \mathcal{U}_X^*$, $U(m) \leq U^*(m)$. Conversely if $\mathcal{U}^* = \mathcal{U}$, then for any $U \in \mathcal{U}_X$ and $X \in \mathcal{C}$, $m \leq U(n) \Leftrightarrow m \leq U^*(n) \Leftrightarrow n \leq U(m)$. \square

We have seen that $\text{QUNIF}(\mathcal{C}, \mathcal{M})$, the conglomerate of all quasi-uniform structures on \mathcal{C} is ordered as follows: $\mathcal{U} \leq \mathcal{V}$ if for all $U \in \mathcal{U}_X$ there is $V \in \mathcal{V}_X$ such that $V(m) \leq U(m)$

for any $m \in \text{sub}X$. We next prove that this order confers to $QUNIF(\mathcal{C}, \mathcal{M})$ the structure of a complete lattice.

Theorem 3.2.10. *Let $\mathcal{A} = \{\mathcal{U}^i \mid i \in I\} \subseteq QUNIF(\mathcal{C}, \mathcal{M})$. Then $\mathcal{B} = \{\mathcal{B}_X \mid X \in \mathcal{C}\}$ with $\mathcal{B}_X = \{U^1 \wedge \dots \wedge U^n \mid \text{for every } 1 \leq i \leq n, U^i \in \mathcal{U}_X^i \text{ for some } \mathcal{U}^i \in \mathcal{A} \text{ and } n \in \mathbb{N}\}$ is a base for the supremum $\mathcal{U} = \bigvee \mathcal{A}$ of \mathcal{A} . If each \mathcal{U}^i is a uniformity (resp transitive quasi-uniformity) on \mathcal{C} then \mathcal{U} is also a uniformity (resp. transitive quasi-uniformity).*

Proof. (U1) and (U4) are clearly satisfied. For (U2), let $U = U^1 \wedge \dots \wedge U^n \in \mathcal{B}_X$, for any $1 \leq i \leq n$, $U^i \in \mathcal{U}_X^i$ for some $\mathcal{U}^i \in \mathcal{A}$. Then there are V^1, \dots, V^n such that $V^1 \circ V^1 \leq U^1, \dots, V^n \circ V^n \leq U^n$. Now, $V = V^1 \wedge \dots \wedge V^n \in \mathcal{B}_X$ and $V \circ V \leq (V^1 \circ V^1) \wedge \dots \wedge (V^n \circ V^n) \leq U$. Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and $U = U^1 \wedge \dots \wedge U^n \in \mathcal{B}_X$. Then there are V^1, \dots, V^n such that $f(U^1(m)) \leq V^1(f(m)), \dots, f(U^n(m)) \leq V^n(f(m))$. Thus $f(U(m)) = f(U^1(m) \wedge \dots \wedge U^n(m)) \leq f(U^1(m)) \wedge \dots \wedge f(U^n(m)) \leq V^1(f(m)) \wedge \dots \wedge V^n(f(m)) \leq (V^1 \wedge \dots \wedge V^n)(f(m)) = V(f(m))$. It is clear that \mathcal{U} is finer than each \mathcal{U}^i and if \mathcal{V} is another quasi-uniformity on \mathcal{C} that is finer than each \mathcal{U}^i , then \mathcal{U} is coarser than \mathcal{V} . Let each \mathcal{U}^i be a uniformity and $p \leq U(m)$ for any $U \in \mathcal{B}_X$ with $p, m \in \text{sub}X$. Then $p \leq (U^1 \wedge \dots \wedge U^n)(m)$ for any $1 \leq i \leq n$, $U^i \in \mathcal{U}^i$, for some $\mathcal{U}^i \in \mathcal{A}$ and $n \in \mathbb{N}$ and there are V^1, \dots, V^n with $m \leq U^1(p), \dots, m \leq U^n(p)$. Hence $m \leq (V^1 \wedge \dots \wedge V^n)(p) = V(p)$. Assume that for each i , $\mathcal{U}^i \in TQUNIF(\mathcal{C}, \mathcal{M})$ and $U \in \mathcal{B}_X$. Then $U = U^1 \wedge \dots \wedge U^n$ for any $1 \leq i \leq n$, $U^i \in \mathcal{U}_X^i$ for some $\mathcal{U}^i \in \mathcal{A}$. Since $U(U(m)) \leq U^1, \dots, U(U(m)) \leq U^n$, $U(U(m)) \leq (U^1 \wedge \dots \wedge U^n)(m) = U(m)$. \square

Corollary 3.2.11. *$QUNIF(\mathcal{C}, \mathcal{M})$ is complete lattice.*

Proof. The least element is $\mathcal{U}_X = \{1_X\}$ for any $X \in \mathcal{C}$ while the greatest is the quasi-uniformity \mathcal{U}_X consisting of all $U \in \mathcal{F}(\text{sub}X)$ satisfying (U1). For any $\mathcal{A} = \{\mathcal{U}^i \mid i \in I\} \subseteq QUNIF(\mathcal{C}, \mathcal{M})$, $\mathcal{U} = \bigvee \mathcal{A}$ of \mathcal{A} is constructed as in Theorem 4.1.17. Thus $QUNIF(\mathcal{C}, \mathcal{M})$ is a complete lattice since the meet can be constructed as the join of all upper bounds of \mathcal{A} . \square

Corollary 3.2.12. *$UNIF(\mathcal{C}, \mathcal{M})$ and $TQUNIF(\mathcal{C}, \mathcal{M})$ are complete sublattices of $QUNIF(\mathcal{C}, \mathcal{M})$.*

Theorem 3.2.13. *The conglomerates $UNIF(\mathcal{C}, \mathcal{M})$ and $TQUNIF(\mathcal{C}, \mathcal{M})$ are coreflective in $QUNIF(\mathcal{C}, \mathcal{M})$.*

While our intention is to study a quasi-uniformity on a category, we have found it more fruitful to express many of our proofs in terms of a syntopogenous structure. The next two theorems that describe the clear relationship between the two structures shall lead us to the achievement of this goal. Let $\mathcal{S} \in \text{SYNT}(\mathcal{C}, \mathcal{M})$, $\mathcal{B}^{\mathcal{S}}$ will denote the base for a quasi-uniformity induced by \mathcal{S} . If \mathcal{B} is a base for a quasi-uniformity, $\mathcal{S}^{\mathcal{B}}$ denotes the syntopogenous structure induced by \mathcal{B} .

Theorem 3.2.14.

The assignments $\mathcal{U} \mapsto \mathcal{S}^{\mathcal{U}}$ and $\mathcal{S} \mapsto \mathcal{U}^{\mathcal{S}}$ given by

$$\mathcal{S}_X^{\mathcal{B}} = \{\sqsubset_X^U \mid U \in \mathcal{B}_X\} \text{ where } m \sqsubset_X^U n \Leftrightarrow U(m) \leq n, \text{ and}$$

$$\mathcal{B}_X^{\mathcal{S}} = \{U^{\sqsubset} \mid \sqsubset_X \in \mathcal{S}_X\} \text{ where } U^{\sqsubset}(m) = \bigwedge \{n \mid m \sqsubset_X n\}$$

for all $X \in \mathcal{C}$ and $m, n \in \text{sub}X$ define an adjunction between $QUNIF(\mathcal{C}, \mathcal{M})$ and $\text{SYNT}(\mathcal{C}, \mathcal{M})$ with $\mathcal{S} \mapsto \mathcal{U}^{\mathcal{S}}$ being the right adjoint.

Proof. For all $X \in \mathcal{C}$ and \mathcal{B} a base of a quasi-uniformity, we first show that $\mathcal{S}^{\mathcal{B}}$ is a syntopogenous structure on \mathcal{C} . (S1) follows from (U1) and the fact that each $U \in \mathcal{B}_X$ is a monotone map while (S2) follows from (U3). For (S3), Let $m \sqsubset_X^U n$ for $U \in \mathcal{B}_X$ and $m, n \in \text{sub}X$. Then $U(m) \leq n$. By (U2), there is $U' \in \mathcal{U}_X$ such that $U'(U'(m)) \leq U(m) \leq n \Rightarrow U'(m) \leq U(U'(m)) \leq n \Leftrightarrow m \sqsubset^{U'} U'(m) \sqsubset^{U'} n$. If $f : X \rightarrow Y$ is a \mathcal{C} -morphism and $U \in \mathcal{B}_Y$, then by (U4) there is $U' \in \mathcal{B}_X$ such that $f(U'(m)) \leq U(f(m)) \leq n$. Now $f(m) \sqsubset_Y^U n \Leftrightarrow U(f(m)) \leq n \Rightarrow f(U'(m)) \leq U(f(m)) \leq n \Rightarrow f(U'(m)) \leq n \Rightarrow U'(m) \leq f^{-1}(n) \Leftrightarrow m \sqsubset_X^{U'} f^{-1}(n)$.

On the other hand, one uses (S1) and (S2) to see that $\mathcal{B}^{\mathcal{S}}$ satisfies (U1) and (U3) respectively. For (BU2), let $U^{\sqsubset} \in \mathcal{B}_X^{\mathcal{S}}$ for $\sqsubset_X \in \mathcal{S}_X$. Then by (S3) there is $\sqsubset_X \in \mathcal{S}_X$ such that $\sqsubset_X \sqsubset \sqsubset_X \circ \sqsubset_X$. This implies that $U^{\sqsubset} \circ U^{\sqsubset} \leq U^{\sqsubset}$. Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and $U_Y^{\sqsubset} \in \mathcal{B}_Y^{\mathcal{S}}$. Then by (S3), there is \sqsubset_Y such that $f(m) \sqsubset_Y n \Rightarrow m \sqsubset_X f^{-1}(n)$. Thus $U^{\sqsubset_X}(m) = \bigwedge \{n \mid m \sqsubset_X n\} \leq \bigwedge \{f^{-1}(p) \mid f(m) \sqsubset_Y p\} = f^{-1}(U^{\sqsubset_Y}(f(m))) \leq f^{-1}(U^{\sqsubset_Y}(f(m))) \Rightarrow U^{\sqsubset_X}(m) \leq f^{-1}(U^{\sqsubset_Y}(f(m))) \Leftrightarrow f(U^{\sqsubset_X}(m)) \leq U^{\sqsubset_Y}(f(m))$.

Assume $\mathcal{B}_X \leq \mathcal{B}'_X$ for all $X \in \mathcal{C}$ and $\sqsubset_X^U \in \mathcal{S}^{\mathcal{B}}$ for some $U \in \mathcal{U}_X$. By assumption, there is $V \in \mathcal{B}'_X$ such that $V \leq U \Rightarrow \sqsubset_X^U \sqsubset \sqsubset_X^V$. Thus $\mathcal{S}_X^{\mathcal{B}} \leq \mathcal{S}_X^{\mathcal{B}'}$.

Furthermore, if $\mathcal{S}_X \leq \mathcal{S}'_X$ and $U^{\sqsubset} \in \mathcal{B}_X^{\mathcal{S}}$ for all $\sqsubset_X \in \mathcal{S}_X$, there is $\sqsubset'_X \in \mathcal{S}'_X$ such that

$\sqsubset_X \subseteq \sqsubset'_X \Rightarrow U_X^{\sqsubset'} \leq U_X^{\sqsubset}$. Hence $\mathcal{B}^{\mathcal{S}} \leq \mathcal{B}^{\mathcal{S}'}$.

Lastly $\mathcal{B} \leq \mathcal{B}^{\mathcal{S}^{\mathcal{B}}}$ and $\mathcal{S}^{\mathcal{B}^{\mathcal{S}}} \leq \mathcal{S}$. For any $U \in \mathcal{B}_X$ and $\sqsubset \in \mathcal{S}_X$, $m \sqsubset_X^{U^{\sqsubset}} n \Leftrightarrow U^{\sqsubset}(m) \leq n \Leftrightarrow \bigwedge \{p \mid m \sqsubset_X p\} \leq n \Rightarrow m \sqsubset_X p$ and $\mathcal{B}^{\mathcal{S}^{\mathcal{B}}} = \mathcal{B}$ since $U^{\sqsubset^U}(m) = \bigwedge \{n \mid U(m) \leq n\} = U(m)$.

□

The fact that $\mathcal{U} \mapsto \mathcal{S}^{\mathcal{U}}$ has a right inverse leads us to a condition on the syntopogenous structure under which the other inverse exists, that is a subconglomerate of $SYNT(\mathcal{C}, \mathcal{M})$ which is order isomorphic to $QUNIF(\mathcal{C}, \mathcal{M})$.

Theorem 3.2.15. *$QUNIF(\mathcal{C}, \mathcal{M})$ is order isomorphic to $CSYNT(\mathcal{C}, \mathcal{M})$*

Proof. $\mathcal{B} = \mathcal{B}^{\mathcal{S}^{\mathcal{B}}}$ by Theorem 3.2.15, we only need to show that $\mathcal{S}^{\mathcal{B}^{\mathcal{S}}} = \mathcal{S}$. It suffices to prove that for any $\sqsubset_X \in \mathcal{S}_X$, $\sqsubset_X^{U^{\sqsubset_X}} = \sqsubset_X$. So for $m \sqsubset_X^{U^{\sqsubset_X}} n \Leftrightarrow U^{\sqsubset_X}(m) \leq n \Leftrightarrow \bigwedge \{p \mid m \sqsubset_X p\} \leq n \Leftrightarrow m \sqsubset_X n$ for all $X \in \mathcal{C}$. □

Since $\mathcal{S}_X \subseteq \bigwedge -TORD(\mathcal{C}, \mathcal{M})$ for each $\mathcal{S} \in CSYNT(\mathcal{C}, \mathcal{M})$, it follows from Theorem 3.2.15 that a quasi-uniformity on \mathcal{C} is a collection of families of closure operators.

By Corollary 2.1.6, $\bigwedge -INTORD(\mathcal{C}, \mathcal{M})$ is isomorphic to the conglomerate of idempotent closure operators and from Theorem 3.2.15, $CSYNT(\mathcal{C}, \mathcal{M}) \cong QUNIF(\mathcal{C}, \mathcal{M})$. Thus by Corollary 3.1.5 every idempotent closure operator on \mathcal{C} is a base for a quasi-uniformity.

From Propositions 2.1.4, 2.1.2 and Theorem 3.2.15, we obtain the interior and neighbourhood operators associated to a quasi-uniformity as one can see in the following proposition.

Proposition 3.2.16. *Let \mathcal{U} be a quasi-uniformity on \mathcal{C} and $X \in \mathcal{C}$.*

(i) *If for any \mathcal{C} -morphism $f : X \rightarrow Y$, f^{-1} commutes with the joins of subobjects, then $i^{\mathcal{U}}(m) = \bigvee \{p \in \text{sub}X \mid U(p) \leq m \text{ for some } U \in \mathcal{U}_X\}$ is an interior operator on \mathcal{C} .*

(ii) *$\nu^{\mathcal{U}}(m) = \{n \in \text{sub}X \mid U(m) \leq n \text{ for some } U \in \mathcal{U}_X\}$ is a neighbourhood operator on \mathcal{C} .*

Proof. (i) (I1) and (I2) are easily seen to be satisfied. To check (I3), we let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and $U \in \mathcal{U}_Y$, then by Proposition 3.2.2 there is $U' \in \mathcal{U}_X$ such that $U'(f^{-1}(m)) \leq f^{-1}(U(m))$ for any $m \in \text{sub}Y$. Thus $f^{-1}(i_Y^U(m)) = f^{-1}(\bigvee \{p \mid U(p) \leq m\}) = \bigvee \{f^{-1}(p) \mid U(p) \leq m\} \leq \bigvee \{l \mid U'(f^{-1}(p)) \leq m\} = i_X^{U'}(f^{-1}(m))$.

(ii) (N1) to (N3) are clear. For (N4), let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and $n \in \nu^{\mathcal{M}}(f(m))$. Then $U(f(m)) \leq n$ for some $U \in \mathcal{U}_Y$. By (U5) there is $U' \in \mathcal{U}_X$ such that $f(U'(m)) \leq U(f(m)) \leq n \Leftrightarrow U'(m) \leq f^{-1}(n) \Leftrightarrow f^{-1}(n) \in \nu^{\mathcal{M}}(m)$.

□

Let us denote by $INTCSYNT(\mathcal{C}, \mathcal{M})$ the conglomerate of all interpolative co-perfect syntopogenous structures on \mathcal{C} . We next prove that interpolative co-perfect syntopogenous structures are exactly the transitive quasi-uniformities.

Proposition 3.2.17. *$INTCSYNT(\mathcal{C}, \mathcal{M})$ is order isomorphic to $TQUNIF(\mathcal{C}, \mathcal{M})$*

Proof. Let \mathcal{B} be a transitive base and $m \sqsubset_X^U n$ for any $U \in \mathcal{B}_X$ and $X \in \mathcal{C}$, then $U(m) = U(U(m)) \sqsubset_X^U U(m) \leq n$, hence $m \sqsubset_X^U U(m) \sqsubset_X^U n$. Conversely if $\mathcal{S} \in INTCSYNT$ and $\sqsubset_X \in \mathcal{S}_X, m \sqsubset_X n \Rightarrow (\exists p) \mid m \sqsubset_X p \sqsubset_X n \Rightarrow U^{\sqsubset_X}(m) \leq p \sqsubset_X n \Rightarrow U^{\sqsubset_X}(m) \leq n$. This implies that $\bigwedge \{l \mid U^{\sqsubset_X}(m) \sqsubset_X l\} \leq \bigwedge \{q \mid m \sqsubset q\}$ that is $U^{\sqsubset_X}(U^{\sqsubset_X}(m)) \leq U^{\sqsubset_X}(m)$ and $U^{\sqsubset_X}(m) \leq U^{\sqsubset_X}(U^{\sqsubset_X}(m))$ follows from (U1) so that $U^{\sqsubset_X}(m) = U^{\sqsubset_X}(U^{\sqsubset_X}(m))$. □

Theorem 3.2.15 provides a bridge from quasi-uniformity to syntopogenous structure and back. This allows us to always compare the results we obtain for the two structures and work at the side where the proofs are easier to manipulate.

Proposition 3.2.18. *Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and $\mathcal{S} \in CSYNT(\mathcal{C}, \mathcal{M})$. Then f is \mathcal{S} -initial if and only if for every $U \in \mathcal{B}_X$ there is $U' \in \mathcal{B}_Y$ such that $f^{-1}(U'(f(m))) \leq U(m)$ for all $m \in \text{sub}X$.*

Proof. Assume that for any $U \in \mathcal{B}_X$ there is $U' \in \mathcal{B}_Y$ such that $f^{-1}(U'(f(m))) \leq U(m)$ for all $m \in \text{sub}X$ and $m \sqsubset_X n$ for some $\sqsubset_X \in \mathcal{S}_X$ and $m, n \in \text{sub}X$. Then there is $U \in \mathcal{B}_X$ such that $\sqsubset^U = \sqsubset_X$ and there is $U' \in \mathcal{B}_Y$ such that $f^{-1}(U'(f(m))) \leq U(m) \Rightarrow f^{-1}(U'(f(m))) \leq U(m) \leq n \Rightarrow f^{-1}(U'(f(m))) \leq n \Leftrightarrow U'(f(m)) \leq f_*(n) \Leftrightarrow f(m) \sqsubset_Y^U f_*(n)$. Conversely if $\mathcal{S} \in CSYNT(\mathcal{C}, \mathcal{M})$ and f is \mathcal{S} -initial, then for all $U \in \mathcal{B}_X$ and $m, n \in \text{sub}X, U(m) \leq n \Leftrightarrow (\exists \sqsubset_X \in \mathcal{S}_X) \mid m \sqsubset_X n \Leftrightarrow (\exists \sqsubset_Y \in \mathcal{S}_Y) \mid f(m) \sqsubset_Y f_*(n) \Leftrightarrow (\exists U' \in \mathcal{B}_Y) \mid U'(f(m)) \leq f_*(n) \Leftrightarrow f^{-1}(U'(f(m))) \leq n$. Thus $f^{-1}(U'(f(m))) \leq U(m)$. □

Proposition 3.2.19. *Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism, $\mathcal{S} \in CSYNT(\mathcal{C}, \mathcal{M})$ and \mathcal{S} is simple. Then f is \mathcal{S} -initial if and only if $f^{-1}(c_Y^{\sqsubset}(f(m))) \leq c_X^{\sqsubset}(m)$ for all $m \in \text{sub}X$.*

Proof. The proof follows from Proposition 2.1.5 and the fact that $\mathcal{S} \in CSYNT(\mathcal{C}, \mathcal{M})$ and \mathcal{S} is simple means that $\mathcal{S} = \{\sqsubset\} \in \bigwedge -INTORD$ for all $X \in \mathcal{C}$. \square

Proposition 3.2.20. *Assume that for any \mathcal{C} -morphism $f : X \rightarrow Y$, f^{-1} commutes with the join of subobjects and $\mathcal{S} \in CSYNT$. If f is $\mathcal{U}^{\mathcal{S}}$ -initial, then*

- (1) f is $\nu^{\mathcal{U}^{\mathcal{S}}}$ -initial;
- (2) f is $i^{\mathcal{U}^{\mathcal{S}}}$ -initial.

Proof. If f is $\mathcal{U}^{\mathcal{S}}$ -initial and $U \in \mathcal{B}_X^{\mathcal{S}}$, then there is $V \in \mathcal{B}_Y^{\mathcal{S}}$ such that $f^{-1}(V(f(m))) \leq U(m)$ for all $m \in \text{sub}X$. Thus,

(1) $f_*(\nu_X^{\mathcal{U}^{\mathcal{S}}}(m)) = \{f_*(n) \mid U(m) \leq n\} \subseteq \{p \mid V(f(m)) \leq p\} = \nu^{\mathcal{U}^{\mathcal{S}}}(f(m)) \Rightarrow \nu^{\mathcal{U}^{\mathcal{S}}}(m) \subseteq f^{-1}(\nu^{\mathcal{U}^{\mathcal{S}}}(f(m)))$ that is f is $\nu^{\mathcal{U}^{\mathcal{S}}}$ -initial.

(2) $n \in \{n \mid U(n) \leq m\} \Rightarrow f(n) \in \{p \mid V(p) \leq f_*(m)\} \Rightarrow n \in \{f^{-1}(p) \mid V(p) \leq f_*(m)\} \Rightarrow \{n \mid U(n) \leq m\} \subseteq \{f^{-1}(p) \mid V(p) \leq f_*(m)\} \Rightarrow i^{\mathcal{U}^{\mathcal{S}}}(m) = \bigvee \{n \mid U(n) \leq m\} \leq \bigvee \{f^{-1}(p) \mid V(p) \leq f_*(m)\} = f^{-1}(i^{\mathcal{U}^{\mathcal{S}}}(f_*(m)))$ that is f is $i^{\mathcal{U}^{\mathcal{S}}}$ -initial. \square

Proposition 3.2.21. *Let $(f : X \rightarrow X_i)_{i \in I}$ be a source in \mathcal{C} and $\mathcal{S} \in CSYNT(\mathcal{C}, \mathcal{M})$. Then f_i is \mathcal{S} -initial if and only if for any $U \in \mathcal{B}_X^{\mathcal{S}}$ there is $i \in I$ and $U \in \mathcal{B}_{X_i}^{\mathcal{S}}$ such that $(f_i)^{-1}(U'(f_i(m))) \leq U(m)$ for all $m \in \text{sub}X$.*

Proof. Assume that for any $U \in \mathcal{B}_X^{\mathcal{S}}$ there is $i \in I$ and $U \in \mathcal{B}_{X_i}^{\mathcal{S}}$ such that $(f_i)^{-1}(U'(f_i(m))) \leq U(m)$ and $m \sqsubset_X n$ for some $\sqsubset_X \in \mathcal{S}_X$. Then there is $U \in \mathcal{B}_X^{\mathcal{S}}$ which determines \sqsubset_X and there is $i \in I$ and $U' \in \mathcal{B}_{X_i}^{\mathcal{S}}$ such that $(f_i)^{-1}(U'(f_i(m))) \leq U(m) \leq n \Rightarrow (f_i)^{-1}(U'(f_i(m))) \leq n \Rightarrow U'(f_i(m)) \leq (f_i)_*(n) \Rightarrow f_i(m) \sqsubset_{X_i} (f_i)_*(n)$. On the other hand if $\mathcal{S} \in CSYNT(\mathcal{C}, \mathcal{M})$ and f_i is \mathcal{S} -initial. Then for any $U \in \mathcal{B}_X^{\mathcal{S}}$, $U(m) \leq n \Leftrightarrow (\exists \sqsubset_X \in \mathcal{S}_X) \mid m \sqsubset_X n \Leftrightarrow (\exists \sqsubset_{X_i} \in \mathcal{S}_{X_i}) \mid f_i(m) \sqsubset_{X_i} (f_i)_*(n) \Leftrightarrow (\exists U' \in \mathcal{B}_{X_i}^{\mathcal{S}}) \mid U'(f_i(m)) \leq (f_i)_*(n) \Leftrightarrow (f_i)^{-1}(U'(f_i(m))) \leq n$. \square

Corollary 3.2.22. *Let $X = \prod_{i \in I} X_i$ be a product in \mathcal{C} and $\mathcal{S} \in CSYNT(\mathcal{C}, \mathcal{M})$. Then $(p_i : X \rightarrow X_i)_{i \in I}$ is \mathcal{S} -initial if for any $U \in \mathcal{B}_X^{\mathcal{S}}$ there is $i \in I$ and $U \in \mathcal{B}_{X_i}^{\mathcal{S}}$ such that $(p_i)^{-1}(U'(p_i(m))) \leq U(m)$ for all $m \in \text{sub}X$.*

Definition 3.2.23. *Let sub^+X be the class of all atomic elements of $\text{sub}X$ for any $X \in \mathcal{C}$ and $\mathcal{S} \in SYNT$. Then $X \in \mathcal{C}$ is said to be \mathcal{S} -separated if for any pair $m, n \in \text{sub}^+X$*

such that $m \wedge n = 0_X$, there are $p, q \in \text{sub}X$ and $\sqsubset_X \in \mathcal{S}_X$ such that $m \sqsubset_X p$ and $n \sqsubset_X q$ with $p \wedge q = 0_X$.

Proposition 3.2.24. *Let $\mathcal{S} \in \text{CSYNT}$ and $X \in \mathcal{C}$. X is \mathcal{S} -separated if and only if for any $n, m \in \text{sub}^+X$ such that $m \wedge n = 0_X$, there is $U \in \mathcal{B}_X^{\mathcal{S}}$ such that $U(m) \wedge U(n) = 0_X$.*

Proof. Assume that X is \mathcal{S} - T_2 and $m, n \in \text{sub}^+X$ such that $m \wedge n = 0_X$. Then there is $\sqsubset_X \in \mathcal{S}_X$ and $p, q \in \text{sub}X$ such that $m \sqsubset_X p$ and $n \sqsubset_X q$ with $p \wedge q = 0_X$. By Theorem 3.2.15 there is $U \in \mathcal{B}_X^{\mathcal{S}}$ such that $\sqsubset_X^U = \sqsubset_X$. Thus $U(m) \leq p$ and $U(n) \leq q \Rightarrow U(m) \wedge U(n) \leq p \wedge q = 0_X \Rightarrow U(m) \wedge U(n) = 0_X$. Conversely if for $n, m \in \text{sub}X$ with $m \wedge n = 0_X$, there $U \in \mathcal{B}_X^{\mathcal{S}}$ such that $U(m) \wedge U(n) = 0_X$. Then, since $\mathcal{S} \in \text{CSYNT}$, $m \sqsubset_X U(m)$, $n \sqsubset_X U(n)$. One puts $p = U(m)$, $q = U(n)$ to complete the proof. \square

Proposition 3.2.25. *If \mathcal{S} is simple, then X is \mathcal{S} -separate if and only if for any pair $m, n \in \text{sub}_oX$ such that $m \wedge n = 0_X$, there are $p, q \in \text{sub}X$ and $\sqsubset_X \in \mathcal{S}_X$ such that $p \in \nu_X^{\mathcal{S}}(m)$ and $q \in \nu_X(m)$ with $p \wedge q = 0_X$.*

Proof. The proof follows from Proposition 2.1.2 and the fact that $\mathcal{S} \in \text{SYNT}(\mathcal{C}, \mathcal{M})$ and \mathcal{S} is simple means that $\mathcal{S} = \{\sqsubset\} \in \text{INTORD}$ for all $X \in \mathcal{C}$. \square

3.3 Quasi-uniform structures determined by closure and interior operators

We have already observed in the previous section that every quasi-uniformity in a category induces an idempotent closure operator (interior). In this section, we first prove a one to one correspondence between idempotent closure operators and the so-called saturated quasi-uniformities. We then define what it means for a quasi-uniformity to be compatible with a closure operator (interior) on \mathcal{C} . With the help of categorical topogenous structures, we show that for any idempotent closure operator c (interior i) on \mathcal{C} , there is at least a transitive quasi-uniformities compatible with c (i). We find a condition under which a topogenous order is compatible with a transitive quasi-uniformity. This allows us to characterize a closure operator (interior) that is compatible with a transitive quasi-uniformity. In particular, when \mathcal{C} is the category **Top** of topological spaces and continuous maps, the coarsest transitive quasi-uniformity compatible with the Kuratowski interior

operator corresponds to the Császár-Pervin quasi-uniformity and the one compatible with the Kuratowski closure is the inverse of the Császár-Pervin quasi-uniformity while in **Grp** it allows to generate a family of idempotent closure operators on **Grp** determined by the normal closure.

Definition 3.3.1. Let $\mathcal{U} \in QUNIF(\mathcal{C}, \mathcal{M})$. We shall say that \mathcal{U} is compatible with \sqsubset (or \sqsubset admits \mathcal{U}) if $\sqsubset_X = \bigcup \{\sqsubset_X^U : U \in \mathcal{U}_X\}$ for all $X \in \mathcal{C}$.

Thinking of a topology as a particular topogenous order, one sees that Definition 3.3.1 carries the idea of a quasi-uniformity compatible with a topology. With this understanding, the classical reality that every topological space admit at least a quasi-uniformity can now be treated in a categorical setting.

Because of Proposition 2.1.2, we can say that $\mathcal{U} \in QUNIF(\mathcal{C}, \mathcal{M})$ is compatible with a closure operator c (respectively an interior i) if $c_X(m) = \bigwedge \{U(m) : U \in \mathcal{U}_X\}$ (respectively $i_X(m) = \bigvee \{n \mid U(n) \leq m \text{ for some } U \in \mathcal{U}_X\}$) for any $m \in \text{sub}X$.

Definition 3.3.2. A quasi-uniform structure \mathcal{U} on \mathcal{C} is said to be saturated if for any $X \in \mathcal{C}$, $\bigwedge \{U : U \in \mathcal{U}_X\} \in \mathcal{U}_X$.

We shall denote by $SQUNIF(\mathcal{C}, \mathcal{M})$ the conglomerate of all saturated quasi-uniform structures on \mathcal{C} .

Proposition 3.3.3. Let $\mathcal{U} \in QUNIF(\mathcal{C}, \mathcal{M})$ and $X \in \mathcal{C}$. Then $\mathcal{U} \in SQUNIF(\mathcal{C}, \mathcal{M})$ if and only if there is a unique base \mathcal{B} for \mathcal{U} such that \mathcal{B}_X has a single member.

Proof. Sufficiency is clear. Let $V(m) = \bigwedge \{U(m) : U \in \mathcal{U}_X\}$ for any $m \in \text{sub}X$. By assumption, $V \in \mathcal{U}_X$. Now, let $\mathcal{B}_X = \{V\}$. We must show that $V \circ V \leq V$ and satisfies (U5). Since $V \in \mathcal{U}_X$, there is $U \in \mathcal{U}_X$ such that $U \circ U \leq V$. But $V \leq U$ and so $V(V(m)) \leq V(U(m)) \leq U(U(m)) \leq V(m)$. Let $f : X \rightarrow Y$ and $V_Y \in \mathcal{B}_Y$. Then $V_Y \in \mathcal{U}_Y$ and there is $U' \in \mathcal{U}_X$ such that $f(U'(m)) \leq V_Y(f(m))$ for all $m \in \text{sub}X$. Since $V_X \leq U'$, $f(V_X(m)) \leq f(U'(m)) \leq V_Y(f(m))$. The uniqueness of \mathcal{B} is easily seen. \square

As mentioned earlier, every idempotent closure operator is a base for a quasi-uniform structure on \mathcal{C} . Proposition 3.3.3, allows now to identify those quasi-uniform structures that are in one to one corespondance with idempotent closure operators.

Theorem 3.3.4. $SQUNIF(\mathcal{C}, \mathcal{M})$ is order isomorphic to $ICL(\mathcal{C}, \mathcal{M})$. The inverse as-

seignments of each other $\mathcal{U} \mapsto \mathcal{C}^{\mathcal{U}}$ and $c \mapsto \mathcal{U}^c$ are given by

$$\mathcal{U}_X^c = \{U \in \mathcal{F}(\text{sub}X) : c \leq U\} \text{ and}$$

$$\mathcal{C}^{\mathcal{U}} = \bigwedge \{U : U \in \mathcal{U}_X\}$$

for all $X \in \mathcal{C}$.

Proof. The assignment $c \mapsto \mathcal{U}^c$ is clearly well defined. For $\mathcal{U} \mapsto \mathcal{C}^{\mathcal{U}}$, we only need to show that for any $\mathcal{U} \in \text{SQUNIF}(\mathcal{C}, \mathcal{M})$, $\mathcal{C}^{\mathcal{U}} \circ \mathcal{C}^{\mathcal{U}} = \mathcal{C}^{\mathcal{U}}$. So for all $U \in \mathcal{U}_X$, there is $V \in \mathcal{U}_X$ such that $V \circ V \leq U$. Now, $V(\mathcal{C}^{\mathcal{U}}(m)) \leq V(V(m)) \leq U(m)$. Thus $\mathcal{C}^{\mathcal{U}}(\mathcal{C}^{\mathcal{U}}(m)) = \bigwedge \{V(\mathcal{C}^{\mathcal{U}}(m)) : V \in \mathcal{U}_X\} \leq \bigwedge \{U(m) : U \in \mathcal{U}_X\} = \mathcal{C}_X^{\mathcal{U}}(m)$. Let $c \in \text{ICL}(\mathcal{C}, \mathcal{M})$, $\mathcal{C}_X^{\mathcal{U}^c}(m) = \bigwedge \{U(m) : U \in \mathcal{U}^c\} = c_X(m)$ and for any $\mathcal{U} \in \text{SQUNIF}(\mathcal{C}, \mathcal{M})$, $\mathcal{U}^{\mathcal{C}^{\mathcal{U}}} = \{U \in \mathcal{F}(\text{sub}X) : \mathcal{C}^{\mathcal{U}} \leq U\} = \mathcal{U}$. \square

Proposition 3.3.5. *Let $\mathcal{U} \in \text{SQUNIF}(\mathcal{C}, \mathcal{M})$ and $f : X \rightarrow Y$ be a \mathcal{C} -morphism. Then*

- (1) *f is \mathcal{U} -initial if and only if $f^{-1}(\mathcal{C}_Y^{\mathcal{U}}(f(m))) \leq \mathcal{C}_X^{\mathcal{U}}(m)$ for any $m \in \text{sub}Y$.*
- (2) *f is $\mathcal{C}^{\mathcal{U}}$ -closed if and only if for any $U \in \mathcal{U}_X$ there is $V \in \mathcal{U}_Y$ such that $V(f(m)) \leq f(U(m))$ for any $m \in \text{sub}Y$.*

Proof. (1) Assume f is \mathcal{U} -initial. Then for any $U \in \mathcal{U}_X$, there is $V \in \mathcal{U}_Y$ such that $f^{-1}(V(f(m))) \leq U(m)$. Thus $f^{-1}(\bigwedge \{V(f(m)) : V \in \mathcal{U}_Y\}) \leq U(m) \Rightarrow f^{-1}(\mathcal{C}_Y^{\mathcal{U}}(f(m))) \leq \bigwedge \{U(m) : U \in \mathcal{U}_X\} = \mathcal{C}_X^{\mathcal{U}}(m)$. Conversely if $f^{-1}(\mathcal{C}_Y^{\mathcal{U}}(f(m))) \leq \mathcal{C}_X^{\mathcal{U}}(m)$ and $U \in \mathcal{U}_X$, then $f^{-1}(\bigwedge \{V(f(m)) : V \in \mathcal{U}_Y\}) \leq \bigwedge \{U(m) : U \in \mathcal{U}_X\} \leq U(m)$. Since $\mathcal{U} \in \text{SQUNIF}(\mathcal{C}, \mathcal{M})$, $V = \mathcal{C}_Y^{\mathcal{U}} \in \mathcal{U}_Y$ and $f(U(f(m))) \leq U(m)$.

- (2) If f is $\mathcal{C}^{\mathcal{U}}$ -closed and $U \in \mathcal{U}_X$, then $\mathcal{C}_Y^{\mathcal{U}}(f(m)) \leq f(\mathcal{C}_Y^{\mathcal{U}}(m)) \leq f(U(m))$. Since $\mathcal{U} \in \text{SQUNIF}(\mathcal{C}, \mathcal{M})$, there is $V = \mathcal{C}_Y^{\mathcal{U}} \in \mathcal{U}_Y$ such that $V(f(m)) \leq f(U(m))$. On the other hand if for any $U \in \mathcal{U}_X$, there is $V \in \mathcal{U}_Y$ such that $V(f(m)) \leq f(U(m))$, then $\mathcal{C}_Y^{\mathcal{U}}(f(m)) \leq f(U(m)) \Rightarrow \mathcal{C}_Y^{\mathcal{U}}(f(m)) \leq f(\bigwedge \{U(m) : U \in \mathcal{U}_X\}) = f(\mathcal{C}_X^{\mathcal{U}}(m))$. \square

Our next theorem was motivated by the observation that a topology on a set is a particular interpolative topogenous order and the fact that each member of a base for Pervin quasi-uniformity depends on a finite number of open sets.

Theorem 3.3.6. *Let $\sqsubset \in \wedge -INTORD(\mathcal{C}, \mathcal{M})$. Then there is a coarsest transitive quasi-uniformity \mathcal{U} on \mathcal{C} compatible with \sqsubset .*

Proof. Let $\sqsubset \in \wedge -INTORD(\mathcal{C}, \mathcal{M})$. For any $X \in \mathcal{C}$, put $\mathcal{A}^\sqsubset = \{m \in \text{sub}X \mid m \text{ is } \sqsubset\text{-strict}\}$. Then \mathcal{A} is a complete sublattice of $\text{sub}X$. If $F(\mathcal{A}^\sqsubset)$ is the collection of all finite sublattices of \mathcal{A}^\sqsubset , then $\mathcal{S}_X = \{\sqsubset_X^L \mid L \in F(\mathcal{A}^\sqsubset)\}$ where $m \sqsubset_X^L n \Leftrightarrow \exists p \in L \mid m \leq p \leq n$ is an interpolative co-perfect syntopogenous structure. By Theorem 3.2.14, $\mathcal{B}_X^\mathcal{S} = \{U^{\sqsubset^L} \mid L \in F(\mathcal{A}^\sqsubset)\}$ is a base for a transitive quasi-uniformity $\mathcal{U}^\mathcal{S}$ on \mathcal{C} . Now, let $\mathcal{U} = \mathcal{U}^\mathcal{S}$ then $m \sqsubset_X^\mathcal{U} n \Leftrightarrow \exists L \in F(\mathcal{A}^\sqsubset) \mid U^{\sqsubset^L}(m) \leq n \Leftrightarrow m \sqsubset_X^L n \Leftrightarrow \exists p \in L \mid m \leq p \sqsubset_X p \leq n \Rightarrow m \sqsubset_X n$. On the other hand, $m \sqsubset_X n \Leftrightarrow c_X(m) \leq n \Rightarrow m \sqsubset_X c_X(m) \sqsubset_X n$. Since $c_X(m) \in \mathcal{A}^\sqsubset$, put $\mathcal{L} = \{c_X(m), 1_X\}$ to have that $m \sqsubset_X^L n \Leftrightarrow U^{\sqsubset^L}(m) \leq n \Rightarrow m \sqsubset_X^\mathcal{U} n$. Thus $\sqsubset^\mathcal{U} = \sqsubset$. Let $\mathcal{U}' \in QUNIF(\mathcal{C}, \mathcal{M})$ such that $\sqsubset = \sqsubset^{\mathcal{U}'}$. We must show that $\mathcal{U} \leq \mathcal{U}'$ i.e if $L \in F(\mathcal{A}^\sqsubset)$ then there is $U' \in \mathcal{U}'_X$ such that $U'(m) \leq U^{\sqsubset^L}(m)$ for any $m \in \text{sub}X$. For any $m \in \text{sub}X$, $m \sqsubset_X^L U^{\sqsubset^L}(m) \Rightarrow m \sqsubset_X U^{\sqsubset^L}(m)$. Since $\sqsubset^{\mathcal{U}'} = \sqsubset$, there is $U' \in \mathcal{U}'_X$ such that $U'(m) \leq U^{\sqsubset^L}(m)$. □

In a similar way to the above, we prove the next Theorem.

Theorem 3.3.7. *Let $\sqsubset \in \vee -INTORD(\mathcal{C}, \mathcal{M})$. Then there is a coarsest transitive quasi-uniformity \mathcal{U} on \mathcal{C} compatible with \sqsubset .*

Theorems 3.3.6 and 3.3.7 are very important. On the one hand they allow us to conclude that for any $c \in ICL(\mathcal{C}, \mathcal{M})$ (respectively $iINT(\mathcal{C}, \mathcal{M})$), there is a coarsest transitive quasi-uniformity \mathcal{U} on \mathcal{C} compatible with c (i). On the other hand they present a categorical version of the well known Császár-Pervin ([Csá63, Per62]) quasi-uniformity.

Furthermore, the analysis of the proof of Theorem 3.3.6, allows to obtain a categorical generalization of A. Cászár's Theorem (see [Csá00]), which characterizes those topogenous orders that are compatible with a transitive quasi-uniformity.

Theorem 3.3.8. *Let $\sqsubset \in TORD(\mathcal{C}, \mathcal{M})$. Then \sqsubset is compatible with $\mathcal{U} \in TQUNIF(\mathcal{C}, \mathcal{M})$ if and only if $\sqsubset = \sqsubset^{\mathcal{A}}$ for some complete sublattice \mathcal{A} of $\text{sub}X$, where $m \sqsubset_X^{\mathcal{A}} n \Leftrightarrow \exists p \in \mathcal{A} \mid m \leq p \leq n$.*

Proof. Assume that $\mathcal{U} \in TQUNIF(\mathcal{C}, \mathcal{M})$ and $\sqsubset = \sqsubset^\mathcal{U}$. For any $X \in \mathcal{C}$, let $\mathcal{A} = \{m \in \text{sub}X \mid m \text{ is } \sqsubset\text{-strict}\}$. If $m \sqsubset_X n$ then there is $U \in \mathcal{U}_X$ such that $U(m) \leq n$. By assumption, $U(m) \sqsubset U(m) \Rightarrow U(m) \in \mathcal{A}$ so that $m \sqsubset_X^{\mathcal{A}} n$. On the other hand, $m \sqsubset_X^{\mathcal{A}} n$

$n \Rightarrow \exists p \in \mathcal{A}$ such that $m \leq p \sqsubset_X p \leq n \Rightarrow m \sqsubset_X n$. Conversely let \mathcal{A} be a complete sublattice of $\text{sub}X$ for any $X \in \mathcal{C}$ and $\sqsubset = \sqsubset^{\mathcal{A}}$. Let $\mathcal{F}(\mathcal{A})$ be the collection of all finite sublattices L of \mathcal{A} . Then $\mathcal{S} \in \bigwedge -\text{INCSYNT}(\mathcal{C}, \mathcal{M})$ where $\mathcal{S}_X = \{\sqsubset_X^L \mid L \in \mathcal{F}(\mathcal{A})\}$. Now $\mathcal{U}_X^{\mathcal{A}} = \{U^{\sqsubset^L} \mid L \in \mathcal{F}(\mathcal{A})\}$ is a base for a transitive quasi-uniformity on \mathcal{C} and $\sqsubset = \sqsubset^{\mathcal{U}^{\mathcal{A}}}$ since $m \sqsubset_X n \Rightarrow \exists p \in \mathcal{A}$ such that $m \leq p \leq n$. Put $L = \{0_X, p, 1_X\}$ to have $m \sqsubset_X^{\mathcal{U}^{\mathcal{A}}} n$. On the other hand if $m \sqsubset_X^L n$ for some $L \in \mathcal{F}(\mathcal{A})$, then there is $p \in L$ such that $m \leq p \leq n \Rightarrow m \sqsubset_X^{\mathcal{A}} n$. Since $\sqsubset^{\mathcal{U}} = \bigcup \{\sqsubset^U : U \in \mathcal{U}_X\}$. \square

From Theorem 4.2.4 and Proposition 2.1.2, we can characterize the closure (interior) operators compatible with a transitive quasi-uniformity.

Corollary 3.3.9. *Let $i \in \text{INT}(\mathcal{C}, \mathcal{M})$. Then i is compatible with $\mathcal{U} \in \text{TQUNIF}(\mathcal{C}, \mathcal{M})$ if and only if for any $X \in \mathcal{C}$ and $m \in \text{sub}X$, $i_X(m) = \bigvee \{n \mid \exists p \in \mathcal{A} : n \leq p \leq m\}$ for some complete sublattice of $\text{sub}X$.*

Corollary 3.3.10. *Let $c \in \text{CL}(\mathcal{C}, \mathcal{M})$. Then c is compatible with $\mathcal{U} \in \text{TQUNIF}(\mathcal{C}, \mathcal{M})$ if and only if for any $X \in \mathcal{C}$ and $m \in \text{sub}X$, $c_X(m) = \bigwedge \{n \mid \exists p \in \mathcal{A} : m \leq p \leq n\}$ for some complete sublattice of $\text{sub}X$.*

Proposition 3.3.11. *Let $\sqsubset \in \text{TORD}(\mathcal{C}, \mathcal{M})$. If there is $\mathcal{U} \in \text{QUNIF}(\mathcal{C}, \mathcal{M})$ such that \mathcal{U} is compatible with \sqsubset , then \mathcal{A}^{\sqsubset} is a complete sublattice of $\text{sub}X$ for any $X \in \mathcal{C}$.*

Proof. Follows from the fact that for any $\mathcal{U} \in \text{QUNIF}(\mathcal{C}, \mathcal{M})$, $\sqsubset^{\mathcal{U}} \in \bigwedge -\text{TORD}(\mathcal{C}, \mathcal{M})$. \square

3.4 Examples

By Theorem 3.2.15, which establishes an equivalence between co-perfect syntopogenous structures and quasi-uniformities on a category, it will be enough to define a co-perfect syntopogenous structure when exhibiting examples of a categorical quasi-uniformity. Already our first example shows that the classical quasi-uniformity is a particular co-perfect syntopogenous structure and hence a particular categorical quasi-uniformity.

1. Let X be a (non empty) set. A filter \mathcal{D} on $X \times X$ is called a quasi-uniformity on X provided each member of \mathcal{U} is a reflexive relation and for each member $D \in \mathcal{D}$ there is $D' \in \mathcal{D}$ such that $D' \circ D' \subseteq D$. A function $f : (X, \mathcal{D}) \rightarrow (Y, \mathcal{D}')$ between quasi-uniform spaces is continuous if $\forall D' \in \mathcal{D}' \exists D \in \mathcal{D} \mid f^2(D) \subseteq D'$ where

$f^2(D) = \{(f(x), f(y)) \mid xDy\}$. Let \mathcal{C} be **Qunif**, the category of quasi-uniform spaces and quasi-uniformly continuous maps with the (surjective, quasi-uniform embeddings)-factorization structure. Every $(X, \mathcal{D}) \in \mathbf{Qunif}$ gives a co-perfect syntopogenous structure, $\mathcal{S}_{(X, \mathcal{D})} = \{\sqsubset_{(X, \mathcal{D})}^D \mid D \in \mathcal{D}\}$ on **Qunif** where $A \sqsubset_{(X, \mathcal{D})}^D B \Leftrightarrow D[A] \subseteq B$ and $A, B \subseteq X$. On the other hand, the co-perfect syntopogenous structure that describes a quasi-uniformity is the one satisfying the property that $A_i \sqsubset B_i \ (i \in I) \Rightarrow \bigcup_{i \in I} A_i \sqsubset \bigcup_{i \in I} B_i$ for any index set I . This is known as the biperfect syntopogenous structures (see [Csá63]). A morphism $f : (X, \mathcal{D}) \rightarrow (Y, \mathcal{D}')$ is \mathcal{S} -initial provided \mathcal{D} is the initial quasi-uniformity induced by f on X . Let \mathcal{B} and \mathcal{B}' be bases for \mathcal{D}' and \mathcal{D} respectively and $f_*(B) = Y \setminus f(X \setminus B)$. For $D \in \mathcal{D}$ and $A, B \subseteq X$, $A \sqsubset_{(X, \mathcal{D})} B \Leftrightarrow D[A] \subseteq B \Leftrightarrow \exists D' \in \mathcal{D} \mid (f \times f)^{-1}(D')[A] \subseteq B \Leftrightarrow f^{-1}(D'[f(A)]) \subseteq B \Leftrightarrow X \setminus B \subseteq X \setminus f^{-1}(D'[f(A)]) \Leftrightarrow X \setminus B \subseteq f^{-1}(Y \setminus D'[f(A)]) \Leftrightarrow f(X \setminus B) \subseteq Y \setminus D'[f(A)] \Leftrightarrow D'[f(A)] \subseteq Y \setminus f(X \setminus B) \Leftrightarrow f(A) \sqsubset_{(Y, \mathcal{D}')}^D f_*(B)$. $\text{Sub}^+ X$ is the class of single element sets of X , thus being \mathcal{S} -separated is equivalent to the conditions in Proposition 3.2.24.

2. In the category **TopGrp** of topological groups and continuous group homomorphisms, let $(\mathcal{E}, \mathcal{M})$ be the (surjective, injective)-factorization structure. For any $X \in \mathbf{TopGrp}$, let $\beta(e)$ be the neighbourhood filter of the identity element e .

For all $U \in \beta(e)$, put

$$U_l = \{(x, y) \in X \times X : x^{-1}y \in U\}$$

$$U_r = \{(x, y) \in X \times X : yx^{-1} \in U\}$$

One defines two co-perfect syntopogenous structures on **TopGrp**;

$$\mathcal{S}_X^l = \{\sqsubset_X^{U_l} \mid U \in \beta(e)\}$$

$$\mathcal{S}_X^r = \{\sqsubset_X^{U_r} \mid U \in \beta(e)\}$$

where $A \sqsubset_X^{U_l} B \Leftrightarrow U \cdot A \subseteq B$ and $A \sqsubset_X^{U_r} B \Leftrightarrow A \cdot U \subseteq B$. Of course, $\mathcal{S}_X^l = \mathcal{S}_X^r$ if X is abelian.

We have proved that interpolative topogenous orders are equivalent to single syntopogenous structures. Below we present a number of examples derived from idempotent closure and interior operators. In some cases we derive the categorical quasi-uniform structure determined by the order.

1. Let $\mathcal{C} = \mathbf{Grp}$ be the category of groups and group homomorphisms with (surjective, injective)-factorization system. For any $A, B \leq G$, let $\mathcal{S}_G = \{\sqsubset_G \mid G \in \mathbf{Grp}\}$ with

$A \sqsubset_G B \Leftrightarrow A \leq N \leq B$ with $N \triangleleft G$ is a co-perfect simple syntopogenous structure (meet preserving topogenous order) on **Grp**. Then $\mathcal{P}(\mathcal{A}^\square) = \{N \leq G \mid N \triangleleft G\}$ is a complete lattice so that $\mathcal{S}_X = \{\sqsubset_X^L \mid L \in \mathcal{P}(\mathcal{A}^\square)\}$ where $A \sqsubset_X^L B \Leftrightarrow \exists N \in L \mid A \leq N \leq B$ i.e each $U^{\sqsubset_X^L}(A) = \bigcap \{B \leq G \mid \exists N \in L : A \leq N \leq B\}$ is an idempotent closure operator on **Grp**. Now

$$\mathcal{B}_X^{\mathcal{S}} = \{U^{\sqsubset_X^L} \mid L \in \mathcal{P}(\mathcal{A}^\square)\}$$

is a family of idempotent closure operators on **Grp** with $\bigcap \{U^{\sqsubset_X^L}(A) \mid L \in \mathcal{P}(\mathcal{A}^\square)\} = \mathcal{N}_G(A)$ where \mathcal{N} is the normal closure.

2. In the category **AbGrp** of abelian groups and abelian group homomorphisms, let \mathcal{M} be the class of injective abelian group homomorphisms and \mathcal{E} be the class of the surjectives ones. Then for any $G \in \mathbf{AbGrp}$ and $A, B \leq G$,

$\mathcal{S}_G = \{\sqsubset_G \mid G \in \mathbf{AbGrp}\}$ where $A \sqsubset_G B \Leftrightarrow t(A) \leq B$ with $t(A) = \{a \in A : (\exists n \in \mathbb{Z}) \ n > 0 \text{ and } na = 0\}$ the torsion part of A is single co-perfect syntopogenous structures.

3. Let $\mathcal{C} = \mathbf{Top}$ the category of topological spaces and continuous maps with \mathcal{M} the class of embeddings and (of course) \mathcal{E} the class of surjective continous maps. For $A, B \subseteq X$,

- (a) $\mathcal{S}_{(X, \mathcal{T})} = \{\sqsubset_{(X, \mathcal{T})} \mid (X, \mathcal{T}) \in \mathbf{Top}\}$ with $A \sqsubset_X B \Leftrightarrow A \subseteq O \subseteq B$ for some $O \in \mathcal{T}$ is a single co-perfect syntopogenous structure (meet preserving topogenous order). Our defintions for \mathcal{S} -separated corresponds to the usual definitions in topology of Hausdorff topological space.

Now, $\mathcal{A}^\square = \{O \subseteq X \mid O \in \mathcal{T}_X\}$ is a complete lattice so that $\mathcal{S}_X = \{\sqsubset_X^L \mid L \in F(\mathcal{A}^\square)\}$ with $A \sqsubset_X^L B \Leftrightarrow \exists O \in L \mid A \subseteq O \subseteq B$ is an interpolative co-perfect syntopogenous structure by Theorem 3.3.7, $\mathcal{B}_X^{\mathcal{S}} = \{U^{\sqsubset_X^L} \mid L \in F(\mathcal{A}^\square)\}$ is a base for a transitive quasi-uniformity \mathcal{U} on **Top**. Since \mathcal{S}_X is biperfect (see [Csá63]), $y \in U^{\sqsubset_X^L}[x]$ if and only if $\{x\} \sqsubset_X^L X \setminus \{y\}$ is not true. We claim that $\mathcal{B}^{\mathcal{S}}$ is equivalent to a base for Császár-Pervin quasi-uniformity \mathcal{P} on X . If $\mathcal{B}^{\mathcal{P}}$ is a base for \mathcal{P} , then $\mathcal{B}^{\mathcal{P}} = \{\bigcap_{i=1}^n S_{O_i} \mid O_i \in \mathcal{T}\}$ where $S_{O_i} = ((X \setminus O_i) \times X) \cup (X \times O_i)$.

Let $V \in \mathcal{B}^{\mathcal{P}}$, then

$$\begin{aligned}
(x, y) \in V &\Leftrightarrow (x, y) \in S_{O_i} \text{ for each } i, 1 \leq i \leq n \text{ and } O_i \in \mathcal{T}, \\
&\Leftrightarrow (x, y) \in (X \setminus O_i) \times X \text{ or } (x, y) \in X \times O_i, \\
&\Leftrightarrow x \in X \setminus O_i \text{ or } y \in O_i, \\
&\Leftrightarrow \{x\} \sqsubset_X^L X \setminus \{y\} \text{ is not true, } L = \{O_i \mid 1 \leq i \leq n\}, \\
&\Leftrightarrow y \in U^{\square^L}[x], \\
&\Leftrightarrow (x, y) \in U^{\square^L}.
\end{aligned}$$

(b) $\mathcal{S}_{(X, \mathcal{T})} = \{\sqsubset_{(X, \mathcal{T})} \mid (X, \mathcal{T}) \in \mathbf{Top}\}$ with $A \sqsubset_X B \Leftrightarrow A \subseteq C \subseteq B$ for some closed $C \subseteq X$ is a single co-perfect syntopogenous structure (meet preserving topogenous order). A continous map $f : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ is \mathcal{S} -initial if and only if \mathcal{T}_X is the initial topology induced by f on X . Since $\mathcal{A}^{\square} = \{C \subseteq X \mid C \text{ is closed in } \mathcal{T}_X\}$ is a complete lattice, $\mathcal{S}_X = \{\sqsubset_X^L \mid L \in F(\mathcal{A}^{\square})\}$ with $A \sqsubset_X^L B \Leftrightarrow \exists C \in L \mid A \subseteq C \subseteq B$ is an interpolative biperfect syntopogenous structure (see [Csá63]) and so

$$\mathcal{B}^{\mathcal{S}} \cong \{U^{\square^L} \mid L \in F(\mathcal{A}^{\square})\}$$

where $(x, y) \notin U^{\square^L} \Leftrightarrow \{x\} \sqsubset_X^L X \setminus \{y\}$ is a base for a transitive quasi-uniformity \mathcal{U} on \mathbf{Top} . Since \mathcal{P}^{-1} is generated by $\{S_C \mid C \text{ is closed in } \mathcal{T}\}$, $\mathcal{U} = \mathcal{P}^{-1}$.

Chapter 4

Completeness

We have established in the previous chapter a satisfactory theory of quasi-uniform structures on a category using syntopogenous structures. Here we wish to study completeness of objects relative to these structures. This extends to an arbitrary category the usual theory (see e.g.[Csá63, SP66, FL82]) on the one hand, on the other hand it produces a much simpler theory of (Cauchy completeness) even when restricted to spaces. Purely categorical proofs of classical results are obtained, providing a possibility for our theory of completeness to be directly applied to categories in other branches of mathematics. We start with the convergence of filters that leads to the \mathcal{S} -Cauchy filters. For a co-perfect syntopogenous structure, various notions of Cauchy filters are defined and the relationship between them is studied. This shall naturally lead us to the study of different notions of complete objects.

4.1 The \mathcal{S} -Cauchy filters

Since for any $X \in \mathcal{C}$, $\text{sub}X$ is a complete lattice, by a filter on X we shall always understand a filter on $\text{sub}X$.

Lemma 4.1.1. *Let $X \in \mathcal{C}$ and \mathcal{F} be a filter on X . Then \mathcal{F} is an ultrafilter on X if and only if $m \wedge n > 0_X$ for all $n \in \mathcal{F}$ implies that $m \in \mathcal{F}$.*

Proof. Assume \mathcal{F} is an ultrafilter on X and $m \wedge n > 0_X$ for all $n \in \mathcal{F}$. Then $\{m \wedge n \mid n \in \mathcal{F}\}$ is a filter base for a filter \mathcal{F}' on X that is finer than \mathcal{F} and contains m . Consequently

$\mathcal{F} = \mathcal{F}'$ and thus $m \in \mathcal{F}$. Conversely if $m \wedge n > 0_X$ for all $n \in \mathcal{F} \Rightarrow m \in \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{F}'$ with \mathcal{F}' an filter on X . Then for all $p' \in \mathcal{F}'$, $p' \wedge p > 0_X$ for any $p \in \mathcal{F}$, which implies that $p' \in \mathcal{F}$. Thus \mathcal{F} is an maximal filter on X . \square

We shall also need the following lemma known as the ultrafilter lemma.

Lemma 4.1.2. *For every filter \mathcal{F} on $X \in \mathcal{C}$, there is an ultrafilter \mathcal{F}' on X such that $\mathcal{F} \subseteq \mathcal{F}'$.*

In realm of spaces, the above lemma is equivalent to the Axiom of Choice (see e.g.[Her06]). Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and \mathcal{F} a filter on $\text{sub}X$ then $f(\mathcal{F})$ is the filter defined by $m \in f(\mathcal{F}) \Leftrightarrow f^{-1}(m) \in \mathcal{F}$. If $f \in \mathcal{E}$ and \mathcal{F} is an ultrafilter then $f(\mathcal{F})$ is also an ultrafilter. If \mathcal{H} is an ultrafilter on Y and f reflects 0 then $f^{-1}(\mathcal{H})$ is also an ultrafilter on X where $f^{-1}(\mathcal{H})$ is the filter defined by $m \in f^{-1}(\mathcal{H}) \Leftrightarrow f(m) \in \mathcal{H}$.

For any $X \in \mathcal{C}$, we shall denote by $\text{sub}_o X$, the class of all $m \in \text{sub}X$ such that $m > 0_X$ and for all $m \in \text{sub}_o X$, $\nu_X^{\mathcal{S}}(m) = \{n \mid m \sqsubset_X n \text{ for some } \sqsubset_X \in \mathcal{S}\}$. We note that $\nu^{\mathcal{S}}(m)$ does not form a filter in general unless \mathcal{S} is co-perfect.

Definition 4.1.3. *Let $\mathcal{S} \in \text{CSYNT}$, $X \in \mathcal{C}$ and \mathcal{F} a filter on X . We say that \mathcal{F} converges to a subobject $m \in \text{sub}_o X$ with respect to \mathcal{S} and write $\mathcal{F} \xrightarrow{\mathcal{S}} m$ if for any $\sqsubset_X \in \mathcal{S}_X$, $m \sqsubset_X n \Rightarrow n \in \mathcal{F}$ for $n \in \text{sub}X$. \mathcal{F} is an \mathcal{S} -Cauchy if for any $\sqsubset_X \in \mathcal{S}_X$ there is $m \in \text{sub}_o X$ such that $m \sqsubset_X n \Rightarrow n \in \mathcal{F}$.*

Definition 4.1.4. *Let $X \in \mathcal{C}$, $m \in \text{sub}_o X$, \mathcal{F} a filter on X and $\mathcal{S} \in \text{CSYNT}$. Then m is a clustering of \mathcal{F} with respect to \mathcal{S} if $n \wedge p > 0_X$ for any $n \in \nu_X^{\mathcal{S}}(m)$ and $p \in \mathcal{F}$.*

The following is an easy observation.

Proposition 4.1.5. *Let $\mathcal{S} \in \text{CSYNT}$, $X \in \mathcal{C}$ and \mathcal{F} a filter on X .*

1. $\mathcal{F} \xrightarrow{\mathcal{S}} m$ and $\mathcal{F} \subseteq \mathcal{F}' \Rightarrow \mathcal{F}' \xrightarrow{\mathcal{S}} m$.
2. $\mathcal{S} \leq \mathcal{S}'$ and $\mathcal{F} \xrightarrow{\mathcal{S}'} m \Rightarrow \mathcal{F} \xrightarrow{\mathcal{S}} m$.
3. For any $m \in \text{sub}_o X$, $\nu_X^{\mathcal{S}}(m) \xrightarrow{\mathcal{S}} m$.
4. \mathcal{F} is \mathcal{S} -Cauchy and $\mathcal{F} \subseteq \mathcal{F}'$ then so is \mathcal{F}' .
5. Every \mathcal{S} -convergent filter on X is \mathcal{S} -Cauchy.

Definition 4.1.6. *Let $\mathcal{S} \in \text{SYNT}$ and $X \in \mathcal{C}$. A filter \mathcal{F} on X is said to be \mathcal{S} -Cauchy if it is Cauchy with respect to any $\mathcal{S}^* \in \text{CSYNT}$ that is coarser than \mathcal{S} . It converges with*

respect to \mathcal{S} if it converges with respect to any $\mathcal{S}^* \in \text{CSYNT}$ such that $\mathcal{S}^* \leq \mathcal{S}$.

One easily sees that Proposition 4.1.5 holds for any \mathcal{S}' , $\mathcal{S} \in \text{SYNT}$.

For the rest of this chapter, it is assumed that the class \mathcal{E} is stable under pullbacks along \mathcal{M} -morphisms and that any $f \in \mathcal{C}$ reflects 0.

Proposition 4.1.7. *Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism, \mathcal{F} a filter on X and $\mathcal{S} \in \text{CSYNT}$.*

- (a) *If $\mathcal{F} \xrightarrow{\mathcal{S}} m$ then $f(\mathcal{F}) \xrightarrow{\mathcal{S}} f(m)$. The converse implication holds if f is \mathcal{S} -initial.*
- (b) *If \mathcal{F} is \mathcal{S} -Cauchy on X then so is $f(\mathcal{F})$ on Y . The converse holds if f is \mathcal{S} -initial and belongs to \mathcal{E} .*
- (c) *If m is a clustering of \mathcal{F} then $f(m)$ is also a clustering of $f(\mathcal{F})$.*

Proof. (a) Assume that $\mathcal{F} \xrightarrow{\mathcal{S}} m$ with $m \in \text{sub}_o X$. Then $f(m) \in \text{sub}_o Y$. Let $f(m) \sqsubset_Y p$ for any $\sqsubset_Y \in \mathcal{S}_Y$ and $p \in \text{sub} Y$. Then by (S4) there is $\sqsubset_X \in \mathcal{S}_X$ such that $m \sqsubset_X f^{-1}(p)$ and $f^{-1}(p) \in \mathcal{F} \Leftrightarrow p \in f(\mathcal{F})$. Conversely if $f(\mathcal{F}) \xrightarrow{\mathcal{S}} f(m)$ and $f(m) \in \text{sub}_o Y$. Then $m \in \text{sub}_o X$. Let $m \sqsubset_X n$ for any $\sqsubset_X \in \mathcal{S}_X$. Then there is $\sqsubset_Y \in \mathcal{S}_Y$ such that $f(m) \sqsubset_Y f_*(n)$ and $f_*(n) \in f(\mathcal{F})$. Thus $f^{-1}(f_*(n)) \in \mathcal{F}$ which implies that $n \in \mathcal{F}$.

(b) Let \mathcal{F} be \mathcal{S} -Cauchy and $\sqsubset_Y \in \mathcal{S}_Y$. Then by (S4) there is $\sqsubset_X \in \mathcal{S}_X$ such that $f(p) \sqsubset_Y q \Rightarrow p \sqsubset_X f^{-1}(q)$ and there is $m \in \text{sub}_o X$ such that $m \sqsubset_X n \Rightarrow n \in \mathcal{F}$. Thus $f(m) \in \text{sub}_o Y$ and $f(m) \sqsubset_Y l \Rightarrow m \sqsubset_X f^{-1}(l) \Rightarrow f^{-1}(l) \in \mathcal{F} \Leftrightarrow l \in f(\mathcal{F})$. Conversely if $f(\mathcal{F})$ is \mathcal{S} -Cauchy and $\sqsubset_X \in \mathcal{S}_X$, then there is $\sqsubset_Y \in \mathcal{S}_Y$ such that $m \sqsubset_X n \Rightarrow f(m) \sqsubset_Y f_*(n)$ and there is $p \in \text{sub}_o Y$ such that $p \sqsubset_Y q \Rightarrow q \in f(\mathcal{F})$. Now $f^{-1}(p) \in \text{sub}_o X$ since $f \in \mathcal{E}$ and $f^{-1}(p) \sqsubset_X n \Rightarrow p = f(f^{-1}(p)) \sqsubset_Y f_*(n) \Rightarrow f_*(n) \in f(\mathcal{F}) \Leftrightarrow f^{-1}(f_*(n)) \in \mathcal{F} \Rightarrow n \in \mathcal{F}$.

(c) Let m be a clustering of \mathcal{F} . If $n \in \nu_Y^{\mathcal{S}}(f(m))$ and $p \in f(\mathcal{F})$, then $f(m) \sqsubset_Y n$ for some $\sqsubset_Y \in \mathcal{S}_Y$. By assumption, $f^{-1}(n) \wedge f^{-1}(p) > 0_X \Rightarrow 0_Y < f(f^{-1}(n) \wedge f^{-1}(p)) \leq n \wedge p$. Thus $n \wedge p > 0_X$ and $f(m)$ is a clustering of \mathcal{F} .

□

Corollary 4.1.8. *Let $f : X \rightarrow Y$ be an \mathcal{E} -morphism and \mathcal{F} a filter on Y and $\mathcal{S} \in \text{CSYNT}$. Assume that f is \mathcal{S} -initial. Then*

1. $\mathcal{F} \xrightarrow{\mathcal{S}} m \Leftrightarrow f^{-1}(\mathcal{F}) \xrightarrow{\mathcal{S}} f^{-1}(m)$.

2. \mathcal{F} is \mathcal{S} -Cauchy if and only if $f^{-1}(\mathcal{F})$ is \mathcal{S} -Cauchy.

Proposition 4.1.9. Let $X = \prod_{i \in I} X_i$ be a product in \mathcal{C} and \mathcal{F} a filter on X . Assume $(p_i : X \rightarrow X_i)_{i \in I}$ is an \mathcal{S} -initial source.

(1) $\mathcal{F} \xrightarrow{\mathcal{S}} m$ if and only if $p_i(\mathcal{F}) \xrightarrow{\mathcal{S}} p_i(m)$ for each i .

(2) If for each $i \in I$ the projections belong to \mathcal{E} , then \mathcal{F} is \mathcal{S} -Cauchy if and only if $p_i(\mathcal{F})$ is \mathcal{S} -Cauchy for each i .

Proof. (1) If $\mathcal{F} \xrightarrow{\mathcal{S}} m$ then $\text{pr}_i(\mathcal{F}) \xrightarrow{\mathcal{S}} \text{pr}_i(m)$ for each i by Proposition 4.1.7(a). Let $p_i(\mathcal{F}) \xrightarrow{\mathcal{S}} p_i(m)$ with $p_i(m) \in \text{sub}_o X_i$. Then $m \in \text{sub}_o X$ for each $i \in I$. Now, if $m \sqsubset_X l$ for any $\sqsubset_X \in \mathcal{S}_X$. Then there is $i \in I$ such that $p_i(m) \sqsubset_{X_i} (p_i)_*(l)$ and $(p_i)_*(l) \in p_i(\mathcal{F})$. So $(p_i)^{-1}((p_i)_*(l)) \in \mathcal{F} \Rightarrow l \in \mathcal{F}$.

(2) If \mathcal{F} is \mathcal{S} -Cauchy on X , then by Proposition 4.1.7(b), $p_i(\mathcal{F})$ is \mathcal{S} -Cauchy on X_i for each $i \in I$. Conversely if $p_i(\mathcal{F})$ is \mathcal{S} -Cauchy on X_i for each $i \in I$ and $\sqsubset_X \in \mathcal{S}_X$, then there is $i \in I$ and $\sqsubset_{X_i} \in \mathcal{S}_{X_i}$ such that $m \sqsubset_X n \Rightarrow p_i(m) \sqsubset_{X_i} (p_i)_*(n)$ and there is $l \in \text{sub}_o X_i$ such that $l \sqsubset_{X_i} p \Rightarrow p \in p_i(\mathcal{F})$. Since $p_i \in \mathcal{E}$ for each $i \in I$, $p_i^{-1}(l) \in \text{sub}_o X$ and $p_i(l) \sqsubset_X n \Rightarrow l = p_i(p_i^{-1}(l)) \sqsubset_{X_i} (p_i)_*(n) \Rightarrow (p_i)_*(n) \in p_i(\mathcal{F}) \Leftrightarrow p_i^{-1}((p_i)_*(n)) \in \mathcal{F} \Rightarrow n \in \mathcal{F}$. \square

From Proposition 4.1.7 and Definition 4.1.6 we have the following.

Proposition 4.1.10. Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism, \mathcal{F} a filter on X and $\mathcal{S} \in \text{SYNT}$.

(1) If $\mathcal{F} \xrightarrow{\mathcal{S}} m$ then $f(\mathcal{F}) \xrightarrow{\mathcal{S}} f(m)$. The converse implication holds if f is \mathcal{S} -initial.

(2) \mathcal{F} is \mathcal{S} -Cauchy on X then so is $f(\mathcal{F})$ on Y . The converse holds if f is \mathcal{S} -initial and belongs to \mathcal{E} .

Proof. (1) If $\mathcal{F} \xrightarrow{\mathcal{S}} m$ then by Definition 4.1.6, $\mathcal{F} \xrightarrow{\mathcal{S}^*} m$ for any $\mathcal{S}^* \in \text{CSYNT}$ such that $\mathcal{S}^* \leq \mathcal{S}$. By Proposition 4.1.7, $f(\mathcal{F}) \xrightarrow{\mathcal{S}^*} f(m)$. Thus $f(\mathcal{F}) \xrightarrow{\mathcal{S}} f(m)$. Conversely if $f(\mathcal{F}) \xrightarrow{\mathcal{S}} f(m)$ then $f(\mathcal{F}) \xrightarrow{\mathcal{S}^*} f(m)$ for any $\mathcal{S}^* \in \text{CSYNT}$ such that $\mathcal{S}^* \leq \mathcal{S}$. Proposition 4.1.7 implies that $\mathcal{F} \xrightarrow{\mathcal{S}^*} m$. Hence $\mathcal{F} \xrightarrow{\mathcal{S}} m$.

(2) Let \mathcal{F} be \mathcal{S} -Cauchy on X , then \mathcal{F} is \mathcal{S}^* -Cauchy for any $\mathcal{S} \in \text{CSYNT}$ such that $\mathcal{S}^* \leq \mathcal{S}$. By Proposition 4.1.7(b), $f(\mathcal{F})$ is \mathcal{S} -Cauchy and so $f(\mathcal{F})$ is \mathcal{S} -Cauchy. On the other hand if $f(\mathcal{F})$ is \mathcal{S} -Cauchy, then $f(\mathcal{F})$ is \mathcal{S}^* -Cauchy for any $\mathcal{S}^* \in \text{CSYNT}$ such that $\mathcal{S}^* \leq \mathcal{S}$. Proposition 4.1.7(b) implies that \mathcal{F} is \mathcal{S}^* -Cauchy and thus \mathcal{F} is \mathcal{S} -Cauchy. \square

Analogous to Corollary 4.1.8 we have the Corollary.

Corollary 4.1.11. *Let $f : X \rightarrow Y$ be an \mathcal{E} -morphism, \mathcal{F} a filter on Y and $\mathcal{S} \in \text{SYNT}$. Assume that f is \mathcal{S} -initial. Then*

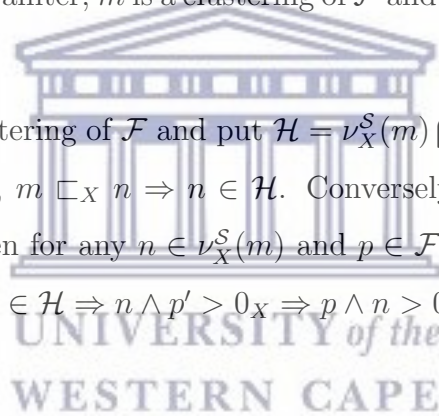
1. $\mathcal{F} \xrightarrow{\mathcal{S}} m \Leftrightarrow f^{-1}(\mathcal{F}) \xrightarrow{\mathcal{S}} f^{-1}(m)$.
2. \mathcal{F} is \mathcal{S} -Cauchy if and only if $f^{-1}(\mathcal{F})$ is \mathcal{S} -Cauchy.

Proposition 4.1.12. *Let \mathcal{F} be a filter on X and $m \in \text{sub}_o X$.*

- (1) *If $\mathcal{F} \xrightarrow{\mathcal{S}} m$ then m is a clustering of \mathcal{F} . The converse holds if \mathcal{F} is an ultrafilter.*
- (2) *m is a clustering of \mathcal{F} if and only if there is a filter \mathcal{H} on X such that $\mathcal{F} \leq \mathcal{H}$ and $\mathcal{H} \xrightarrow{\mathcal{S}} m$.*

Proof. (1) If $\mathcal{F} \xrightarrow{\mathcal{S}} m$, $n \in \nu_X^{\mathcal{S}}(m)$ and $p \in \mathcal{F}$, then $n \wedge p > 0_X$ since \mathcal{F} is a filter. Conversely if \mathcal{F} is an ultrafilter, m is a clustering of \mathcal{F} and for any $\sqsubset_X \in \mathcal{S}_X$, $m \sqsubset_X n$, then $n \in \mathcal{F}$.

- (2) Assume that m is a clustering of \mathcal{F} and put $\mathcal{H} = \nu_X^{\mathcal{S}}(m) \cap \mathcal{F}$. We have that $\mathcal{F} \leq \mathcal{H}$ and if for any $\sqsubset_X \in \mathcal{S}_X$, $m \sqsubset_X n \Rightarrow n \in \mathcal{H}$. Conversely if \mathcal{H} is a filter such that $\mathcal{F} \leq \mathcal{H}$ and $\mathcal{H} \xrightarrow{\mathcal{S}} m$ then for any $n \in \nu_X^{\mathcal{S}}(m)$ and $p \in \mathcal{F}$, there is $p' \in \mathcal{H}$ such that $p' \leq p$. Since $\mathcal{H} \xrightarrow{\mathcal{S}} m$, $n \in \mathcal{H} \Rightarrow n \wedge p' > 0_X \Rightarrow p \wedge n > 0_X$.



□

Theorem 4.1.13. *Let $\mathcal{S} \in \text{CSYNT}$ and \mathcal{F} be a filter on X . Then X is \mathcal{S} -separated if and only if $\mathcal{F} \xrightarrow{\mathcal{S}} m$ and $\mathcal{F} \xrightarrow{\mathcal{S}} n$ implies that $m = n$ for all $m, n \in \text{sub}^+ X$ and $X \in \mathcal{C}$.*

Proof. Assume that X is \mathcal{S} -separated, $\mathcal{F} \xrightarrow{\mathcal{S}} m$ and $\mathcal{F} \xrightarrow{\mathcal{S}} n$ with $m, n \in \text{sub}^+ X$. If $m \sqsubset_X p$ and $n \sqsubset_X q$ for some $\sqsubset_X \in \mathcal{S}_X$, then $p \wedge q > 0_X$. Thus $m \wedge n > 0_X$ that is $m = n$. Conversely if X is not \mathcal{S} - T_2 , then there are $m, n \in \text{sub} X$ with $m \wedge n = 0_X$ such that $p \wedge q > 0_X$ for all $m \sqsubset_X p$, $n \sqsubset_X q$ and $\sqsubset_X \in \mathcal{S}_X$. Now $\mathcal{F} = \{p \mid m \sqsubset_X p\} \cup \{q \mid n \sqsubset_X q\}$ for all $\sqsubset_X \in \mathcal{S}_X$ so that $\mathcal{F} \xrightarrow{\mathcal{S}} m$ and $\mathcal{F} \xrightarrow{\mathcal{S}} n$ simultaneously. □

Definition 4.1.14. *Let \mathcal{U} be a quasi-uniformity on \mathcal{C} and \mathcal{F} filter on X . We shall say that \mathcal{F} converges to m ($m \in \text{sub}_o X$) with respect to \mathcal{U} and write $\mathcal{F} \xrightarrow{\mathcal{U}} m$ if for any $U \in \mathcal{U}_X$, $U(m) \in \mathcal{F}$. \mathcal{F} is \mathcal{U} -cauchy if for each $U \in \mathcal{U}_X$ there is $m \in \text{sub}_o X$ such that $U(m) \in \mathcal{F}$.*

Let \mathcal{U} and \mathcal{V} be quasi-uniform structures on \mathcal{C} . For any $X \in \mathcal{C}$, if $\mathcal{U}_X \leq \mathcal{V}_X$ then every \mathcal{V} -cauchy filter on X is \mathcal{U} -Cauchy. In particular, every \mathcal{U} -Cauchy filter is \mathcal{U}^* -Cauchy.

Proposition 4.1.15. *Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism. If \mathcal{F} is \mathcal{U} -Cauchy filter then so is $f(\mathcal{F})$. The converse holds if f is \mathcal{U} -initial and belongs to \mathcal{E} .*

Proof. Let \mathcal{F} be \mathcal{U} -cauchy and $U \in \mathcal{U}_Y$. Then by (U5) there is $U' \in \mathcal{U}_X$ such that $f(U'(n)) \leq U(f(n))$ for all $n \in \text{sub}X$ and there is $m \in \text{sub}_oX$ with $U'(m) \in \mathcal{F}$. This implies that $f(U'(m)) \in f(\mathcal{F})$. Thus $f(U'(m)) \leq U(f(m)) \Rightarrow U(f(m)) \in f(\mathcal{F})$ and $f(m) \in \text{sub}_oY$. Assume that $f(\mathcal{F})$ is \mathcal{U} -cauchy and $U \in \mathcal{U}_X$. Since f is \mathcal{U} -initial, there is $U' \in \mathcal{U}_Y$ such that $f^{-1}(U'(f(m))) \leq U(m)$ and there is $n \in \text{sub}_oY$ such that $U'(n) \in f(\mathcal{F}) \Leftrightarrow f^{-1}(U'(n)) \in \mathcal{F}$. Since $f \in \mathcal{E}$, $f^{-1}(n) \in \text{sub}_oY$ and $f^{-1}(U'(n)) = f^{-1}(U'(f(f^{-1}(n)))) \leq U(f^{-1}(n)) \Rightarrow U(f^{-1}(n)) \in \mathcal{F}$. \square

Proposition 4.1.16. *Let $\mathcal{S} \in \text{CSYNT}$ and \mathcal{F} a filter on X . Then $\mathcal{F} \xrightarrow{\mathcal{S}} m \Leftrightarrow \mathcal{F} \xrightarrow{\mathcal{U}^{\mathcal{S}}} m$*

Proof. Let $m \in \text{sub}_oX$ and $\mathcal{F} \xrightarrow{\mathcal{S}} m$. If $U \in \mathcal{B}_X^{\mathcal{S}}$ then there is $\sqsubset_X \in \mathcal{S}_X$ such that $\sqsubset_X = \sqsubset_X^U$ and $m \sqsubset_X n \Rightarrow n \in \mathcal{F}$. But $m \sqsubset_X \bigwedge \{n \mid m \sqsubset_X n\} = U(m)$. Thus $U(m) \in \mathcal{F}$. Conversely if $\mathcal{F} \xrightarrow{\mathcal{U}^{\mathcal{S}}} m$ and $m \sqsubset_X n$ then $U(m) \leq n$ for some $U \in \mathcal{B}_X^{\mathcal{S}}$. Hence $n \in \mathcal{F}$. \square

Theorem 4.1.17. *Let $\mathcal{S} \in \text{CSYNT}$ and \mathcal{F} be a filter on X . Then \mathcal{F} is \mathcal{S} -Cauchy if and only if it is $\mathcal{U}^{\mathcal{S}}$ -Cauchy.*

Proof. Let \mathcal{F} be $\mathcal{U}^{\mathcal{S}}$ -Cauchy and $\sqsubset_X \in \mathcal{S}_X$. Since $\mathcal{S} \in \text{CSYNT}$, by Theorem 3.2.15 there is $U \in \mathcal{B}_X^{\mathcal{S}}$ that determines \sqsubset_X and there is $m \in \text{sub}_oX$ such that $U(m) \in \mathcal{F}$. Now, let $m \sqsubset_X n \Leftrightarrow U(m) \leq n \Rightarrow n \in \mathcal{F}$. Conversely if \mathcal{F} is \mathcal{S} -cauchy and $U \in \mathcal{B}_X^{\mathcal{S}}$ then there is $\sqsubset_X \in \mathcal{S}_X$ such that $\sqsubset_X^U = \sqsubset_X$ and there is $m \in \text{sub}_oX$ such that $m \sqsubset_X n \Rightarrow n \in \mathcal{F}$. Now $m \sqsubset_X U(m)$ since $\mathcal{S} \in \text{CSYNT}$ and $U(m) \in \mathcal{F}$. \square

Left (resp. right) Cauchy filters in quasi-uniform spaces have been investigated in [Rom96] and these lead to left (resp. right) complete quasi-uniform spaces which are closely related to the Smyth complete quasi-uniform spaces ([Sün93]). It seems that our co-perfect syntopogenous structure provides a simple and natural way of stating these notions in categorical language and link them to those obtained above. Let us also note that a categorical approach to convergence has already been studied in the literature (see e.g [Šla96]).

Definition 4.1.18. *Let $\mathcal{S} \in \text{CSYNT}$. A filter \mathcal{F} on an object $X \in \mathcal{C}$ will be called:*

- (1) \mathcal{S} -round if for every $n \in \mathcal{F}$, there is $m \in \mathcal{F}$ and $\sqsubset_X \in \mathcal{S}_X$ such that $m \sqsubset_X n$.
- (2) weakly \mathcal{S} -Cauchy (or simply $w\mathcal{S}$ -Cauchy) if for every $\sqsubset_X \in \mathcal{S}_X$ and for every $n \in \mathcal{F}$, there is $m \in \text{sub}_o X$ with $m \leq n$ such that $m \sqsubset_X p \Rightarrow p \in \mathcal{F}$ for any $p \in \text{sub}X$.
- (3) left \mathcal{S} -Cauchy if for all $\sqsubset_X \in \mathcal{S}_X$ there is $n \in \mathcal{F}$ such that $m \sqsubset_X p \Rightarrow p \in \mathcal{F}$ for any $m \leq n$ and $m \in \text{sub}_o X$.
- (4) Corson \mathcal{S} -filter or simply $c\mathcal{S}$ -filter if for all $\sqsubset_X \in \mathcal{S}_X$, there is $m \in \text{sub}_o X$ such that $m \sqsubset_X n \Rightarrow n \wedge p > 0_X$ for any $p \in \mathcal{F}$.

For any $X \in \mathcal{C}$ and $m \in \text{sub}_o X$, $\nu_X^{\mathcal{S}}(m) = \{n \mid \text{for some } \sqsubset_X \in \mathcal{S}_X, m \sqsubset_X n\}$ is \mathcal{S} -round and an \mathcal{S} -Cauchy filter on X . An \mathcal{S} -round and $w\mathcal{S}$ -Cauchy filter shall simply be called \mathcal{S} -round Cauchy filter.

Proposition 4.1.19. *A filter \mathcal{F} on $X \in \mathcal{C}$ is:*

- (1) is \mathcal{S} -round if and only if for every $n \in \mathcal{F}$, there is $m \in \mathcal{F}$ and there is $U \in \mathcal{B}_X^{\mathcal{S}}$ such that $U(m) \leq n$.
- (2) is $w\mathcal{S}$ -Cauchy if and only if for every $U \in \mathcal{B}_X^{\mathcal{S}}$ and for every $n \in \mathcal{F}$, there is $m \in \text{sub}_o X$ with $m \leq n$ such that $U(m) \in \mathcal{F}$.
- (3) is left \mathcal{S} -Cauchy if and only if every $U \in \mathcal{B}_X^{\mathcal{S}}$ there is $n \in \mathcal{F}$ such that $U(m) \in \mathcal{F}$ for any $m \leq n$ and $m \in \text{sub}_o X$.
- (4) is $c\mathcal{S}$ -filter if and only if for every $U \in \mathcal{B}_X^{\mathcal{S}}$, there is $m \in \text{sub}_o X$ such that $U(m) \wedge p > 0_X$ for any $p \in \mathcal{F}$.

Proof. (1) Assume \mathcal{F} is \mathcal{S} -round and $m \in \mathcal{F}$. Then there are $n \in \mathcal{F}$ and $\sqsubset_X \in \mathcal{S}_X$ such $n \sqsubset_X m$. By Theorem 3.2.15 there is $U \in \mathcal{U}_X$ with $\sqsubset_X^U = \sqsubset_X$. Thus $U(n) \in \mathcal{F}$. Conversely if for any $n \in \mathcal{F}$, there $m \in \mathcal{F}$ and $U \in \mathcal{B}_X^{\mathcal{S}}$ such that $U(n) \leq m$, then Theorem 3.2.15 implies the existence of $\sqsubset_X \in \mathcal{S}_X$ that determines U . Thus $n \sqsubset_X m$.

A similar argument holds for (2) and (3).

(4) Assume \mathcal{F} is a $c\mathcal{S}$ -filter and $U \in \mathcal{B}_X^{\mathcal{S}}$. Then there is $\sqsubset_X \in \mathcal{S}_X$ and there is $m \in \text{sub}_o X$ such that $m \sqsubset_X n \Rightarrow n \wedge p > 0_X$ for all $p \in \mathcal{F}$. Since $m \sqsubset_X U(m)$, we get that $U(m) \wedge p > 0_X$. On the other hand if for any $\sqsubset_X \in \mathcal{S}_X$, then there is $U \in \mathcal{B}_X^{\mathcal{S}}$ such that $U^{\sqsubset_X} = U$ and there is $m \in \text{sub}X$ such that $U(m) \wedge p > 0_X$ for any $p \in \mathcal{F}$. Now $m \sqsubset_X n \Leftrightarrow U(m) \leq n \Rightarrow n \wedge p > 0_X$.

□

We shall sometimes say \mathcal{U} -round (resp. left \mathcal{U} -Cauchy) filters to mean the equivalent expressions of Definition 4.1.18 provided by Proposition 4.1.19.

Definition 4.1.20. A filter \mathcal{F} on an object $X \in \mathcal{C}$ is said to be \mathcal{U} -stable if for any $U \in \mathcal{U}_X$, $\bigwedge\{U(n) \mid n \in \mathcal{F}\} \in \mathcal{F}$. \mathcal{F} is right \mathcal{U} -Cauchy if for any $U \in \mathcal{U}_X$ there is $n \in \mathcal{F}$ such that $U^*(m) \in \mathcal{F}$ for any $m \leq n$ and $m \in \text{sub}_o X$.

Proposition 4.1.21. Let \mathcal{F} be a filter on $X \in \mathcal{C}$. Then \mathcal{F} is \mathcal{U} -stable if and only if for any $U \in \mathcal{U}_X$, there is $m \in \mathcal{F}$ such that $m \leq U(n)$ for any $n \in \mathcal{F}$.

Proof. Assume that \mathcal{F} is \mathcal{U} -stable and $U \in \mathcal{U}_X$. Then $\bigwedge\{U(n) : n \in \mathcal{F}\} \in \mathcal{F}$. One puts $m = \bigwedge\{U(n) : n \in \mathcal{F}\} \in \mathcal{F}$ to have that, $m \leq U(n)$ for any $n \in \mathcal{F}$. Conversely if it holds that for any $U \in \mathcal{U}_X$ there is $m \in \mathcal{F}$ such that $m \leq U(n)$ for any $n \in \mathcal{F}$, then $m \leq \bigwedge\{U(n) : n \in \mathcal{F}\} \Rightarrow \bigwedge\{U(n) : n \in \mathcal{F}\} \in \mathcal{F}$.

□

Proposition 4.1.22. Consider the following for a filter \mathcal{F} on $X \in \mathcal{C}$.

- | | |
|--|---|
| (1) \mathcal{F} is \mathcal{U}^* -stable. | (5) \mathcal{F} is \mathcal{S} -Cauchy. |
| (2) \mathcal{F} is $c\mathcal{S}$ -Cauchy. | (6) \mathcal{F} is \mathcal{U} -stable. |
| (3) \mathcal{F} is left \mathcal{S} -Cauchy. | (7) \mathcal{F} is right \mathcal{U} -Cauchy. |
| (4) \mathcal{F} is $w\mathcal{S}$ -Cauchy. | |

Then (1) \Rightarrow (2), (3) \Rightarrow (4), (3) \Rightarrow (6), (3) \Rightarrow (1) and (5) \Rightarrow (2)

If \mathcal{F} is an ultrafilter then (1) \Rightarrow (3), (6) \Rightarrow (7), (5) \Rightarrow (3) and (2) \Rightarrow (5).

Proof. (1) \Rightarrow (2) If \mathcal{F} is \mathcal{U}^* -stable and $U \in \mathcal{U}_X$, then there is $m \in \mathcal{F}$ such that $m \leq U^*(n)$ for any $n \in \mathcal{F}$. This implies that $U(m) \wedge n > 0_X$. By Proposition 4.1.19, \mathcal{F} is $c\mathcal{S}$ -Cauchy.

(3) \Rightarrow (4) If \mathcal{F} is left \mathcal{S} -Cauchy, $\square_X \in \mathcal{S}_X$ and $n \in \mathcal{F}$ then there is $l \in \mathcal{F}$ such that $m' \square_X p \Rightarrow p \in \mathcal{F}$ for any $m' \leq l$ and $m' \in \text{sub}_o X$. Since $n, l \in \mathcal{F}$, $l \wedge n > 0_X$ and putting $m = l \wedge n$, we get that $m \square_X p \Rightarrow p \in \mathcal{F}$, that is \mathcal{F} is $w\mathcal{S}$ -Cauchy.

(3) \Rightarrow (6) Assume that \mathcal{F} is left \mathcal{S} -Cauchy and $\square_X \in \mathcal{S}_X$. Then there is $n \in \mathcal{F}$ such that $m \square_X p \Rightarrow p \in \mathcal{F}$ for any $m \leq n$ and $m \in \text{sub}_o X$. Thus \mathcal{F} is \mathcal{S} -Cauchy.

(3) \Rightarrow (1) Let \mathcal{F} be left \mathcal{U} -Cauchy and $U \in \mathcal{U}_X$. Then there is $m \in \mathcal{F}$ such that $U(m) \in \mathcal{F}$ for all $n \leq m$ with $n \in \text{sub}_o X$. If $p \in \mathcal{F}$, then $m \wedge p \leq U(m) \Rightarrow m \leq U^*(m \wedge p) \leq U^*(p)$. Thus \mathcal{F} is \mathcal{U}^* -stable.

(5) \Rightarrow (2) If \mathcal{F} is \mathcal{S} -Cauchy and $\sqsubset_X \in \mathcal{S}$, then there is $m \in \text{sub}_o X$ such that $m \sqsubset_X n \Rightarrow n \in \mathcal{F}$, which implies that $n \wedge p > o$ for all $p \in \mathcal{F}$.

Let \mathcal{F} be an ultrafilter.

(1) \Rightarrow (3) If \mathcal{F} is \mathcal{U}^* -stable and $U \in \mathcal{U}_X$, then there is $m \in \mathcal{F}$ such that $m \leq U^*(p)$ for any $p \in \mathcal{F}$. Let $n \leq m$ and $n \in \text{sub}_o X$. Then $n \leq U^*(p) \Leftrightarrow n \wedge U^*(p) > 0_X \Rightarrow n \in \mathcal{F}$ since \mathcal{F} is an ultrafilter and thus $U(n) \in \mathcal{F}$.

(6) \Rightarrow (7) If \mathcal{F} is \mathcal{U} -stable and $U \in \mathcal{U}_X$ then there is $m \in \mathcal{F}$ such that $m \leq U(p)$ for all $p \in \mathcal{F}$. If $n \leq m$ and $n \in \text{sub}_o X$, then $n \leq U(p) \Rightarrow n \wedge U(p) > 0_X \Rightarrow n \in \mathcal{F}$ since \mathcal{F} is an ultrafilter and thus $U(n) \in \mathcal{F}$.

(5) \Rightarrow (3) If \mathcal{F} is \mathcal{S} -Cauchy and $U \in \mathcal{U}_X$. Then there is $m \in \text{sub}_o X$ such that $U(m) \in \mathcal{F}$ by Proposition 4.1.19. Now, $m \wedge U(m) > 0_X \Rightarrow m \in \mathcal{F}$ since \mathcal{F} is an ultrafilter. Let $n \leq m$ and $n \in \text{sub}_o X$, then $U(n) \leq U(m)$ and $U(n) \wedge U(m) > 0_X$. Thus $U(n) \in \mathcal{F}$.

(2) \Rightarrow (5) If \mathcal{F} is $c\mathcal{S}$ -Cauchy and $\sqsubset_X \in \mathcal{S}_X$, then there is $m \in \text{sub}_o X$ such that $m \sqsubset_X n \Rightarrow n \wedge p > 0_X$ for any $p \in \mathcal{F}$. Since \mathcal{F} is an ultrafilter, $n \in \mathcal{F}$. \square

It follows from Propositions 4.1.19 and 4.1.22 that every \mathcal{U} -Cauchy filter is $c\mathcal{S}$ -Cauchy with the converse holding if \mathcal{F} is an ultrafilter.

Proposition 4.1.23. *Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and \mathcal{F} a filter on X . Then if \mathcal{F} is \mathcal{U} -stable on X then so is $f(\mathcal{F})$. The converse holds if f is \mathcal{U} -initial.*

Proof. Let \mathcal{F} be \mathcal{U} -stable on X and $U \in \mathcal{U}_Y$. Then there is $V \in \mathcal{U}_X$ such that $f(V(p)) \leq U(f(p))$ and there is $m \in \mathcal{F}$ such that $m \leq V(n)$ for all $n \in \mathcal{F}$. This implies that $f(m) \leq f(V(m)) \leq U(f(n))$, that is $f(\mathcal{F})$ is \mathcal{U} -stable. Conversely if $f(\mathcal{F})$ is \mathcal{U} -stable and $U \in \mathcal{U}_X$, there is $U' \in \mathcal{U}_Y$ such $f^{-1}(U'(f(p))) \leq U(p)$ and there is $m \in f(\mathcal{F})$ such that $m \leq U'(n)$ for all $n \in f(\mathcal{F})$. In particular $m \leq U'(f(p))$ for all $p \in \mathcal{F}$. Thus $f^{-1}(m) \leq f^{-1}(U'(f(p))) \leq U(p)$, that is \mathcal{F} is \mathcal{U} -stable. \square

From the above proposition, we can prove the following.

Proposition 4.1.24. *Let $X = \prod_{i \in I} X_i$ be a product in \mathcal{C} and \mathcal{F} a filter on X . Assume that $(p_i : X \rightarrow X_i)_{i \in I}$ is a \mathcal{U} -initial source. Then \mathcal{F} is \mathcal{U} -stable if and only if $p_i(\mathcal{F})$ is \mathcal{U} -stable for each i .*

4.2 Variant of completeness

Our different notions of Cauchy filters developed in the previous section allows us to define distinct notions of complete objects.

Definition 4.2.1. *Let $\mathcal{S} \in \text{SYNT}$. A \mathcal{C} -object X is said to be strongly \mathcal{S} -complete if every \mathcal{S} -cauchy filter on X converges with respect to \mathcal{S} .*

It can be seen from Definition 4.1.6 that a \mathcal{C} object X is strongly \mathcal{S} -complete if and only if it is strongly \mathcal{S}^* -complete with $\mathcal{S}^* \in \text{CSYNT}$ such that $\mathcal{S}^* \leq \mathcal{S}$. From Proposition 4.1.16 and Theorem 4.1.17 we have that,

Proposition 4.2.2. *For $\mathcal{S} \in \text{CSYNT}$, an object $X \in \mathcal{C}$ is strongly $\mathcal{U}^{\mathcal{S}}$ -complete if and only if it strongly \mathcal{S} -complete.*

Proof. Assume that X is strongly $\mathcal{U}^{\mathcal{S}}$ -complete and let \mathcal{F} be an \mathcal{S} -Cauchy filter. Then by Theorem 4.1.17, \mathcal{F} is $\mathcal{U}^{\mathcal{S}}$ -Cauchy and by assumption \mathcal{F} is $\mathcal{U}^{\mathcal{S}}$ -convergent. Proposition 4.1.16 implies that \mathcal{F} is \mathcal{S} -convergent. Conversely if X is \mathcal{S} -complete and \mathcal{F} is a $\mathcal{U}^{\mathcal{S}}$ Cauchy filter on X , then \mathcal{F} is \mathcal{S} -Cauchy and \mathcal{S} -convergent. By Propoposition 4.1.16, \mathcal{F} is $\mathcal{U}^{\mathcal{S}}$ -convergent. \square

Proposition 4.2.3. *Let $f : X \rightarrow Y$ be a \mathcal{E} -morphism that is \mathcal{S} -initial. Then X is strongly \mathcal{S} -complete if and only if Y is strongly \mathcal{S} -complete.*

Proof. If Y is strongly \mathcal{S} -complete and \mathcal{F} an \mathcal{S} -Cauchy filter on X , then by Proposition 4.1.7(b), $f(\mathcal{F})$ is \mathcal{S} -Cauchy and by assumption, $f(\mathcal{F}) \xrightarrow{\mathcal{S}} n$ for some $n \in \text{sub}_o Y$. This implies that $\mathcal{F} \xrightarrow{\mathcal{S}} f^{-1}(n)$ since $f \in \mathcal{E}$. By Proposition 4.1.7(a) $\mathcal{F} \xrightarrow{\mathcal{S}} f^{-1}(n)$ and $f^{-1}(n) \in \text{sub}_o X$. Thus X is strongly \mathcal{S} -complete.

On the other hand if X is strongly \mathcal{S} -complete and \mathcal{F} is \mathcal{S} -Cauchy filter on Y , then by Corollary 4.1.8 $f^{-1}(\mathcal{F})$ is \mathcal{S} -Cauchy filter. Since X is strongly \mathcal{S} -complete, $f^{-1}(\mathcal{F}) \xrightarrow{\mathcal{S}} n \Rightarrow f(f^{-1}(\mathcal{F})) \xrightarrow{\mathcal{S}} f(n) \Rightarrow \mathcal{F} \xrightarrow{\mathcal{S}} f(n)$. Thus Y is strongly \mathcal{S} -complete. \square

Theorem 4.2.4. *Let $X = \prod_{i \in I} X_i$ be a product in \mathcal{C} and \mathcal{F} a filter on X and $\mathcal{S} \in \text{CSYNT}$. Assume that $(p_i : X \rightarrow X_i)_{i \in I}$ is an \mathcal{S} -initial source. Let for each $i \in I$ the projections belong to \mathcal{E} . Then X is strongly \mathcal{S} -complete if and only if X_i is strongly \mathcal{S} -complete for each i .*

In light of Proposition 4.2.2 we get

Proposition 4.2.5. *Let $X = \prod_{i \in I} X_i$ be a product in \mathcal{C} and \mathcal{F} a filter on X and $\mathcal{S} \in \text{CSYNT}$. Assume that $(p_i : X \rightarrow X_i)_{i \in I}$ is an \mathcal{S} -initial source. Let for each $i \in I$ the projections belong to \mathcal{E} . Then X is strongly \mathcal{S} -complete if and only if X_i is strongly $\mathcal{U}^{\mathcal{S}}$ -complete for each i .*

One naturally obtains a categorical closure operator from \mathcal{S} -convergent filters.

Proposition 4.2.6. *Let $\mathcal{S} \in \text{CSYNT}$. Then $c_X^{\mathcal{S}}(m) = \bigvee \{n \in \text{sub}_o X \mid \exists \mathcal{F} \text{ on } X \text{ such that } \mathcal{F} \xrightarrow{\mathcal{S}} n \text{ and } m \in \mathcal{F}\}$ is a closure operator on \mathcal{C} .*

Proof. (C1) and (C2) are easily seen to be satisfied. To check (C3), let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and \mathcal{F} be a filter on Y such that $\mathcal{F} \xrightarrow{\mathcal{S}} n$ and $m \in \mathcal{F}$. Then by Proposition 4.1.7 (1), $f(\mathcal{F}) \xrightarrow{\mathcal{S}} f(n)$ and $f(m) \in f(\mathcal{F})$. Thus $f(c_X^{\mathcal{S}}(m)) = \bigvee \{f(n) \mid \exists \mathcal{F} \text{ on } X \text{ such that } \mathcal{F} \xrightarrow{\mathcal{S}} n \text{ and } m \in \mathcal{F}\} \leq \bigvee \{n' \mid \exists \mathcal{F}' \text{ on } Y \text{ such that } \mathcal{F}' \xrightarrow{\mathcal{S}} n' \text{ and } f(m) \in \mathcal{F}'\} = c_Y^{\mathcal{S}}(f(m)). \quad \square$

We shall say that m is \mathcal{S} -closed if $m = c_X^{\mathcal{S}}(m)$.

Proposition 4.2.7. *Let $m : M \rightarrow X$ be \mathcal{S} -initial and X strongly \mathcal{S} -complete. Then M is strongly \mathcal{S} -complete provided m is \mathcal{S} -closed.*

Proof. Let \mathcal{F} be \mathcal{S} -Cauchy filter on M . Then by Proposition 4.1.7 (2), $m(\mathcal{F})$ is \mathcal{S} -Cauchy and $m(\mathcal{F})$ converges to some $n \in \text{sub} X$ because X is strongly \mathcal{S} -complete. Since m is \mathcal{S} -closed, $n \leq m \Leftrightarrow n = m \circ j = m(j)$ for some $j \in \text{sub} M$. Thus $m(\mathcal{F}) \xrightarrow{\mathcal{S}} m(j) \Rightarrow \mathcal{F} \xrightarrow{\mathcal{S}} j$ by Proposition 4.1.7 (1). \square

Definition 4.2.8. *Let $\mathcal{S} \in \text{CSYNT}$. An object $X \in \mathcal{C}$ will be said to be:*

- (1) \mathcal{S} -complete if every \mathcal{S} -round Cauchy filter on X there is a unique $m \in \text{sub}_o X$ such that $\mathcal{F} = \nu_X^{\mathcal{S}}(m)$.
- (2) left \mathcal{S} -complete if every left \mathcal{S} -Cauchy filter on X is \mathcal{S} -convergent.
- (3) $c\mathcal{S}$ -complete if every $c\mathcal{S}$ -filter on X has a clustering with respect to \mathcal{S}

Because of Theorem 3.2.15, Proposition 4.1.19, we equivalently express the above definition as follows.

Definition 4.2.9. *Let $\mathcal{S} \in \text{CSYNT}$. An object $X \in \mathcal{C}$ will be said to be:*

- (1) \mathcal{S} -complete if every $\mathcal{U}^{\mathcal{S}}$ -round Cauchy filter on X there is a unique $m \in \text{sub}_o X$ such that $\mathcal{F} = \nu_X^{\mathcal{U}^{\mathcal{S}}}(m)$.

(2) left \mathcal{S} -complete if every left $\mathcal{U}^{\mathcal{S}}$ -Cauchy filter on X is $\mathcal{U}^{\mathcal{S}}$ -convergent.

(3) $c\mathcal{S}$ -complete if every $c\mathcal{U}^{\mathcal{S}}$ -filter on X has a clustering with respect to $\mathcal{U}^{\mathcal{S}}$.

We immediately get from Proposition 4.1.22 that every strongly \mathcal{S} -complete object is left \mathcal{S} -complete.

Definition 4.2.10. An object of $X \in \mathcal{C}$ is \mathcal{U} -stable if every Cauchy filter on X is \mathcal{U} -complete.

Our next proposition relates left (right) \mathcal{U} -complete objects with \mathcal{U} -stable ultrafilters.

Proposition 4.2.11. For $X \in \mathcal{C}$, the following hold true.

(1) If X is left \mathcal{U} -complete, then every \mathcal{U}^* -stable ultrafilter on X is \mathcal{U} -convergent.

(2) If X is right \mathcal{U} -complete, then every \mathcal{U} -stable ultrafilter on X is \mathcal{U} -convergent.

Proof. (1) Assume X is left \mathcal{U} -complete and \mathcal{F} be \mathcal{U}^* -stable ultrafilter. By Proposition 4.1.22, \mathcal{F} is left \mathcal{U} -Cauchy and hence \mathcal{U} -convergent.

(2) If X is right \mathcal{U} -complete and \mathcal{F} be \mathcal{U} -stable ultrafilter on X . By Proposition 4.1.22, \mathcal{F} is right \mathcal{U} -Cauchy and thus \mathcal{U} -convergent. \square

Proposition 4.2.12. Let $f : X \rightarrow Y$ be a \mathcal{U} -initial \mathcal{C} -morphism. Then X is \mathcal{U} -stable if and only if Y is \mathcal{U} -stable.

Proof. Assume that X is \mathcal{U} -stable and \mathcal{F} is a \mathcal{U} -Cauchy filter on Y . Then by Proposition 4.1.15, $f^{-1}(\mathcal{F})$ is a \mathcal{U} -Cauchy filter and \mathcal{U} -stable. Since $f \in \mathcal{E}$, Proposition 4.1.23 implies that $\mathcal{F} = f(f^{-1}(\mathcal{F}))$ is \mathcal{U} -stable. On the other hand if Y is \mathcal{U} -stable and \mathcal{F} is \mathcal{U} -Cauchy filter on X , then $f(\mathcal{F})$ is \mathcal{U} -Cauchy and \mathcal{U} -stable. By Proposition 4.1.15, $f^{-1}(f(\mathcal{F}))$ is \mathcal{U} -stable. Since $f^{-1}(f(\mathcal{F})) \subseteq \mathcal{F}$, \mathcal{F} is \mathcal{U} -stable. \square

Proposition 4.2.13. Let $X = \prod_{i \in I} X_i$ be a product in \mathcal{C} . Assume that $(p_i : X \rightarrow X_i)_{i \in I}$ is a \mathcal{U} -initial source. Then X is \mathcal{U} -stable if and only if X_i is \mathcal{U} -stable for each i .

Proposition 4.2.14. Let $\mathcal{U} \in \text{SQUNIF}(\mathcal{C}, \mathcal{M})$. Then every \mathcal{C} -object is Strongly \mathcal{U} -complete.

Proof. Let \mathcal{F} be a \mathcal{U} -Cauchy filter on X and $\mathcal{B}_X = \{U\}$ be a base for \mathcal{U} . Since \mathcal{F} is \mathcal{U} -Cauchy, there is $m \in \text{sub}_0 X$ such that $V(m) \in \mathcal{F}$. But $V(m) \leq U(m)$ for any $U \in \mathcal{U}_X$. Thus $U(m) \in \mathcal{F}$ and \mathcal{F} converges to m . \square

4.3 Precompactness

With the theory of completeness of objects of \mathcal{C} already established, one would expect precompactness coming next.

Definition 4.3.1. *Let $X \in \mathcal{C}$ and $\mathcal{S} \in \text{CSYNT}$. X is said to be \mathcal{S} -precompact if every ultrafilter on X is an \mathcal{S} -Cauchy filter.*

It is immediate from the above definition that every \mathcal{U} -precompact object is \mathcal{U}^* -precompact.

Proposition 4.3.2. *Let $X \in \mathcal{C}$ and $\mathcal{S} \in \text{CSYNT}$. X is \mathcal{S} -precompact if and only if every ultrafilter on X is a $\mathcal{U}^{\mathcal{S}}$ -Cauchy filter.*

Proof. Assume that X is \mathcal{S} -precompact and \mathcal{F} be an ultrafilter on X . Let $U \in \mathcal{B}_X^{\mathcal{S}}$. Then by Theorem 3.2.15, there is $\sqsubset_X \in \mathcal{S}_X$ such that $U^{\sqsubset} = U$ and there is $m \in \text{sub}_o X$ such that $m \sqsubset_X n \Rightarrow n \in \mathcal{F}$. Since $m \sqsubset_X U(m)$, $U(m) \in \mathcal{F}$. Conversely if every ultrafilter on X is $\mathcal{U}^{\mathcal{S}}$ -cauchy, let \mathcal{F} be an ultrafilter on X and $\sqsubset_X \in \mathcal{S}_X$. Then there is $U \in \mathcal{B}_X^{\mathcal{S}}$ such that $\sqsubset_X^U = \sqsubset_X$ and there is $m \in \text{sub}_o X$ such that $U(m) \in \mathcal{F}$. Now $m \sqsubset_X n \Leftrightarrow U(m) \leq n \Rightarrow n \in \mathcal{F}$. \square

Proposition 4.3.3. *Let $f : X \rightarrow Y$ be an \mathcal{E} -morphism that is \mathcal{S} -initial. Then X is \mathcal{S} -precompact if and only if Y is \mathcal{S} -precompact.*

Proof. Let X be \mathcal{S} -precompact and \mathcal{F} be an ultrafilter on X . Then $f^{-1}(\mathcal{F})$ is an ultrafilter on X . By assumption, $f^{-1}(\mathcal{F})$ is an \mathcal{S} -Cauchy filter on X . It follows from Proposition 4.1.7 that $\mathcal{F} = f(f^{-1}(\mathcal{F}))$ is \mathcal{S} -Cauchy filter on Y . Conversely let f be \mathcal{S} -initial, belongs to \mathcal{E} and Y is \mathcal{S} -precompact. If \mathcal{F} is an ultrafilter on X , then $f(\mathcal{F})$ is an ultrafilter on Y . By assumption, $f(\mathcal{F})$ is \mathcal{S} -Cauchy. By Proposition 4.1.7, \mathcal{F} is \mathcal{S} -Cauchy. \square

Theorem 4.3.4. *Let $X = \prod_{i \in I} X_i$ be a product in \mathcal{C} . Assume $(p_i : X \rightarrow X_i)_{i \in I}$ be an \mathcal{S} -initial source. Let p_i belong to \mathcal{E} for each i , then X is \mathcal{S} -precompact if and only if X_i is \mathcal{S} -precompact for each i .*

Definition 4.3.5. *An object X is said to be hereditarily precompact (resp. \mathcal{U}^* -precompact) if every ultrafilter on X is left \mathcal{U} -Cauchy (resp. \mathcal{U}^* -Cauchy).*

From the observation that a filter on $X \in \mathcal{C}$ is \mathcal{U}^* -Cauchy if and only if it is right \mathcal{U} -Cauchy, we have the following

Proposition 4.3.6. (1) *X is \mathcal{U}^* -precompact if and only if it is hereditarily \mathcal{U}^* -precompact.*

(2) X is right \mathcal{U} -complete if and only if it is \mathcal{U}^* -complete.

Proposition 4.3.7. *If an object $X \in \mathcal{C}$ is \mathcal{S} -precompact and left \mathcal{S} -complete then every ultrafilter on X is \mathcal{S} -convergent.*

Proof. Let X be \mathcal{S} -precompact, left \mathcal{S} -complete and \mathcal{F} an ultrafilter on X . Then \mathcal{F} is an \mathcal{S} -Cauchy filter and by Proposition 4.1.22, \mathcal{F} is left \mathcal{S} -Cauchy and thus \mathcal{S} -convergent. \square

Proposition 4.3.8. *The following hold true for a quasi-uniformity \mathcal{U} on \mathcal{C} and $X \in \mathcal{C}$.*

(1) X is \mathcal{U} -precompact if every ultrafilter on X is a $c\mathcal{S}$ -Cauchy filter.

(2) X is hereditary \mathcal{U} -precompact, then every ultrafilter on X is \mathcal{U}^* -Cauchy.

Proof. (1) Assume X is \mathcal{U} precompact and \mathcal{F} be an ultrafilter on X . Then \mathcal{F} is \mathcal{U} -Cauchy filter on X . By Proposition 4.1.22 \mathcal{F} is a $c\mathcal{S}$ -Cauchy.

(2) If X is hereditarily \mathcal{U} -precompact and \mathcal{F} is an ultrafilter on X , then \mathcal{F} is left \mathcal{U} -Cauchy filter. By Proposition 4.1.22, \mathcal{F} is \mathcal{U}^* -Cauchy. \square

4.4 The pair completeness

In order to make Cászár's theory of completeness of quasi-uniform spaces, expressed in terms of syntopogenous structures, easily understandable at a level of completeness of uniform spaces, W. F. Lindgren and P. Fletcher ([LF78]) introduced the concept of pair completeness. Having studied various complete objects with respect to our quasi-uniformity, it sounds natural to define pair completeness in these settings and relate it with those obtained in the second section of this chapter.

Definition 4.4.1. *Let $X \in \mathcal{C}$ and $(\mathcal{G}, \mathcal{F})$ be an ordered pair of filters on X . We shall say that $(\mathcal{G}, \mathcal{F}) \xrightarrow{\mathcal{U}} m$ (resp. $(\mathcal{G}, \mathcal{F})$ is \mathcal{U} -stable) if $\mathcal{G} \xrightarrow{\mathcal{U}^*} m$ and $\mathcal{F} \xrightarrow{\mathcal{U}} m$ (resp. \mathcal{G} is \mathcal{U}^* -stable and \mathcal{F} is \mathcal{U} -stable). $(\mathcal{G}, \mathcal{F})$ is \mathcal{U} -Cauchy if for any $U \in \mathcal{U}_X$, there are $m \in \mathcal{G}$ and $n \in \mathcal{F}$ such that $n \leq U(m')$ for all $m' \leq m$ and $m' \in \text{sub}_o X$.*

It is clear from the above Definition that if $(\mathcal{G}, \mathcal{F})$ is a \mathcal{U} -Cauchy pair then \mathcal{F} is \mathcal{U} -Cauchy.

Definition 4.4.2. *A filter \mathcal{F} on $X \in \mathcal{C}$ is said to be Doitchinov \mathcal{U} -Cauchy or simply $d\mathcal{U}$ -Cauchy if there is a filter \mathcal{G} on X such that $(\mathcal{G}, \mathcal{F})$ is \mathcal{U} -Cauchy.*

Proposition 4.4.3. *If $\mathcal{F} \xrightarrow{\mathcal{U}} m$ ($m \in \text{sub}^+ X$) then \mathcal{F} is $d\mathcal{U}$ -Cauchy.*

Proof. Assume $\mathcal{F} \xrightarrow{\mathcal{U}} m$ and $U \in \mathcal{U}_X$. Then $U(m) \in \mathcal{F}$. Now, let $\mathcal{G} = \{l \mid m \leq l\}$, $p \leq m$ and $p \in \text{sub}_o X$, then $U(m) \leq U(p)$. Thus \mathcal{F} is $d\mathcal{U}$ -Cauchy. \square

Proposition 4.4.4. *Let \mathcal{U} be a quasi-uniformity on \mathcal{C} and \mathcal{F} a filter on $X \in \mathcal{C}$. Then*

- (1) $\mathcal{F} \xrightarrow{\mathcal{U}} m$ if and only if $(\mathcal{F}, \mathcal{F}) \xrightarrow{\mathcal{U}} m$.
- (2) If \mathcal{U} is a quasi-uniformity, then \mathcal{F} is \mathcal{U} -Cauchy if and only if $(\mathcal{F}, \mathcal{F})$ is \mathcal{U} -Cauchy.

Proof. (1) If $\mathcal{F} \xrightarrow{\mathcal{U}} m$ and $U \in \mathcal{U}_X$, then $U(m) \in \mathcal{F}$. Since, $U(m) \leq U^*(m)$, $U^*(m) \in \mathcal{F}$ and so $(\mathcal{F}, \mathcal{F}) \xrightarrow{\mathcal{U}} m$. On the other hand if $(\mathcal{F}, \mathcal{F}) \xrightarrow{\mathcal{U}} m$ and $U \in \mathcal{U}_X$, then $U(m), U^*(m) \in \mathcal{F}$. Thus $\mathcal{F} \xrightarrow{\mathcal{U}} m$.

(2) If \mathcal{F} is \mathcal{U} -Cauchy and $U \in \mathcal{U}_X$, then there is $V \in \mathcal{U}_X$ such that $V \circ V \leq U$ and there is $m \in \text{sub}_o X$ such that $V(m) \in \mathcal{F}$. Let $p \leq V(m)$ and $p \in \text{sub}_o X$. Since \mathcal{U} is a uniformity, $m \leq V(p) \Rightarrow m \leq V(m) \leq V(V(p)) \leq U(p)$. Thus $(\mathcal{F}, \mathcal{F})$ is \mathcal{U} -Cauchy. Conversely if $(\mathcal{F}, \mathcal{F})$ is \mathcal{U} -Cauchy, then there are $m, n \in \mathcal{F}$ such that $m \leq U(p)$ for all $p \leq n$ and $p \in \text{sub}_o X$. Thus $U(p) \in \mathcal{F}$. \square

Proposition 4.4.5. *Let $f : X \rightarrow Y$ be a \mathcal{U} -initial \mathcal{C} -morphism that reflects 0 and $(\mathcal{G}, \mathcal{F})$ an ordered pair of filters on X . Then*

- (1) $(\mathcal{G}, \mathcal{F}) \xrightarrow{\mathcal{U}} m$ if and only if $(f(\mathcal{G}), f(\mathcal{F})) \xrightarrow{\mathcal{U}} f(m)$.
- (2) $(\mathcal{G}, \mathcal{F})$ is \mathcal{U} -stable if and only if $(f(\mathcal{G}), f(\mathcal{F}))$ is \mathcal{U} -stable.
- (3) $(\mathcal{G}, \mathcal{F})$ is \mathcal{U} -Cauchy if and only if $(f(\mathcal{G}), f(\mathcal{F}))$ is \mathcal{U} -Cauchy.

Proof. (1) Assume that $(\mathcal{G}, \mathcal{F}) \xrightarrow{\mathcal{U}} m$ and $U \in \mathcal{U}_Y$. Then by (U5), there is $U' \in \mathcal{U}_X$ such that $f(U'(m)) \leq U(f(m))$, $U^*(m) \in \mathcal{G}$ and $U(m) \in \mathcal{F}$. Now, $f(U'(m)) \in f(\mathcal{F}) \Rightarrow U(f(m)) \in \mathcal{F}$ and $f(U^*(m)) \in f(\mathcal{G}) \Rightarrow U^*(f(m)) \in f(\mathcal{G})$. Thus $(f(\mathcal{G}), f(\mathcal{F})) \xrightarrow{\mathcal{U}} f(m)$ and $f(m) \in \text{sub}_o Y$ since f reflects 0.

Conversely, let $(f(\mathcal{G}), f(\mathcal{F})) \xrightarrow{\mathcal{U}} f(m)$ and $U \in \mathcal{U}_X$. Then by \mathcal{U} -initiality of f , there is $U' \in \mathcal{U}_Y$ such that $f^{-1}(U'(f(m))) \leq U(m)$, $U'(f(m)) \in f(\mathcal{F})$ and $U^*(f(m)) \in f(\mathcal{G})$. So $f^{-1}(U'(f(m))) \in \mathcal{F}$ and $f^{-1}(U^*(f(m))) \in \mathcal{G}$ which implies that $U(m) \in \mathcal{F}$ and $U^*(m) \in \mathcal{G}$. Consequently, $(\mathcal{G}, \mathcal{F}) \xrightarrow{\mathcal{U}} m$.

(2) Let $(\mathcal{G}, \mathcal{F})$ be \mathcal{U} -stable and $U \in \mathcal{U}_Y$. Then there is $V \in \mathcal{U}_X$ such that $V(f^{-1}(p)) \leq$

$f^{-1}(U(p))$ for all $p \in \text{sub}Y$ and there is $m \in \mathcal{F}$ and $n \in \mathcal{G}$ such that $m \leq V(m')$ and $n \leq V^*(n')$ for all $m' \in \mathcal{F}$ and $n' \in \mathcal{G}$. If $k \in f(\mathcal{F})$ and $l \in f(\mathcal{G})$, then $m \leq V(f^{-1}(k)) \leq f^{-1}(U(k))$ and $n \leq V^*(f^{-1}(l)) \leq f^{-1}(U^*(l))$. Thus $f(m) \leq U(k)$ and $f(n) \leq U^*(l)$. Consequently $(f(\mathcal{G}), f(\mathcal{F}))$ is \mathcal{U} -stable.

On the other hand if $(f(\mathcal{G}), f(\mathcal{F}))$ is \mathcal{U} -stable and $V \in \mathcal{U}_X$, then there is $U \in \mathcal{U}_Y$ such that $f^{-1}(U(f(p))) \leq V(p)$ for all $p \in \text{sub}X$ and there is $m' \in f(\mathcal{G})$ and $m \in f(\mathcal{F})$ such that $m \leq U(l)$ and $m' \leq U^*(k)$ for all $l \in f(\mathcal{F})$ and $k \in f(\mathcal{G})$. Let $n \in \mathcal{F}$ and $n' \in \mathcal{G}$, then $m \leq U(f(n))$ and $m' \leq U^*(f(n'))$. Thus $f^{-1}(m) \leq f^{-1}(U(f(n))) \leq V(n)$ and $f^{-1}(m')$ and $f^{-1}(m') \leq f^{-1}(U^*(f(n'))) \leq V^*(n')$, that is $(\mathcal{G}, \mathcal{F})$ is \mathcal{U} -stable.

(3) Assume that $(\mathcal{G}, \mathcal{F})$ is \mathcal{U} -Cauchy and $U \in \mathcal{U}_Y$. Then there is $V \in \mathcal{U}_X$ such that $V(f^{-1}(p)) \leq f^{-1}(U(p))$ for all $p \in \text{sub}Y$ and there is $m \in \mathcal{G}$ and $n \in \mathcal{F}$ such that $n \leq V(m')$ for all $m' \leq m$ and $m' \in \text{sub}_oX$. Let $l \in \text{sub}_oY$ and $l \leq f(m)$, then $f(m) \wedge l = l \Leftrightarrow f(m \wedge f^{-1}(l)) = l$ since $f \in \mathcal{E}$. Now $l \in \text{sub}_oY$ implies that $m \wedge f^{-1}(l) \in \text{sub}_oX$ and $n \leq V(m \wedge f^{-1}(l)) \leq V(f^{-1}(l)) \leq f^{-1}(U(l)) \Rightarrow n \leq f^{-1}(U(l)) \Leftrightarrow f(n) \leq U(l)$. Thus $(f(\mathcal{F}), f(\mathcal{G}))$ is \mathcal{U} -Cauchy.

Conversely, let $(f(\mathcal{G}), f(\mathcal{F}))$ be \mathcal{U} -Cauchy and $U \in \mathcal{U}_X$. Then by \mathcal{U} -initiality of f , there is $V \in \mathcal{U}_Y$ such that $f^{-1}(V(f(p))) \leq U(p)$ for all $p \in \text{sub}X$ and there is $m \in f(\mathcal{G})$ and $n \in f(\mathcal{F})$ such that $n \leq V(m')$ for all $m' \leq m$ and $m' \in \text{sub}_oY$. Let $l \in \text{sub}_oX$ and $l \leq f^{-1}(m)$. Then $f(l) \leq m$. Since $f(l) \in \text{sub}_oY$, $n \leq V(f(l)) \Rightarrow f^{-1}(n) \leq f^{-1}(V(f(l))) \leq U(l) \Rightarrow f^{-1}(m) \leq U(l)$. Thus $(\mathcal{G}, \mathcal{F})$ is \mathcal{U} -Cauchy. \square

Because of Proposition 4.4.5, we have that every $d\mathcal{U}$ -Cauchy filter on X is \mathcal{U} -Cauchy.

Proposition 4.4.6. *Let $X = \prod_{i \in I} X_i$ be a product in \mathcal{C} and $(\mathcal{G}, \mathcal{F})$ be an ordered pair of filters on X . Assume that $(p_i : X \rightarrow X_i)_{i \in I}$ is a \mathcal{U} -initial source.*

- (1) $(\mathcal{G}, \mathcal{F}) \xrightarrow{\mathcal{U}} m$ if and only if $(p_i(\mathcal{G}), p_i(\mathcal{F})) \xrightarrow{\mathcal{U}} p_i(m)$ for each i .
- (2) $(\mathcal{G}, \mathcal{F})$ is \mathcal{U} -Cauchy if and only if $(p_i(\mathcal{G}), p_i(\mathcal{F}))$ is \mathcal{U} -Cauchy for each i .
- (3) $(\mathcal{G}, \mathcal{F})$ is \mathcal{U} -stable if and only if $(p_i(\mathcal{G}), p_i(\mathcal{F}))$ is \mathcal{U} -stable for each i .

Proposition 4.4.7. *If each filter pair on X is \mathcal{U} -stable then X is hereditarily \mathcal{U}^* -precompact.*

Proof. Assume that each filter on X is \mathcal{U} -stable and \mathcal{F} is an ultrafilter on X . Then $(\mathcal{F}, \mathcal{F})$ is \mathcal{U} -stable and so \mathcal{F} is \mathcal{U} -stable. Thus \mathcal{F} is right \mathcal{U} -Cauchy. Consequently X is

\mathcal{U}^* -precompact. □

From Propositions 4.4.6 and 4.4.5, we can conclude that:

Proposition 4.4.8. *Let $f : X \rightarrow Y$ be a \mathcal{U} -initial \mathcal{C} -morphism and \mathcal{F} a filter on X . Then \mathcal{F} is $d\mathcal{U}$ -Cauchy if and only if $f(\mathcal{F})$ is $d\mathcal{U}$ -Cauchy.*

Proposition 4.4.9. *Let $X = \prod_{i \in I} X_i$ be a product in \mathcal{C} and \mathcal{F} be filters on X . Assume that $(p_i : X \rightarrow X_i)_{i \in I}$ is a \mathcal{U} -initial source. Then \mathcal{F} is $d\mathcal{U}$ -Cauchy on X if and only if $p_i(\mathcal{F})$ is $d\mathcal{U}$ -Cauchy for each i .*

Definition 4.4.10. *An object $X \in \mathcal{C}$ is $d\mathcal{U}$ -complete if every $d\mathcal{U}$ -Cauchy filter on X is \mathcal{U} -convergent. X is \mathcal{U} -Cauchy bounded if every ultrafilter is $d\mathcal{U}$ -Cauchy.*

It is now clear from Propositions 4.4.6 and 4.4.5 that Propositions 4.2.3 and 4.2.4 remain true for $d\mathcal{U}$ -complete objects.

Proposition 4.4.11. (1) *Every \mathcal{U} -Cauchy bounded object is \mathcal{U} -precompact. The converse holds if \mathcal{U} is a uniformity.*

(2) *If every ultrafilter on X is \mathcal{U} -convergent then X is \mathcal{U} -Cauchy bounded.*

(3) *If X is $d\mathcal{U}$ -complete and \mathcal{U} -Cauchy bounded then every ultrafilter on X is \mathcal{U} -convergent.*

(4) *Every strongly \mathcal{U} -complete object is $d\mathcal{U}$ -complete.*

(5) *If $(\mathcal{G}, \mathcal{F})$ is a \mathcal{U} -Cauchy pair then $(\mathcal{F}, \mathcal{G})$ is a \mathcal{U}^* -Cauchy pair.*

Proof. (1) and (4) follow from the fact that every $d\mathcal{U}$ -Cauchy filter is filter is \mathcal{U} -Cauchy while (2) follows from Proposition 4.4.3.

(3) is immediate from the definitions.

(5) Assume that $(\mathcal{G}, \mathcal{F})$ is a \mathcal{U} -Cauchy pair and $U \in \mathcal{U}_X$. Then there is $m \in \mathcal{G}$ and $n \in \mathcal{F}$ such that $n \leq U(m')$ for all $m' \leq m$ and $m' \in \text{sub}_o X$. If $n' \leq n$ and $n \in \text{sub}_o X$, then $n' \leq U(m) \Rightarrow m \leq U^*(n')$. Thus $(\mathcal{F}, \mathcal{G})$ is a \mathcal{U}^* -Cauchy pair on X . □

Proposition 4.4.12. *Let \mathcal{F} be a filter on $X \in \mathcal{C}$. Then \mathcal{F} is a left \mathcal{U}^* -Cauchy filter if and only if it is a right $d\mathcal{U}$ -Cauchy filter.*

Proof. Let \mathcal{F} be right \mathcal{U} -Cauchy and $U \in \mathcal{U}_X$. Then there is $m \in \mathcal{F}$ such that $U(m) \in \mathcal{F}$ for all $m' \leq m$ and $m' \in \text{sub}_o X$. Now, $p \leq U(m')$ then $m \leq U(p)$. Thus one puts $\mathcal{F} = \mathcal{G}$ to see that \mathcal{F} is $d\mathcal{U}^*$ -Cauchy. Conversely if \mathcal{F} is $d\mathcal{U}^*$ -Cauchy, then there is a filter \mathcal{G} on X such that $(\mathcal{G}, \mathcal{F})$ is \mathcal{U}^* -Cauchy. Let $U \in \mathcal{U}$, there is $m \in \mathcal{G}$, $n \in \mathcal{F}$ such that $n \leq U(m')$

for all $m' \leq m$ and $m' \in \text{sub}_o X$. If $p \leq n$ and $p \in \text{sub}_o X$ then $p \leq U(n)$, thus $n \leq U^*(p)$ and $U^*(p) \in \mathcal{F}$. \square

Every left \mathcal{U} -Cauchy filter is \mathcal{U}^* -Cauchy and \mathcal{U}^* -Cauchy filters are $d\mathcal{U}$ -Cauchy. Consequently, every $d\mathcal{U}$ -complete object is left \mathcal{U} -Complete. However, the proposition above allows one to conclude that an object is $d\mathcal{U}^*$ -complete if and only if it is right \mathcal{U} -complete.

Proposition 4.4.13. *If X is right \mathcal{U} -Complete and $(\mathcal{G}, \mathcal{F})$ is a filter pair on X , then \mathcal{F} has a clustering with respect to \mathcal{U} .*

Proof. Assume that $(\mathcal{G}, \mathcal{F})$ is a \mathcal{U} -Cauchy filter pair on X with X right \mathcal{U} -complete. Let \mathcal{F}' be an ultrafilter on X that is finer than \mathcal{F} . Then $(\mathcal{G}, \mathcal{F})$ is a \mathcal{U} -Cauchy pair which implies that \mathcal{F}' is \mathcal{U}^* -Cauchy since $U^* \leq \mathcal{U}$. Thus \mathcal{F}' is right \mathcal{U} -Cauchy and so \mathcal{F}' is \mathcal{U} -convergent. Consequently \mathcal{F} has a clustering with respect to \mathcal{U} . \square



4.5 Examples

1. Consider the syntopogenous structure in Example 3.3(1). For any $(X, \mathcal{D}) \in \mathbf{Qunif}$, let $\text{sub}_o X$ be the class of singleton subsets of X . According to Proposition 4.1.16, an \mathcal{S} -convergent filter on X is the one converging in the topology induced by \mathcal{D} . \mathcal{S} -cauchy filters are \mathcal{D} -cauchy filters (Theorem 4.1.17). By Proposition 4.1.19, left (resp. right) \mathcal{S} -Cauchy filters become the left K -Cauchy (resp. right) K -Cauchy filters introduced by Romaguera in ([Rom96]) while $w\mathcal{S}$ -Cauchy filters and $c\mathcal{S}$ -Cauchy filters are called weakly hereditarily Cauchy filters ([PPnR99]) and Corson Cauchy filters ([PPnR99]). Thus, strongly \mathcal{S} -complete objects coincide with the convergent complete quasi-uniform spaces ([FL82]). An \mathcal{S} -complete object corresponds to a Smyth complete quasi-uniform space ([Sün93]) and left \mathcal{S} (resp. $c\mathcal{S}$)-complete objects are the left (resp. corson) complete quasi-uniform spaces (see e.g [Kün02, PPnR99]).
2. Let \mathcal{S} be the syntopogenous structure in Example 3.3.5(b). For any $(X, \mathcal{T}) \in \mathbf{Top}$, a filter \mathcal{F} on X is \mathcal{S} -round if for all $A \in \mathcal{F}$, there is $B \in \mathcal{F}$ such that $B \subseteq O \subseteq A$ for some $O \in \mathcal{T}$, that is \mathcal{F} has base of open subsets of X , say $\mathcal{B} = \{O \mid O \in \mathcal{T}\}$. Now \mathcal{F} being \mathcal{S} -round and $w\mathcal{S}$ -Cauchy shall mean that for all $O \in \mathcal{B}$, there is $x \in O$

such that $\mathcal{N}(x) \subseteq \mathcal{B}$ which is equivalent to saying that \mathcal{F} is a completely prime open filter where $\mathcal{N}(x) = \{O \in \mathcal{T} \mid x \in O\}$. Thus, (X, \mathcal{T}) being $w\mathcal{S}$ -complete is equivalent to the fact that every completely prime filter of open sets of X is the neighbourhood filter of a unique point of X i.e (X, \mathcal{T}) is a sober space. In [Smy94] sobriety is established using covers.

3. Let \mathcal{S} be the syntopogenous structure in Examples 3.3(2). For any $(X, \cdot) \in \mathbf{TorGrp}$, a filter \mathcal{F} on X converges to x with respect to \mathcal{S} if for any $U \in \beta(x)$, $U \cdot x \in \mathcal{F}$. \mathcal{F} is \mathcal{S} -Cauchy if for any $U \in \beta(e)$, there is $x \in X$ such that $U \cdot x \in \mathcal{F}$. Thus every complete group (see e.g [Bou66]) is \mathcal{S} -complete.



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Chapter 5

Quasi-uniform structures and Functors

This chapter aims to describe the quasi-uniformities induced by an \mathcal{E} -pointed (respectively an \mathcal{M})-copointed endofunctor and to investigate the continuity of functors between categories supplied with fixed quasi-uniformities. We commence by defining the continuity of a \mathcal{C} -morphism with respect to two syntopogenous structures which permits us to study the syntopogenous structures induced by \mathcal{E} -pointed and \mathcal{M} -copointed endofunctors of \mathcal{C} . Then apply Theorem 3.2.15 to obtain the corresponding quasi-uniformities. The notion of continuity of functors between categories endowed with quasi-uniformities is then introduced. The results proved are shown to yield in particular some of those obtained by D. Diranjan and Tholen in ([DT95]). We conclude the chapter with a few examples that demonstrate our results.

5.1 Quasi-uniform structures induced by (co)pointed endofunctors

For a syntopogenous structure \mathcal{S} on \mathcal{C} and an \mathcal{E} -pointed endofunctor (F, η) , we show, in this section, that there is a coarsest syntopogenous structure $\mathcal{S}^{F, \eta}$ on \mathcal{C} for which every $\eta_X : X \rightarrow FX$ is $(\mathcal{S}^{F, \eta}, \mathcal{S})$ -continuous. This allows us to use Theorem 3.2.15 and obtain the coarsest quasi-uniformity $\mathcal{U}^{\mathcal{S}^{F, \eta}}$ on \mathcal{C} which makes every $\eta_X : X \rightarrow FX$ $(\mathcal{U}^{\mathcal{S}^{F, \eta}}, \mathcal{U}^{\mathcal{S}})$ -

continuous. The dual case of an \mathcal{M} -copointed endofunctor is also studied.

Throughout this section, the class \mathcal{E} will be assumed to be stable under pullback along \mathcal{M} -morphisms.

Definition 5.1.1. *Let \mathcal{S} and \mathcal{S}' be syntopogenous structures on \mathcal{C} . A morphism $f : X \rightarrow Y$ is $(\mathcal{S}, \mathcal{S}')$ -continuous if for all $\sqsubset'_Y \in \mathcal{S}'_Y$, there is $\sqsubset_X \in \mathcal{S}_X$ such that $f(m) \sqsubset'_Y n \Rightarrow m \sqsubset_X f^{-1}(n)$ for all $m \in \text{sub}X$ and $n \in \text{sub}Y$, equivalently $m \sqsubset'_Y n \Rightarrow f^{-1}(m) \sqsubset_X f^{-1}(n)$ for all $n, m \in \text{sub}Y$.*

Since every \mathcal{C} -morphism f is $(\mathcal{S}, \mathcal{S})$ -continuous and $(\mathcal{S}', \mathcal{S}')$ -continuous, f is $(\mathcal{S}, \mathcal{S}')$ -continuous if $\mathcal{S}' \leq \mathcal{S}$. Because \mathcal{S} is simple if $\mathcal{S}_X = \{\sqsubset_X\}$ is an interpolative topogenous order, we obtain the following proposition which is a particular case of Definition 2.5.1 for interpolative topogenous orders.

Proposition 5.1.2. *Assume that \mathcal{S} and \mathcal{S}' are simple syntopogenous structures i.e $\mathcal{S}_X = \{\sqsubset_X\}, \mathcal{S}'_X = \{\sqsubset'_X\} \in \text{INTORD}(\mathcal{C}, \mathcal{M})$. Then f is $(\mathcal{S}, \mathcal{S}')$ -continuous if and only if $f(m) \sqsubset'_Y n \Rightarrow m \sqsubset_X f^{-1}(n)$ for all $m \in \text{sub}X$ and $n \in \text{sub}Y$.*

Proposition 5.1.3. *If $\mathcal{S}, \mathcal{S}' \in \text{CSYNT}(\mathcal{C}, \mathcal{M})$. Then f is $(\mathcal{S}, \mathcal{S}')$ -continuous if and only if for any $V \in \mathcal{B}_Y^{\mathcal{S}'}$ there is $U \in \mathcal{B}_X^{\mathcal{S}}$ such that $f(U(m)) \leq V(f(m))$ for all $m \in \text{sub}X$.*

Proof. Assume that $f : X \rightarrow Y$ is $(\mathcal{S}, \mathcal{S}')$ -continuous and $\mathcal{S}, \mathcal{S}' \in \text{CSYNT}(\mathcal{C}, \mathcal{M})$. Then for any $V \in \mathcal{B}_Y^{\mathcal{S}'}$, there is $\sqsubset'_Y \in \mathcal{S}'_Y$ which determines V and there is $\sqsubset_X \in \mathcal{S}_X$ such that $f(m) \sqsubset'_Y n \Rightarrow m \sqsubset_X f^{-1}(n)$. Now $U(m) = U_X^{\sqsubset}(m) = \bigwedge \{p \mid m \sqsubset_X p\} \leq \bigwedge \{f^{-1}(n) \mid f(m) \sqsubset'_Y n\} = f^{-1}(V(f(m))) \Rightarrow U(m) \leq f^{-1}(V(f(m))) \Leftrightarrow f(U(m)) \leq V(f(m))$. Conversely, assume that for any $V \in \mathcal{B}_Y^{\mathcal{S}'}$ there is $U \in \mathcal{B}_X^{\mathcal{S}}$ such that $f(U(m)) \leq V(f(m))$. Now, for any $\sqsubset'_Y \in \mathcal{S}'_Y$, there is, by Theorem 3.2.15, $V \in \mathcal{B}^{\mathcal{S}}$ such that $\sqsubset_Y = \sqsubset^V$. Thus $f(m) \sqsubset'_Y n \Leftrightarrow V(f(m)) \leq n \Rightarrow f(U(m)) \leq n \Leftrightarrow U(m) \leq f^{-1}(n) \Leftrightarrow m \sqsubset_X^U f^{-1}(n) \Leftrightarrow m \sqsubset_X f^{-1}(n)$. \square

Propositions 5.1.2 and 2.5.2 allow us to prove the following.

Proposition 5.1.4. *Let \mathcal{S} and \mathcal{S}' be simple and co-perfect syntopogenous structures i.e $\mathcal{S}_X = \{\sqsubset_X\}, \mathcal{S}'_X = \{\sqsubset'_X\} \in \bigwedge -\text{INTORD}(\mathcal{C}, \mathcal{M})$. Then f is $(\mathcal{S}, \mathcal{S}')$ -continuous if and only if $f(c_X^{\sqsubset}(m)) \leq c_X^{\sqsubset'}(f(m))$ for all $m \in \text{sub}X$.*

Theorem 5.1.5. *Let (F, η) be an \mathcal{E} -pointed endofunctor of \mathcal{C} and \mathcal{S} a syntopogenous structure on \mathcal{C} with respect to \mathcal{M} . Then $\mathcal{S}_X^{F, \eta} = \{\sqsubset_X^{F, \eta} \mid \sqsubset_{FX} \in \mathcal{S}_X\}$ with $m \sqsubset_X^{F, \eta}$*

$n \Leftrightarrow \eta_X(m) \sqsubset_{FX} p$ and $\eta_X^{-1}(p) \leq n$ for some $p \in \text{sub}FX$ is the coarsest syntopogenous structure on \mathcal{C} with respect to \mathcal{M} for which every $\eta_X : X \rightarrow FX$ is $(\mathcal{S}^{F,\eta}, \mathcal{S})$ -continuous. If \mathcal{S} is interpolative and co-perfect, then so is $\mathcal{S}^{F,\eta}$.

Proof. (S1) follows Theorem 2.5.4 while (S2) is clear. For (S3), let $\sqsubset_X^{F,\eta} \in \mathcal{S}_X^{F,\eta}$ and $m \sqsubset_X^{F,\eta} n$. Then there is $p \in \text{sub}FX$ and $\sqsubset_{FX} \in \mathcal{S}_{FX}$ such that $\eta_X(m) \sqsubset_{FX} p$ and $\eta_X^{-1}(p) \leq n$ and there is $\sqsubset'_{FX} \in \mathcal{S}_{FX}$ such that $\eta_X(m) \sqsubset'_{FX} l \sqsubset'_{FX} p$ and $\eta_X^{-1}(p) \leq n$ for some $l \in \text{sub}FX$. Since each $\eta_X \in \mathcal{E}$, $\eta_X(m) \sqsubset'_{FX} \eta_X(\eta_X^{-1}(l)) \sqsubset'_{FX} p$. Thus $m \sqsubset_X'^{F,\eta} \eta_X^{-1}(l) \sqsubset_X'^{F,\eta} p$.

Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and $f(m) \sqsubset_Y^{F,\eta} n$ for $\sqsubset_Y^{F,\eta} \in \mathcal{S}_Y^{F,\eta}$. Then there is $p \in \text{sub}FY$ and $\sqsubset_{FY} \in \mathcal{S}_{FY}$ such that $\eta_Y(f(m)) \sqsubset_{FY} p$ and $\eta_Y^{-1}(p) \leq n$. By Definition 2.5.3, $Ff \circ \eta_X = \eta_Y \circ f$.

Now, $(Ff)(\eta_X(m)) \sqsubset_{FY} p$ and $\eta_Y^{-1}(p) \leq n \Rightarrow (Ff)(\eta_X(m)) \sqsubset_{FX} p$ and $f^{-1}(\eta_Y^{-1}(p)) \leq f^{-1}(n)$. So $\eta_X(m) \sqsubset_X (Ff)^{-1}(p)$ and $\eta_X^{-1}((Ff)^{-1}(p)) \leq f^{-1}(n)$ which gives $\eta_X(m) \sqsubset_{FX} l$ and $\eta_X^{-1}(l) \leq f^{-1}(n)$ (with $l = g^{-1}(p)$), that is $m \sqsubset_X^{F,\eta} f^{-1}(n)$.

If \mathcal{S} is interpolative and $m \sqsubset_X^{F,\eta} n$, then $\eta_X(m) \sqsubset_{FX} p$ and $\eta_X^{-1}(p) \leq n$ for some $p \in \text{sub}FX$. This implies that there is $l \in \text{sub}FX$ such that $\eta_X(m) \sqsubset_{FX} l \sqsubset_{FX} p$. Thus $\eta_X(m) \sqsubset_{FX} \eta_X(\eta_X^{-1}(l)) \sqsubset_{FX} p$, that is $m \sqsubset_X^{F,\eta} \eta_X^{-1}(l) \sqsubset_X^{F,\eta} n$.

η_X is $(\mathcal{S}^{F,\eta}, \mathcal{S})$ -continuous, since for all $\sqsubset_X \in \mathcal{S}_X$, $\eta_X(m) \sqsubset_X n \Rightarrow \eta_X(m) \sqsubset_{FX} (\eta_X(\eta_X^{-1}(n))) \Leftrightarrow m \sqsubset_X^{F,\eta} \eta_X^{-1}(n)$.

Assume \mathcal{S} is co-perfect, then for each $i \in I$, $m \sqsubset_X^{F,\eta} n_i \Leftrightarrow \eta_X(m) \sqsubset_{FX} p_i$ and $\eta_X^{-1}(p_i) \leq n_i$. By assumption, $\eta_X(m) \sqsubset_{FX} \bigwedge_{i \in I} p_i$ and $\eta_X^{-1}(\bigwedge_{i \in I} p_i) = \bigwedge_{i \in I} \eta_X^{-1}(p_i) \leq \bigwedge_{i \in I} n_i$. Consequently $m \sqsubset_X^{F,\eta} \bigwedge_{i \in I} n_i$. If \mathcal{S}' is another syntopogenous structure on \mathcal{C} such that η_X is $(\mathcal{S}', \mathcal{S})$ -continuous, then for any $\sqsubset_X^{F,\eta} \in \mathcal{S}_X^{F,\eta}$, $m \sqsubset_X^{F,\eta} n \Leftrightarrow \eta_X(m) \sqsubset_{FX} p$ and $\eta_X^{-1}(p) \leq n$. This implies that there is $\sqsubset'_X \in \mathcal{S}'_X$ such that $m \sqsubset'_X \eta_X^{-1}(p) \leq n \Rightarrow m \sqsubset'_X n$. Thus $\mathcal{S}^{F,\eta} \leq \mathcal{S}'$. \square

Viewing a reflector as endofunctor of \mathcal{C} , one obtains the corollary below.

Proposition 5.1.6. *Let \mathcal{A} be an \mathcal{E} -reflective subcategory of \mathcal{C} and \mathcal{S} a syntopogenous structure on \mathcal{A} with respect to \mathcal{M} . Then $\mathcal{S}_X^A = \{\sqsubset_X^A \mid \sqsubset_{FX} \in \mathcal{S}_{FX}\}$ with $m \sqsubset_X^A n \Leftrightarrow \eta_X(m) \sqsubset_{FX} p$ and $\eta_X^{-1}(p) \leq n$ for some $p \in \text{sub}FX$ is the coarsest syntopogenous structure on \mathcal{C} with respect to \mathcal{M} for which every reflection morphism $\eta_X : X \rightarrow FX$ is $(\mathcal{S}^A, \mathcal{S})$ -continuous. If \mathcal{S} is interpolative and co-perfect, then so is \mathcal{S}^A .*

It is important to note that if \mathcal{S} is a simple syntopogenous structure, then Theorem 5.1.5

(resp. Proposition 5.1.6) becomes a particular case of Theorem 2.5.4 (resp. Proposition 2.5.6).

Since $\mathcal{S}^{F,\eta}$ is co-perfect provided \mathcal{S} is co-perfect, we get the next proposition from Theorems 3.2.14 and 3.2.15.

Proposition 5.1.7. *Let (F, η) be a pointed endofunctor of \mathcal{C} and $\mathcal{S} \in \text{CSYNT}(\mathcal{C}, \mathcal{M})$. Then*

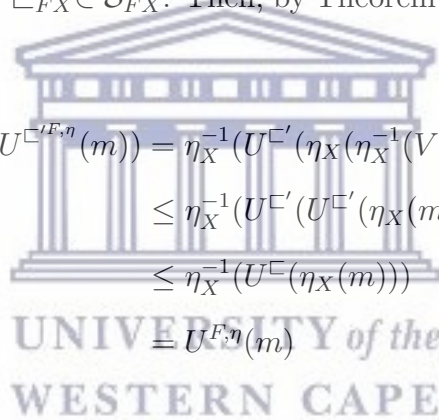
$$\mathcal{B}_X^{\mathcal{S}^{F,\eta}} = \{U^{\square^{F,\eta}} \mid U^\square \in \mathcal{B}_{FX}^{\mathcal{S}}\} \text{ with } U^{\square^{F,\eta}}(m) = \eta_X^{-1}(U^\square(\eta_X(m)))$$

is a base for the coarsest quasi-uniformity on \mathcal{C} with respect to \mathcal{M} for which every $\eta_X : X \rightarrow FX$ is $(\mathcal{U}^{\mathcal{S}^{F,\eta}}, \mathcal{U}^{\mathcal{S}})$ -continuous. $\mathcal{B}^{\mathcal{S}^{F,\eta}}$ is a transitive base provided that \mathcal{S} is interpolative.

Proof. (U1) $m \leq \eta_X^{-1}(\eta_X(m)) \leq \eta_X^{-1}(U(\eta_X(m))) = U^{F,\eta}(m)$.

For (U2), let $U^{\square^{F,\eta}} \in \mathcal{B}^{\mathcal{S}^{F,\eta}}$ for $\square_{FX} \in \mathcal{S}_{FX}$. Then, by Theorem 3.2.15, there is $\square_{FX} \in \mathcal{S}_{FX}$ such that $V^{\square'} \circ V^{\square'} \leq U^\square$.

$$\begin{aligned} \text{Hence } U^{\square^{F,\eta}}(U^{\square'^{F,\eta}}(m)) &= \eta_X^{-1}(U^{\square'}(\eta_X(\eta_X^{-1}(V^{\square'}(\eta_X(m)))))) \\ &\leq \eta_X^{-1}(U^{\square'}(U^{\square'}(\eta_X(m)))) \\ &\leq \eta_X^{-1}(U^\square(\eta_X(m))) \\ &= U^{F,\eta}(m) \end{aligned}$$



(U4) If $U^{\square^{F,\eta}}, U^{\square'^{F,\eta}} \in \mathcal{B}^{\mathcal{S}^{F,\eta}}$ for $\square_X, \square'_X \in \mathcal{S}_{FX}$. $U^{\square^{F,\eta}}(m) \wedge U^{\square'^{F,\eta}}(m) = \eta_X^{-1}(U^\square(\eta_X(m))) \wedge \eta_X^{-1}(U^{\square'}(\eta_X(m))) = \eta_X^{-1}(U^\square(\eta_X(m)) \wedge U^{\square'}(\eta_X(m))) = \eta_X^{-1}(U^{\square \wedge \square'}(\eta_X(m))) = U^{(\square \wedge \square')^{F,\eta}}(m)$.

Thus $U^{\square^{F,\eta}} \wedge U^{\square'^{F,\eta}} \in \mathcal{B}^{\mathcal{S}^{F,\eta}}$.

(U5) Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and $U^{\square^{F,\eta}} \in \mathcal{B}^{\mathcal{S}^{F,\eta}}$ for $\square_{FY} \in \mathcal{S}_{FY}$. Then there is $\square_{FX} \in \mathcal{S}_{FX}$ such that $f(V^{\square_{FX}}(m)) \leq U^{\square_{FY}}(f(m))$.

$$\begin{aligned} \text{Thus } f(V^{\square^{F,\eta}}(m)) &= f(\eta_X^{-1}(V^{\square_{FX}}(\eta_X(m)))) \\ &\leq \eta_Y^{-1}(Ff)(V^{\square_{FX}}(\eta_X(m))) && \text{Lemma 1.2.4} \\ &\leq \eta_Y^{-1}(U^{\square_{FY}}(Ff)(\eta_X(m))) \\ &= \eta_Y^{-1}(U^{\square_{FX}}(\eta_Y(f(m)))) && \text{Definition 2.5.3} \end{aligned}$$

Since, for any $\sqsubset_{FX} \in \mathcal{S}_X$, $U^{\square_{FX}, \eta}(m) = \eta_X^{-1}(U^{\square}(\eta_X(m))) \Rightarrow \eta_X(U^{\square_{FX}, \eta}(m)) \leq U^{\square}(\eta_X(m))$, η_X is $(\mathcal{U}^{\mathcal{S}^{F, \eta}}, \mathcal{U})$ -continuous for all $X \in \mathcal{C}$. If \mathcal{S} is interpolative then

$$\begin{aligned}
U^{\square_{FX}, \eta}(U^{\square_{FX}, \eta}(m)) &= U^{\square_{FX}, \eta}(\eta_X^{-1}(U^{\square}(\eta_X(m)))) \\
&= \eta_X^{-1}(U^{\square}(\eta_X(\eta_X^{-1}(U^{\square}(\eta_X(m))))) \\
&\leq \eta_X^{-1}(U^{\square}(U^{\square}(\eta_X(m)))) \\
&= \eta_X^{-1}(U^{\square}(\eta_X(m))) \\
&= U^{\square_{FX}, \eta}(m).
\end{aligned}$$

Let \mathcal{B}' be a base for another quasi-uniformity on \mathcal{C} such that η_X is $(\mathcal{U}', \mathcal{U}^{\mathcal{S}})$ -continuous, then for any $U^{\square} \in \mathcal{B}_{FX}^{\mathcal{S}}$, there is $U' \in \mathcal{B}'_X$ such that $\eta_X(U(m)) \leq U^{\square}(\eta_X(m)) \Leftrightarrow U'(m) \leq \eta_X^{-1}(U^{\square}(\eta_X(m))) = U^{\square_{FX}, \eta}(m)$. Thus $\mathcal{B}^{\mathcal{S}^{F, \eta}} \leq \mathcal{B}'$. \square

One sees from the above proposition that the condition of (F, η) being \mathcal{E} -pointed is not needed when the syntopogenous structure is co-perfect.

Theorem 5.1.8. *Let (F, η) be a pointed endofunctor of \mathcal{C} and $\mathcal{U} \in \text{QUNIF}(\mathcal{C}, \mathcal{M})$. Then the assignment $\mathcal{U} \mapsto \mathcal{U}^{F, \eta}$ preserves arbitrary joins and uniformity.*

Proof. Let $\mathcal{A} = \{\mathcal{U}^i \mid i \in I\} \subseteq \text{QUNIF}(\mathcal{C}, \mathcal{M})$. Theorem 4.1.17, $\mathcal{B}_X = \{U^1 \wedge \dots \wedge U^n \mid \text{for each } 1 \leq i \leq n, U^i \in \mathcal{U}_X^i \text{ for some } \mathcal{U}^i \in \text{QUNIF}(\mathcal{C}, \mathcal{M})\}$ is a base for $\bigvee \mathcal{S}$. We must show that $\mathcal{B}_X^\eta = \{(U^1 \wedge \dots \wedge U^n)^\eta \mid \text{for each } 1 \leq i \leq n, U^i \in \mathcal{U}_X^i \text{ for some } \mathcal{U}^i \in \text{QUNIF}(\mathcal{C}, \mathcal{M})\} = \{(U^1)^\eta \wedge \dots \wedge (U^n)^\eta \mid \text{for each } 1 \leq i \leq n, U^i \in \mathcal{U}_X^i \text{ for some } \mathcal{U}^i \in \text{QUNIF}(\mathcal{C}, \mathcal{M})\}$. Now, $(U^1 \wedge \dots \wedge U^n)^\eta = (\eta_X^{-1}(U^1 \wedge \dots \wedge U^n)(\eta_X(m))) = \eta_X^{-1}(U^1(\eta_X(m))) \wedge \dots \wedge \eta_X^{-1}(U^n(\eta_X(m))) = (U^1)^\eta \wedge \dots \wedge (U^n)^\eta$. Assume that \mathcal{U} is a uniformity on \mathcal{C} . For any $X \in \mathcal{C}$, $m, n \in \text{sub}X$ and $U \in \mathcal{U}_{FX}$, $n \leq U^\eta(m) \Leftrightarrow n \leq \eta_X^{-1}(U(\eta_X(m))) \Leftrightarrow \eta_X(n) \leq U(\eta_X(m))$. This implies that there is $V \in \mathcal{U}_{FX}$ such that $\eta_X(m) \leq V(\eta_X(n)) \Leftrightarrow m \leq \eta_X^{-1}(V(\eta_X(n))) = V^\eta(n)$. \square

Proposition 5.1.9. *Let (F, η) be a pointed endofunctor of \mathcal{C} and $\mathcal{U} \in \text{QUNIF}(\mathcal{C}, \mathcal{M})$. Assume that η_X reflects 0 for each X in \mathcal{C} .*

- (1) *If \mathcal{F} is a \mathcal{U}^η -Cauchy filter on X , then $\eta_X(\mathcal{F})$ is \mathcal{U} -Cauchy on FX . The converse implication holds if (F, η) is \mathcal{E} -pointed and \mathcal{E} is pullback stable.*
- (2) *$\mathcal{F} \xrightarrow{\mathcal{U}^\eta} m$ ($m \in \text{sub}_o X$) if and only if $\eta_X(\mathcal{F}) \xrightarrow{\mathcal{U}} \eta_X(m)$.*

(3) X is \mathcal{U}^η -precompact if and only if FX is \mathcal{U} -precompact.

Proof. (1) Assume that \mathcal{F} is \mathcal{U}^η -Cauchy and $U \in \mathcal{U}_{FX}$. Since η_X is $(\mathcal{U}^\eta, \mathcal{U})$ -continuous, $\eta_X(U^{\eta_X}(n)) \leq U(\eta_X(n))$ for all $n \in \text{sub}X$ and there is $m \in \text{sub}_oX$ such that $U^\eta(m) \in \mathcal{F}$. Thus $\eta_X^{-1}(U(\eta_X(m))) \in \mathcal{F} \Leftrightarrow U(\eta_X(m)) \in \eta_X(\mathcal{F})$, that is $\eta_X(\mathcal{F})$ is \mathcal{U} -Cauchy. Conversely if $\eta_X(\mathcal{F})$ is \mathcal{U} -Cauchy on FX and $U^\eta \in \mathcal{U}_X^\eta$. Then there is $m \in \text{sub}_oFX$ such that $U(m) \in \eta_X(\mathcal{F}) \Leftrightarrow \eta_X^{-1}(U(m)) \in \mathcal{F}$. Since (F, η) is \mathcal{E} -pointed and \mathcal{E} is pullback stable, $\eta_X^{-1}(m) \in \text{sub}_oX$ and $\eta_X^{-1}(U(m)) = \eta_X^{-1}(U(\eta_X(\eta_X^{-1}(m)))) = U^\eta(\eta_X^{-1}(m))$. Thus $U^\eta(\eta_X^{-1}(m)) \in \mathcal{F}$.

(2) Let $\mathcal{F} \xrightarrow{\mathcal{U}^\eta} m$ ($m \in \text{sub}_oX$) and $U \in \mathcal{U}_{FX}$. Since η_X is $(\mathcal{U}^\eta, \mathcal{U})$ -continuous, $\eta_X(U^\eta(m)) \leq U(\eta_X(m)) \Leftrightarrow U^\eta(m) \leq \eta_X^{-1}(U(\eta_X(m)))$. Hence $\eta_X^{-1}(U(\eta_X(m))) \in \mathcal{F} \Leftrightarrow U(\eta_X(m)) \in \eta_X(\mathcal{F})$. Conversely assume that $\eta_X(\mathcal{F}) \xrightarrow{\mathcal{U}} \eta_X(m)$ ($\eta_X(m) \in \text{sub}_oFX$) and $U^\eta \in \mathcal{U}^\eta$. Then $m \in \text{sub}_oX$ and $U(\eta_X(m)) \in \eta_X(\mathcal{F}) \Leftrightarrow \eta_X^{-1}(U(\eta_X(m))) \in \mathcal{F}$.

(3) The proof of (3) follows from (1) and (2). □

According to [Hol09], if $c, c' \in \mathcal{CL}(\mathcal{C}, \mathcal{M})$, a \mathcal{C} -morphism $f : X \rightarrow Y$ is said to be (c, c') -preserving if $f(c_X(m)) = c'_Y(f(m))$.

Proposition 5.1.10. *Let (F, η) be an \mathcal{E} -pointed endofunctor of \mathcal{C} and $\mathcal{U} \in \mathcal{QUNIF}(\mathcal{C}, \mathcal{M})$. Then for every $X \in \mathcal{J}$, η_X is $(c^{\mathcal{M}^{F, \eta}}, c^{\mathcal{M}})$ -preserving.*

Proof.

$$\begin{aligned} \text{For any } X \in \mathcal{C}, \quad \eta_X(c_X^{U^{F, \eta}}(m)) &= \eta_X(\bigwedge \{\eta_X^{-1}(U(\eta_X(m))) : U \in \mathcal{U}_{FX}\}) \\ &= \eta_X(\eta_X^{-1}(\bigwedge \{U(\eta_X(m)) : U \in \mathcal{U}_{FX}\})) \\ &= \bigwedge \{U(\eta_X(m)) : U \in \mathcal{U}_{FX}\} = c_{FX}^{\mathcal{U}}(\eta_X(m)). \end{aligned}$$

□

Proposition 5.1.11. *Let \mathcal{A} be a reflective subcategory of \mathcal{C} and \mathcal{S} a co-perfect syntopogenous structure on \mathcal{A} . Then*

$$\mathcal{B}_X^{\mathcal{A}} = \{U^{\square^{\mathcal{A}}} \mid U^{\square} \in \mathcal{B}_{FX}^{\mathcal{S}}\} \text{ with } U^{\square^{\mathcal{A}}}(m) = \eta_X^{-1}(U^{\square}(\eta_X(m)))$$

is a base for the coarsest quasi-uniformity on \mathcal{C} with respect to \mathcal{M} for which every reflection morphism $\eta_X : X \rightarrow FX$ is $(\mathcal{U}^{\mathcal{S}^A}, \mathcal{U}^{\mathcal{S}})$ -continuous. $\mathcal{B}^{\mathcal{S}^A}$ is a transitive base provided that $\mathcal{S}^{F,\eta}$ is interpolative.

While the syntopogenous structure $\mathcal{S}^{F,\eta}$ was obtained with the help of the \mathcal{S} -initial morphism, a generalization of the notion of weakly \square -final morphism to the case of syntopogenous structures gives the syntopogenous structure $\mathcal{S}^{G,\varepsilon}$ induced by a copointed endofunctor (G, ε) on \mathcal{C} .

Theorem 5.1.12. *Let (G, ε) be a \mathcal{M} -copointed endofunctor of \mathcal{C} and \mathcal{S} a syntopogenous structure on \mathcal{C} , then $\mathcal{S}_X^{G,\varepsilon} = \{\square_X^{G,\varepsilon} \mid \square_{GX} \in \mathcal{S}_{GX}\}$ with $m \square_X^{G,\varepsilon} n \Leftrightarrow \varepsilon_X^{-1}(n) \sqsubseteq_{GX} \varepsilon_X^{-1}(m)$ for all $m \in \text{sub}X$ and $n \geq m$, is the finest syntopogenous structure on \mathcal{C} for which every $\varepsilon_X : GX \rightarrow X$ is $(\mathcal{S}, \mathcal{S}^{G,\varepsilon})$ -continuous.*

Proof. (S1) and (S2) are easily seen to be satisfied.

(S3) If $\square_X^{G,\varepsilon} \in \mathcal{S}_X^{G,\varepsilon}$ then there is $\square_{GX} \in \mathcal{S}_{GX}$ such that $\square_{GX} \subseteq \square'_{GX} \circ \square_{GX}$. Now for all $m \leq n$, $m \square_X^{G,\varepsilon} n \Leftrightarrow \varepsilon_X^{-1} \square_{GX} \varepsilon_X^{-1}(n) \Rightarrow \exists p \in \text{sub}GX \mid \varepsilon_X^{-1}(m) \square'_{GX} p \square_{GX} \varepsilon_X^{-1}(n) \Rightarrow \varepsilon_X^{-1}(m) \square'_{GX} \varepsilon_X^{-1}(\varepsilon_X(p)) \square_{GX} \varepsilon_X^{-1}(n) \Leftrightarrow m \square_X^{G,\varepsilon} \varepsilon_X(p) \square_X^{G,\varepsilon} n$. Thus $\square_X^{G,\varepsilon} \subseteq \square_X^{G,\varepsilon} \circ \square_X^{G,\varepsilon}$. Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and $\square_Y^{G,\varepsilon} \in \mathcal{S}_Y^{G,\varepsilon}$. Then for all $m \in \text{sub}X$ and $n \in \text{sub}Y$ such that $f(m) \leq n$, $f(m) \square_Y^{G,\varepsilon} n \Leftrightarrow \varepsilon_Y^{-1}(f(m)) \square_{GY} \varepsilon_Y^{-1}(n) \Rightarrow (Gf)(\varepsilon_X^{-1}(m)) \square_{GY} \varepsilon_Y^{-1}(n) \Rightarrow \exists \square_{GX} \in \mathcal{S}_{GX} \mid \varepsilon_X^{-1}(m) \square_{GX} (Gf)^{-1}(\varepsilon_Y^{-1}(n)) \Rightarrow \varepsilon_X^{-1}(m) \square_{GX} \varepsilon_X^{-1}(f^{-1}(n)) \Rightarrow m \square_X^{G,\varepsilon} f^{-1}(n)$.

For all $X \in \mathcal{C}$, $\varepsilon_X : GX \rightarrow X$ is $(\mathcal{S}, \mathcal{S}^{G,\varepsilon})$ -continuous, since for any $\square_X^{G,\varepsilon} \in \mathcal{S}_X^{G,\varepsilon}$ and $m, n \in \text{sub}X$ with $n \leq m$, $m \square_X^{G,\varepsilon} n \Rightarrow \varepsilon_X^{-1}(n) \square_{GX} \varepsilon_X^{-1}(m)$.

If \mathcal{S}' is another syntopogenous structure on \mathcal{C} such that ε_X is $(\mathcal{S}, \mathcal{S}')$ -continuous, then for any $\square_X \in \mathcal{S}'_X$, $m \square_X n \Rightarrow \varepsilon_X(\varepsilon_X^{-1}(m)) \square_X n \Rightarrow \exists \square_{GX} \in \mathcal{S}_{GX} \mid \varepsilon_X^{-1}(m) \square_X \varepsilon_X^{-1}(n) \Leftrightarrow m \square_X^{G,\varepsilon} n$. \square

Corollary 5.1.13. *Let \mathcal{A} be an \mathcal{M} -coreflective subcategory of \mathcal{C} and \mathcal{S} a syntopogenous structure on \mathcal{A} , then $\mathcal{S}_X^{\mathcal{A}} = \{\square_X^{\mathcal{A}} \mid \square_X \in \mathcal{S}_X\}$ with $m \square_X^{\mathcal{A}} n \Leftrightarrow \varepsilon_X^{-1}(n) \square_{GX} \varepsilon_X^{-1}(m)$ for all $m \in \text{sub}X$ and $n \geq m$, is the finest syntopogenous structure on \mathcal{C} for which every coreflection $\varepsilon_X : GX \rightarrow X$ is $(\mathcal{S}, \mathcal{S}^{\mathcal{A}})$ -continuous.*

Proposition 5.1.14. *Assume that f^{-1} commutes with the join of subobjects for any $f \in \mathcal{C}$ and (G, ε) be an \mathcal{M} -copointed endofunctor of \mathcal{C} and $\mathcal{S} \in \text{CSYNT}(\mathcal{C}, \mathcal{M})$. Then*

$$\mathcal{B}_X^{\mathcal{S}^{G,\varepsilon}} = \{V^{\square_{GX}} \mid V^{\square} \in \mathcal{B}_{GX}^{\mathcal{S}}\} \text{ with } V_X^{\square_{GX}}(m) = m \vee \varepsilon_X(V^{\square}(\varepsilon_X^{-1}(m)))$$

is a base for finest quasi-uniformity on \mathcal{C} which makes every ε_X $(\mathcal{V}, \mathcal{V}^{G,\varepsilon})$ -continuous.

Proof. (U1) is clear.

(U2) Let $V^{\square^{G,\varepsilon}} \in \mathcal{B}_X^{S^{G,\varepsilon}}$, then there is $\square'_{GX} \in \mathcal{S}_{GX}$ such that $\square \subseteq \square' \circ \square'$.

$$\begin{aligned}
\text{Now, } V^{\square'^{G,\varepsilon}}(V^{\square'^{G,\varepsilon}}(m)) &= V^{\square'^{G,\varepsilon}}(m \vee \varepsilon_X(V^{\square'}(\varepsilon_X^{-1}(m)))) \\
&= m \vee \varepsilon_X(V^{\square'}(\varepsilon_X^{-1}(m \vee \varepsilon_X(V^{\square'}(\varepsilon_X^{-1}(m))))) \\
&= m \vee \varepsilon_X(V^{\square'}(\varepsilon_X^{-1}(m) \vee \varepsilon_X^{-1}(\varepsilon_X(V^{\square'}(\varepsilon_X^{-1}(m))))) \\
&= m \vee \varepsilon_X(V^{\square'}(\varepsilon_X^{-1}(m) \vee V^{\square'}(\varepsilon_X^{-1}(m)))) \\
&= m \vee \varepsilon_X(V^{\square'}(V^{\square'}(\varepsilon_X^{-1}(m)))) \\
&\leq m \vee \varepsilon_X(V^{\square}(\varepsilon_X^{-1}(m))) \\
&= V^{\square^{G,\varepsilon}}(m)
\end{aligned}$$

(U4) If $V^{\square^{G,\varepsilon}}, V^{\square'^{G,\varepsilon}} \in \mathcal{B}_X^{S^{G,\varepsilon}}$, then

$$\begin{aligned}
V^{\square^{G,\varepsilon}} \wedge V^{\square'^{G,\varepsilon}} &= [m \vee \varepsilon_X(V^{\square}(\varepsilon_X^{-1}(m)))] \wedge [m \vee \varepsilon_X(V^{\square'}(\varepsilon_X^{-1}(m)))] \\
&= m \vee (\varepsilon_X(V^{\square}(\varepsilon_X^{-1}(m))) \wedge \varepsilon_X(V^{\square'}(\varepsilon_X^{-1}(m)))) && \text{Lemma 1.2.13} \\
&= m \vee (\varepsilon_X(V^{\square}(\varepsilon_X^{-1}(m))) \wedge V^{\square'}(\varepsilon_X^{-1}(m))) \\
&= m \vee (\varepsilon_X((V^{\square} \wedge V^{\square'})(\varepsilon_X^{-1}(m)))) \\
&= V^{(\square \wedge \square')^{G,\varepsilon}}(m)
\end{aligned}$$

Since $V^{\square \wedge \square'} \in \mathcal{B}_X^S, V^{\square^{G,\varepsilon}} \wedge V^{\square'^{G,\varepsilon}} \in \mathcal{B}_X^{S^{G,\varepsilon}}$.

(U5) Let $f : X \rightarrow Y$ be a \mathcal{C} -morphism and $V^{\square^{G,\varepsilon}} \in \mathcal{B}_X^{S^{G,\varepsilon}}$. Then, there is $\square_Y \in \mathcal{S}_Y$ such

that $f(V^{\square_X}(m)) \leq V^{\square_Y}(f(m))$ for all $m \in \text{sub}X$.

$$\begin{aligned}
\text{Now, } f(V^{\square_X^{G,\varepsilon}}(m)) &= f(m \vee \varepsilon_X(V^{\square}(\varepsilon_X^{-1}(m)))) \\
&= f(m) \vee f(\varepsilon_X(V^{\square}(\varepsilon_X^{-1}(m)))) \\
&= f(m) \vee \varepsilon_Y(Gf)(V^{\square}(\varepsilon_X^{-1}(m))) && \text{Definition 2.5.3} \\
&\leq f(m) \vee \varepsilon_Y V^{\square}(Gf)(\varepsilon_X^{-1}(m)) && \mathcal{U}\text{-continuity} \\
&\leq f(m) \vee \varepsilon_Y(V^{\square}(\varepsilon_X^{-1}(f(m)))) && \text{Lemma 1.2.4} \\
&= V^{\square_Y^{G,\varepsilon}}(f(m)).
\end{aligned}$$

Since $\varepsilon_X(V^{\square}(\varepsilon_X^{-1}(m))) \leq V^{\square^{G,\varepsilon}}(m) \Leftrightarrow V^{\square}(\varepsilon_X^{-1}(m)) \leq \varepsilon_X^{-1}(V^{\square^{G,\varepsilon}}(m))$, ε_X is $(\mathcal{V}, \mathcal{V}^{\mathcal{S}^{G,\varepsilon}})$ -continuous.

Let \mathcal{B}' be base for another quasi-uniformity \mathcal{V}' on \mathcal{C} such that ε_X is $(\mathcal{V}, \mathcal{V}')$ -continuous. Then for all $V' \in \mathcal{V}'_X$, there is $V \in \mathcal{V}_{GX}^{\mathcal{S}}$ such $V(\varepsilon_X^{-1}(m)) \leq \varepsilon_X^{-1}(V'(m)) \Leftrightarrow \varepsilon_X(V(\varepsilon_X^{-1}(m))) \leq V'(m) \Rightarrow m \vee \varepsilon_X(V(\varepsilon_X^{-1}(m))) \leq V'(m) \Leftrightarrow V^{\square^{G,\varepsilon}}(m) \leq V'(m)$. Thus $\mathcal{B}' \leq \mathcal{B}^{G,\varepsilon}$. \square

Proposition 5.1.15. *Assume that f^{-1} commutes with the join of subobjects for any $f \in \mathcal{C}$, \mathcal{A} be an \mathcal{M} -coreflective subcategory of \mathcal{C} and \mathcal{S} a syntopogenous on \mathcal{A} . Then*

$$\mathcal{B}_X^{\mathcal{A}} = \{V^{\square^{\mathcal{A}}} \mid V^{\square} \in \mathcal{B}_{GX}^{\mathcal{S}} \text{ with } V^{\square^{\mathcal{A}}}(m) = m \vee \varepsilon_X(V^{\square}(\varepsilon_X^{-1}(m)))\}$$

is a base for finest quasi-uniformity on \mathcal{C} which makes every coreflection morphism ε_X $(\mathcal{V}, \mathcal{V}^{\mathcal{A}})$ -continuous.

5.2 The $(\mathcal{U}, \mathcal{V})$ -continuity of functors

Let \mathcal{A} be a category endowed with an $(\mathcal{E}', \mathcal{M}')$ -factorization system for morphisms and \mathcal{A} be \mathcal{M}' -complete.

Definition 5.2.1. [DT95] *A functor $F : \mathcal{A} \rightarrow \mathcal{C}$ is said to preserve subobjects provided that Fm is an \mathcal{M} -subobject for every \mathcal{M}' -subobject m .*

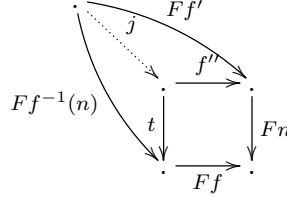
If F preserves subobjects, then for every $X \in \mathcal{A}$, F induces a monotone map $\text{sub}X \rightarrow \text{sub}FX$. Assuming the preservation of subobjects by F allows us to prove the Lemma below.

Lemma 5.2.2. [DT95] *Let $f : X \rightarrow Y$ be an \mathcal{A} -morphism. Then*

(1) $Ff^{-1}(n) \leq (Ff)^{-1}(Fn)$ for all $n \in \text{sub}Y$.

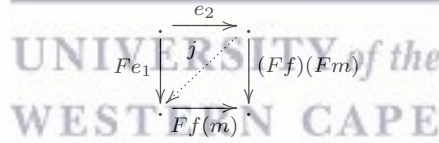
(2) $(Ff)(Fm) \leq F(f(m))$ for all $m \in \text{sub}X$.

Proof. (1) By Definition 1.2.1, $Ff \circ (Ff)^{-1}(n) = Fn \circ f''$ and $f \circ f^{-1}(n) = n \circ f'$ so that $Ff \circ (Ff)^{-1}(n) = F(f \circ f^{-1}(n)) = F(n \circ f') = Fn \circ Ff'$. Thus there is a morphism j which makes the diagram



commute with $t = (Ff)^{-1}(Fn)$. Now $(Ff)^{-1}(Fn) \circ j = Ff^{-1}(n)$, that is $Ff^{-1}(n) \leq (Ff)^{-1}(Fn)$.

(2) From Definition 1.2.2, $f \circ m = f(m) \circ e_1$ for $e_1 \in \mathcal{E}'$, and $Ff \circ Fm = (Ff)(Fm) \circ e_2$ with $e_2 \in \mathcal{E}$. So $Ff \circ Fm = F(f \circ m) = F(f(m) \circ e_1) = Ff(m) \circ Fe_1$. The diagonalization property implies the existence of a morphism j making the the diagram below commute.



$Ff \circ j = (Ff)(Fm)$, that is $(Ff)(Fm) \leq Ff(m)$.

□

It is clear from the proof of the above Lemma that $Ff^{-1}(n) = (Ff)^{-1}(Fn)$ if F preserves pullbacks along \mathcal{M} -morphisms. In this case we say that F preserves inverse images. Similarly, $(Ff)(Fm) = Ff(m)$ if $Fe \in \mathcal{E}$. In this case, we say that F preserves images.

Definition 5.2.3. Let $F : \mathcal{A} \rightarrow \mathcal{C}$ be a functor that preseves subobjects, $\mathcal{U} \in QUNIF(\mathcal{A}, \mathcal{M}')$ and $\mathcal{V} \in QUNIF(\mathcal{C}, \mathcal{M})$. Then F is $(\mathcal{U}, \mathcal{V})$ -continuous if for all $V \in \mathcal{V}_{FX}$, there is $U \in \mathcal{U}_X$ such that $FU(m) \leq V(Fm)$ for all $m \in \text{sub}X$, $X \in \mathcal{A}$.

It can be easily seen that our definition for $(\mathcal{U}, \mathcal{V})$ -continuity of F is a generalization of \mathcal{U} -continuity of morphisms to functors. Using Theorem 3.2.14, we can formulate an equivalent definition of the $(\mathcal{U}, \mathcal{V})$ -continuity of F in terms of co-perfect syntopogenous

structures so that F is $(\mathcal{S}, \mathcal{S}')$ -continuous will mean that F is continuous with respect to the quasi-uniform structures associated with \mathcal{S} and \mathcal{S}' .

Definition 5.2.4. Let $F : \mathcal{A} \rightarrow \mathcal{C}$ be a functor that preseves subobjects, $\mathcal{S} \in CSYNT(\mathcal{A}, \mathcal{M}')$ and $\mathcal{S} \in CSYNT(\mathcal{C}, \mathcal{M})$. Then F is $(\mathcal{S}, \mathcal{S}')$ -continuous if for all $\sqsubset'_{FX} \in \mathcal{S}'_{FX}$, there is $\sqsubset_X \in \mathcal{S}_X$ such that $FU^{\sqsubset}(m) \leq U^{\sqsubset'}(Fm)$ for all $m \in \text{sub}X$, $X \in \mathcal{A}$.

Continuity of a functor between categories supplied with fixed closure operators has been studied in [DT95]. We next use the above proposition together with Corollary 2.1.6 and the fact that $\bigwedge -INTORD(\mathcal{C}, \mathcal{M})$ is equivalent to the simple co-perfect syntopogenous structures to produce the $(\mathcal{U}, \mathcal{V})$ -continuity of F in terms of idempotent closure operators.

Proposition 5.2.5. Let $F : \mathcal{A} \rightarrow \mathcal{C}$ be a functor that preseves subobjects, $\mathcal{S} \in CSYNT(\mathcal{A}, \mathcal{M}')$ and $\mathcal{S} \in CSYNT(\mathcal{C}, \mathcal{M})$ with \mathcal{S} and \mathcal{S}' being simple i.e $\mathcal{S}_X = \{\sqsubset_X\}$ and $\mathcal{S}'_{FX} = \{\sqsubset'_{FX}\}$. Then F is $(\mathcal{S}, \mathcal{S}')$ -continuous if and only if for all $Fc_X^{\sqsubset}(m) \leq c_{FX}^{\sqsubset'}(Fm)$ for all $m \in \text{sub}X$, $X \in \mathcal{A}$.

We next prove some properties for the $(\mathcal{U}, \mathcal{V})$ -continuity of F that will be useful in what follows.

Proposition 5.2.6. (1) For any $\mathcal{U} \in QUNIF(\mathcal{A}, \mathcal{M}')$, $Id_{\mathcal{A}}$ is $(\mathcal{U}, \mathcal{U})$ -continuous.

(2) If F is $(\mathcal{U}, \mathcal{V})$ -continuous and $G : \mathcal{C} \rightarrow \mathcal{D}$ a $(\mathcal{V}, \mathcal{W})$ -continuous functor that preserves subobjects where \mathcal{W} is a quasi-uniformity on \mathcal{D} with respect to a class \mathcal{L} of monomorphisms of \mathcal{D} , then GF is $(\mathcal{U}, \mathcal{W})$ -continuous.

(3) Let $\mathcal{U} \leq \mathcal{U}'$ in $QUNIF(\mathcal{A}, \mathcal{M}')$ and $\mathcal{V}' \leq \mathcal{V}$ in $QUNIF(\mathcal{C}, \mathcal{M})$. Then F is $(\mathcal{U}, \mathcal{V})$ -continuous implies that it is $(\mathcal{U}', \mathcal{V}')$ -continuous.

Proof. (1) is obvious.

(2) Let $W \in \mathcal{W}_{GFx}$ for $X \in \mathcal{A}$. By $(\mathcal{V}, \mathcal{W})$ -continuity of G , there is $V \in \mathcal{V}_{FX}$ such that $GV(n) \leq W(Gn)$ for all $n \in \text{sub}FX$. Since F is $(\mathcal{U}, \mathcal{V})$ -continuous, there is $U \in \mathcal{U}_X$ such that $FU(m) \leq V(Fm)$ for all $m \in \text{sub}X$. Thus $GFU(m) \leq GV(Fm) \leq W(GFm)$, that is GF is $(\mathcal{U}, \mathcal{W})$ -continuous.

(3) Assume F is $(\mathcal{U}, \mathcal{V})$ -continuous and $V' \in \mathcal{V}'_{FX}$. Since $\mathcal{V}' \leq \mathcal{V}$, there is $V \in \mathcal{V}_{FX}$ such that $V \leq V'$ and there is $U \in \mathcal{U}_X$ such $FU(m) \leq V(Fm) \leq V'(Fm)$ for all $m \in \text{sub}X$. From $\mathcal{U} \leq \mathcal{U}'$, we get $U' \in \mathcal{U}_X$ such that $U' \leq U$ and $FU'(m) \leq FU(m) \leq V(Fm) \leq V'(Fm)$. Thus F is $(\mathcal{U}', \mathcal{V}')$ -continuous. \square

5.3 Lifting a quasi-uniformity along an \mathcal{M} -fibration

We prove, in this section, that if $F : \mathcal{A} \rightarrow \mathcal{C}$ is an \mathcal{M} -fibration and \mathcal{B} is a base for a quasi-uniformity on \mathcal{C} , there is a coarsest quasi-uniformity \mathcal{U}^F on \mathcal{A} such that F is $(\mathcal{U}^F, \mathcal{U})$ -continuous. We then use the syntopogenous structure to deduce the lifted idempotent closure operator, which turns out to be the largest one for which F is $(c^{\square^F}, c^{\square})$ -continuous.

Proposition 5.3.1. *Let $F : \mathcal{A} \rightarrow \mathcal{C}$ be a faithful \mathcal{M} -fibration and \mathcal{S} be a syntopogenous structure on \mathcal{C} with respect to \mathcal{M} . Then*

$$\mathcal{S}_X^F = \{\square_X^F \mid \square_{FX} \in \mathcal{S}_{FX}\} \text{ where } m \square_X^F n \Leftrightarrow Fm \square_{FX} \gamma_X(n)$$

is a syntopogenous structure on \mathcal{A} with respect to \mathcal{M}_F which is interpolative, co-perfect provided \mathcal{S} has the same property. Moreover, an \mathcal{A} -morphism is \mathcal{S}^F -initial provided Ff is \mathcal{S} -initial.

Proof. (S1) follows Proposition 2.4.2. For (S2), we let $\square^F, \square'^F \in \mathcal{S}^F$ for $\square_{FX}, \square'_{FX} \in \mathcal{S}_{FX}$. Then there is \square''_{FX} such that $\square_{FX} \subseteq \square''_{FX}$ and $\square'_{FX} \subseteq \square''_{FX}$. Thus $\square_X^F \subseteq \square_X''^F$ and $\square_X'^F \subseteq \square_X''^F$. (S3) Let $m \square_X^F n$ for $\square_{FX} \in \mathcal{S}_{FX}$ and $m, n \in \text{sub}X$. Then there is $p \in \text{sub}FX$ and $\square'_{FX} \in \mathcal{S}_{FX}$ such that $Fm \square'_{FX} p \square'_{FX} \gamma_X(n) \Leftrightarrow Fm \square'_{FX} F\delta_X(p) \square'_{FX} Fn \Leftrightarrow m \square_X'^F \delta_X(p) \square'^F n$. Let $f : X \rightarrow Y$ and $\square_Y^F \in \mathcal{S}_Y^F$. Then $f(m) \square_Y^F n \Leftrightarrow F(f(m)) \square_{FY} \gamma_Y(n) \Leftrightarrow (Ff)(Fm) \square_{FY} \gamma_Y(n) \Leftrightarrow Fm \square_{FX} (Ff)^{-1}(\gamma_Y(n)) = \gamma_X(f^{-1}(n)) \Leftrightarrow m \square_X^F f^{-1}(n)$.

If \mathcal{S} is co-perfect, then for all $i \in I, m \square_X^F n_i \Leftrightarrow Fm \square_{FX} \gamma_X(n_i) \Rightarrow Fm \square_{FX} \bigwedge_{i \in I} \gamma_X(n_i) \Leftrightarrow Fm \square_{FX} \gamma(\delta_X(\bigwedge_{i \in I} \gamma_X(n_i))) = \gamma_X(\bigwedge_{i \in I} \delta_X(\gamma_X(n_i))) = \gamma_X(\bigwedge_{i \in I} n_i)$.

Interpolation of \mathcal{S}^F follows from Proposition 2.4.2

Assume that Ff is \mathcal{S} -initial and $\square_X^F \in \mathcal{S}_X$. Then for all $m, n \in \text{sub}X, m \square_X^F n \Leftrightarrow Fm \square_{FX} Fn$. This implies that there is $\square_{FY} \in \mathcal{S}_{FY}$ such that $(Ff)(Fm) \square_{FY} p$ and $(Ff)^{-1}(p) \leq Fn$. Since F preserves images, $Ff(m) \square_{FY} p$ and $\delta_X(Ff)^{-1}(p) \leq n$. Now, F preserves preimages, $Ff(m) \square_{FY} F(\delta_Y(p))$ and $f^{-1}(\delta_Y(p)) \leq n$. Thus $f(m) \square_Y^F \delta_Y(p)$ and $f^{-1}(\delta_Y(p)) \leq n$, that is f is \mathcal{S}^F -initial. \square

Although our next proposition can be obtained from Theorem 3.2.15 and Proposition 5.3.1, we provide a direct proof (without use of syntopogenous structures) as we want to

describe \mathcal{U}^F .

Theorem 5.3.2. *Let $F : \mathcal{A} \longrightarrow \mathcal{C}$ be a faithful \mathcal{M} -fibration and \mathcal{B} be a base for a quasi-uniform structure on \mathcal{C} with respect to \mathcal{M} . Then*

(i) $\mathcal{B}_X^F = \{U^F \mid U \in \mathcal{U}_{FX}\}$ where $U^F(m) = \delta_X(U(Fm))$ is a base for a quasi-uniformity on \mathcal{A} with respect to \mathcal{M}_F .

(ii) \mathcal{B}^F is transitive provided \mathcal{B} is a transitive base.

(iii) \mathcal{B}^F is a base for the coarsest quasi-uniformity for which F is $(\mathcal{U}^F, \mathcal{U})$ -continuous.

(iv) An \mathcal{A} -morphism f is \mathcal{U}^F -initial provided Ff is \mathcal{S} -initial.

Proof. (i) (U1) $m \leq \gamma_X(U(Fm))$ since $m = \gamma_X(Fm) \leq \gamma_X(U(Fm))$.

(U2) If $U_X^F \in \mathcal{B}_X^F$ for $U \in \mathcal{U}_{FX}$, then there is $V \in \mathcal{B}_{FX}$ such $V \circ V \leq U$.

$$\begin{aligned} \text{Now } V^F(V^F(m)) &= V^F(\delta_X(V(Fm))) \\ &= \delta_X(V(F(\delta_X V(Fm)))) \\ &= \delta_X(V(V(Fm))) \\ &\leq \delta_X(U(Fm)) \\ &= U^F(m). \end{aligned}$$

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(U4) If $U^F, V^F \in \mathcal{B}_{FX}$ for $U, V \in \mathcal{B}_{FX}$, then $U \wedge V \in \mathcal{B}_{FX}$. So $U^F(m) \wedge V^F(m) = \delta_X(U(Fm)) \wedge \delta_X(V(Fm)) = \delta_X(U(Fm) \wedge V(Fm)) = \delta_X((U \wedge V)(Fm))$.

Thus $U^F \wedge V^F \in \mathcal{B}_X^F$.

(U5) Let $f : X \longrightarrow Y$ be a \mathcal{A} -morphism and $U^F \in \mathcal{B}_Y^F$ for $U \in \mathcal{B}_{FY}$.

$$\begin{aligned} \text{Thus } f(U^F(m)) &= f(\delta_X(U(Fm))) \\ &= \delta_Y((Ff)(U(Fm))) \\ &\leq \delta_Y(V(Ff)(Fm)) && \text{for some } V \in \mathcal{B}_{FX} \\ &= \delta_Y(V(Ff(m))) \\ &= V^F(f(m)) \end{aligned}$$

(ii) If \mathcal{B} is a transitive base, then for all $U \in \mathcal{B}_{FX}$

$$\begin{aligned}
U^F(U^F(m)) &= U^F(\delta_X(U(Fm))) \\
&= \delta_X(U(F(\delta_X(U(Fm)))) \\
&= \delta_X(U(U(Fm))) \\
&= \delta_X(U(Fm)) \\
&= U^F(m)
\end{aligned}$$

(iii) F is $(\mathcal{U}^F, \mathcal{U})$ -continuous, since for any $U \in \mathcal{B}_{FX}$, $U^F(m) = \delta_X(U(Fm))$

$$\Leftrightarrow \gamma_X(U^F(m)) = U(Fm) \Leftrightarrow F(U^F(m)) = U(Fm).$$

If \mathcal{B}' is a base for another quasi-uniformity \mathcal{U}' on \mathcal{A} such that F is $(\mathcal{U}', \mathcal{U})$ -continuous, then for all $U^F \in \mathcal{B}_X^F$, there is $U' \in \mathcal{B}'$ such that $FU'(m) \leq U(Fm) = FU^F(m)$. Thus $U'(m) = \delta_X(FU'(m)) \leq \delta_X(FU^F(m)) = U^F(m)$, that is $\mathcal{B}^F \leq \mathcal{B}'$.

(iv) If Ff is \mathcal{U} -initial and $U^F \in \mathcal{U}_X^F$, there is $U' \in \mathcal{U}_{FY}$ such that $(Ff)^{-1}(U'(Ff)(p)) \leq U(p)$ for all $p \in \text{sub}FX$. Now $f^{-1}(\delta_Y(U'(Ff(m)))) = \delta_X((Ff)^{-1}(U'(Ff(m)))) = \delta_X((Ff)^{-1}(U'((Ff)(Fm)))) \leq \delta_X(U(Fm)) = U^F(m)$ for all $m \in \text{sub}X$. \square

Corollary 5.3.3. *Under the assumptions of Theorem 5.3.2 and F is essentially surjective on objects, \mathcal{B} is the base of the finest quasi-uniformity on \mathcal{C} for which F is $(\mathcal{U}^F, \mathcal{U})$ -continuous.*

Proof. By essential surjectivity of F on objects, we have that for all $Y \in \mathcal{C}$, $Y \cong FX$ for some $X \in \mathcal{A}$. Thus if \mathcal{B}' is another quasi-uniformity on \mathcal{C} such that F is $(\mathcal{U}^F, \mathcal{U}')$ -continuous, then for all $Y \in \mathcal{C}$ and $U' \in \mathcal{U}'_Y$, there is $X \in \mathcal{A}$ and $U^F \in \mathcal{B}^F$ such that $Y = FX$ and $FU^F(m) \leq U'(Fm) \Leftrightarrow U(Fm) = F\delta_X(U(Fm)) \leq U'(Fm) = U'(Fm)$. Thus $\mathcal{B}' \leq \mathcal{B}$. \square

Proposition 5.3.4. *Let $F : \mathcal{A} \rightarrow \mathcal{C}$ be faithful \mathcal{M} -fibration and \mathcal{S} be a simple co-perfect syntopogenous structure on \mathcal{C} with respect to \mathcal{M} i.e $\mathcal{S} = \{\square_X\} \in \bigwedge -\text{INTORD}$. Then $c^{\square^F}(m) = \delta_X(c^{\square}(Fm))$ is an idempotent closure operator on \mathcal{A} with respect to \mathcal{M}_F . It is the largest closure operator on \mathcal{A} for which F is $(c^{\square^F}, c^{\square})$ -continuous.*

Proof. (C1) and (C2) are easily seen to be satisfied. For (C3), let $f : X \rightarrow Y$ be an \mathcal{A} -morphism and $m \in \text{sub}Y$.

$$\begin{aligned}
\text{Then } f(c_X^{\square^F}(m)) &= f(\delta_X(c_{FX}^{\square}(Fm))) \\
&= \delta_Y(Ff)(c_{FX}^{\square}(Fm)) \\
&\leq \delta_Y(c_{FY}^{\square}(Ff(m))) \\
&= \delta_Y(c_{FY}^{\square}(Ff(m))) \\
&= c_Y^{\square^F}(f(m))
\end{aligned}$$

F is $(c^{\square^F}, c^{\square})$ -continuous since, $\gamma_X(c^{\square^F}(m)) = c^{\square}(Fm) \Leftrightarrow Fc^{\square^F}(m) = c^{\square}(Fm)$.

$$\begin{aligned}
\text{Now, } c^{\square^F}(c^{\square^F}(m)) &= c^{\square^F}(\delta_X(c_{FX}^{\square}(Fm))) \\
&= \delta_X(c_{FX}^{\square}(F\delta_X(c^{\square}(Fm)))) \\
&= \delta_X(c_{FX}^{\square}(c_{FX}^{\square}(Fm))) \\
&= \delta_X(c_{FX}^{\square}(Fm)) = c^{\square^F}(m)
\end{aligned}$$

If c' is another closure operator on \mathcal{A} such that F is (c', c^{\square}) -continuous, then $Fc_X(m) \leq c^{\square}(Fm)$. Thus $c'_X(m) = \delta_X(F(c_X(m))) \leq \delta_X(c_{FX}^{\square}(Fm)) = c_X^{\square^F}(m)$. \square

5.4 Quasi-uniform structures and adjoint functors

We start by recalling the definition of adjoint functors.

Let \mathcal{A} and \mathcal{C} be categories. An adjunction from \mathcal{A} to \mathcal{C} consists of functors $F : \mathcal{A} \rightarrow \mathcal{C}$, $G : \mathcal{C} \rightarrow \mathcal{A}$ and a natural transformation $\eta : 1_{\mathcal{A}} \rightarrow GF$ such that for all $X \in \mathcal{A}$, $\langle FX, \eta_X \rangle$ is a G -universal arrow with domain X i.e for any $Y \in \mathcal{C}$ and $f : X \rightarrow GY \in \mathcal{A}$, there is a unique $\bar{f} : FX \rightarrow Y$ such that the following diagram commutes.

$$\begin{array}{ccc}
X & \xrightarrow{\eta_X} & GFX \\
& \searrow f & \swarrow G\bar{f} \\
& & GY
\end{array}$$

Thus the correspondance

$$\frac{X \rightarrow GY}{FX \rightarrow Y}$$

is inverse to the map $\phi_{X,Y} : \mathcal{C}(FX, Y) \rightarrow \mathcal{A}(X, GY)$ given by $g \mapsto G\bar{f} \circ \eta_X$ and the existence of a G -universal arrow for X therefore gives a natural isomorphism $\phi_{X,-} : \mathcal{C}(FX, -) \rightarrow \mathcal{A}(X, G-)$ so that $\mathcal{A}(X, G-)$ is a representable functor. Conversely, the representability of $\mathcal{A}(X, G-)$ gives a G -universal arrow for X with FX the representing object and with an isomorphism as above, $\eta_X = \phi_{X, FX}(1_{FX})$.

F is called the left adjoint, G the right adjoint and for the rest of the paper we shall simply write $F \dashv G : \mathcal{C} \rightarrow \mathcal{A}$.

Lemma 5.4.1. [DT95] *Let $F \dashv G : \mathcal{C} \rightarrow \mathcal{A}$ be adjoint functors. Then $GM \subseteq \mathcal{M}'$ if and only if $F\mathcal{E}' \subseteq \mathcal{E}$.*

Theorem 5.4.2. *Let $F \dashv G : \mathcal{A} \rightarrow \mathcal{C}$ be adjoint functors and \mathcal{B} be a base for a quasi-uniformity $\mathcal{U} \in QUNIF(\mathcal{A}, \mathcal{M}')$. Assume that G and F preserves subobjects. Then $\mathcal{B}_X^\eta = \{U^\eta \mid U \in \mathcal{B}_{FX}\}$ with $U^\eta(m) = \eta_X^{-1}(GU(Fm))$ for any $X \in \mathcal{C}$ is a base for quasi-uniformity on \mathcal{C} . \mathcal{B}^η is a base for the coarsest quasi-uniformity for which F is $(\mathcal{U}^\eta, \mathcal{U})$ -continuous.*

Proof. Let us first note that for any $U \in \mathcal{U}_{FX}$, we have the diagram below.

$$\begin{array}{ccc}
 M & \xrightarrow{\eta_M} & GFm \\
 \downarrow k & & \downarrow j \\
 U^\eta(M) & \xrightarrow{g} & GU(Fm) \\
 \downarrow m & & \downarrow GFm \\
 X & \xrightarrow{\eta_X} & GFX \\
 & \swarrow U^\eta(m) & \searrow GU(Fm)
 \end{array}$$

By adjointness, $g : U^\eta(M) \rightarrow GU(Fm)$ corresponds to a morphism $\bar{g} : FU^\eta(M) \rightarrow U(Fm)$ such that the following diagram commutes.

$$\begin{array}{ccc}
 FU^\eta M & \xrightarrow{\bar{g}} & U(Fm) \\
 \downarrow FU^\eta(m) & & \downarrow U(Fm) \\
 FX & & FX
 \end{array}$$

So $U(Fm) \circ \bar{g} = FU^\eta(m) \Leftrightarrow FU^\eta(m) \leq U(Fm)$. (U1) follows from the diagram below. For (U2), let $U^\eta \in \mathcal{U}_X^\eta$, then $U^\eta(m) = \eta_X^{-1}(GU(Fm))$ for some $U \in \mathcal{B}_{FX}$ and there is $V \in \mathcal{U}_{FX}$ such that $V \circ V \leq U$. Since, $FV^\eta(m) \leq V(Fm)$; $V(FV^\eta(m)) \leq V(V(Fm)) \leq U(Fm) \Rightarrow V(FV^\eta(m)) \leq U(Fm) \Rightarrow G(V(FV^\eta(m))) \leq G(U(Fm)) \Rightarrow \eta_X^{-1}(GV(FV^\eta(m))) \leq \eta_X^{-1}(U(Fm)) \Leftrightarrow V^\eta(V^\eta(m)) \leq U^\eta(m)$.

(U3) Let $U^\eta, V^\eta \in \mathcal{U}^\eta$ and $m \in \text{sub}X$. Since, G preserves inverse images,

$$\begin{aligned}
U^\eta(m) \wedge V^\eta(m) &= \eta_X^{-1}(GU(Fm)) \wedge \eta_X^{-1}(GV(Fm)) \\
&= \eta_X^{-1}(GU(Fm) \wedge GV(Fm)) \\
&= \eta_X^{-1}(G(U(Fm) \wedge V(Fm))) \\
&= \eta_X^{-1}(G(U \wedge V)(Fm)).
\end{aligned}$$

Since, $U \wedge V \in \mathcal{U}_X, U^\eta \wedge V^\eta \in \mathcal{U}^\eta$.

(U5) Let $X \rightarrow Y$ be a \mathcal{C} -morphism and $U^\eta \in \mathcal{U}^\eta$ for any $U \in \mathcal{U}_Y$. Then there is $V \in \mathcal{U}_X$ such that $f(V(m)) \leq U(f(m))$.

$$\begin{aligned}
\text{Thus } f(V^\eta(m)) &= f(\eta_X^{-1}(GV(Fm))) \\
&\leq \eta_Y^{-1}(GFf)(GV(Fm)) && \text{Lemma 1.2.4} \\
&\leq \eta_X^{-1}(G(Ff)(V(Fm))) \\
&\leq \eta_Y^{-1}(GU((Ff)(Fm))) && \mathcal{U}\text{-continuity of } Ff \\
&= \eta_Y^{-1}(GU(Ff(m))) && \text{Lemma 5.4.1} \\
&= U^\eta(f(m)).
\end{aligned}$$

F is $(\mathcal{U}^\eta, \mathcal{U})$ -continuous, since for any $U \in \mathcal{U}_{FX}, FU^\eta(m) \leq U(Fm)$ for any $X \in \mathcal{C}$. Let \mathcal{B}' be a base for another quasi-uniformity \mathcal{U}' on \mathcal{C} such that F is $(\mathcal{U}', \mathcal{U})$ -continuous. Then for any $U^\eta \in \mathcal{B}'_X$, there is $U' \in \mathcal{B}'_X$ such that $FU'(m) \leq U(Fm)$. Thus $\eta_X(U'(m)) \leq GFU'(m) \leq GU(Fm) \Rightarrow \eta_X(U'(m)) \leq GU(Fm) \Leftrightarrow U'(m) \leq \eta_X^{-1}(GU(Fm)) = U^\eta(m)$, that is $\mathcal{U}^\eta \leq \mathcal{U}'$. \square

Theorem 5.4.3. *Let $F \dashv G : \mathcal{C} \rightarrow \mathcal{A}$ be adjoint functors and \mathcal{B} be a base for $\mathcal{U} \in \text{QUNIF}(\mathcal{C}, \mathcal{M})$. Then the assignment $\mathcal{U} \mapsto \mathcal{U}^\eta$ preserves all joins and transitivity. Moreover,*

$$\text{QUNIF}(\mathcal{A}, \mathcal{M}') \begin{array}{c} \xleftarrow{(-)^\eta} \\ \perp \\ \xrightarrow{(-)_\eta} \end{array} \text{QUNIF}(\mathcal{C}, \mathcal{M})$$

with $\mathcal{V}_{(\eta)} = \bigvee \{ \mathcal{U} \in \text{QUNIF}(\mathcal{C}, \mathcal{M}) : \mathcal{U}^\eta \leq \mathcal{V} \}$. In particular, F is $(\mathcal{U}^\eta, \mathcal{U})$ and $(\mathcal{V}, \mathcal{U}^\eta)$ -continuous for all $\mathcal{U} \in \text{QUNIF}(\mathcal{C}, \mathcal{M})$ and $\mathcal{V} \in \text{QUNIF}(\mathcal{A}, \mathcal{M}')$.

Proof. One proceeds as in the proof of Theorem 5.1.8 to show that $\mathcal{U} \mapsto \mathcal{U}^\eta$ preserves all joins. Assume that \mathcal{U} is transitive. Since $FU^\eta(m) \leq U(Fm)$ for any $U \in \mathcal{B}_{FX}$, $U^\eta(U^\eta(m)) = \eta_X^{-1}(GU(FU^\eta(m))) \leq \eta_X^{-1}(GU(U(Fm))) = \eta_X^{-1}(GU(Fm)) = U^\eta(m)$. F is clearly $(\mathcal{U}^\eta, \mathcal{U})$ -continuous. Now, let $\mathcal{V} \in QUNIF(\mathcal{A}, \mathcal{M}')$ and $\mathcal{U} \in QUNIF(\mathcal{C}, \mathcal{M})$ such that $\mathcal{U}^\eta \leq \mathcal{V}$. Then F is $(\mathcal{V}, \mathcal{U})$ -continuous and so by Lemma 5.2.6(3), F is $(\mathcal{V}, \mathcal{V}_\eta)$ -continuous since $\mathcal{U} \leq \mathcal{V}_\eta$. □

Proposition 5.4.4. *Let $F \dashv G : \mathcal{A} \rightarrow \mathcal{C}$ be adjoint functors and $\mathcal{S} \in CSYNT(\mathcal{A}, \mathcal{M}')$. Assume that G and F preserve subobjects.*

Then $\mathcal{S}^\eta = \{\sqsubset_X^\eta \mid \sqsubset_{FX} \in \mathcal{S}_{FX}\}$ with $m \sqsubset_X^\eta n \Leftrightarrow \eta_X^{-1}(GU^\square(Fm)) \leq n$ is a co-perfect syntopogenous structure. It is the coarsest syntopogenous structure for which F is $(\mathcal{S}^\eta, \mathcal{S})$ -continuous.

Proposition 5.4.5. *Under the assumptions of Proposition above, if $\mathcal{S} \in CSYNT(\mathcal{A}, \mathcal{M}')$ and simple i.e $\mathcal{S} = \{\sqsubset_X\} \in \bigwedge -INTORD(\mathcal{A}, \mathcal{M}') \cong ICL(\mathcal{A}, \mathcal{M}')$. Then $c_X^{\sqsubset_X^\eta}(m) = \eta_X^{-1}(Gc_{FX}^\square(Fm))$ is an idempotent closure operator. It is the largest closure operator c for which F is (c, c^\square) -continuous.*

One obtains quasi-uniform structures induced by pointed and copointed endofunctors (F, η) and (G, ε) as follows. $\bar{\mathcal{U}}_X^{F, \eta} = \{\bar{U}^{F, \eta} \mid U \in \mathcal{U}_{FX}\}$ where $\bar{U}^{F, \eta}(m) = \eta_X^{-1}(U(Fm))$ and $\bar{\mathcal{U}}_X^{G, \varepsilon} = \{\bar{U}^{G, \varepsilon} \mid U \in \mathcal{U}_{FX}\}$ with $\bar{U}^{G, \varepsilon}(m) = m \vee \varepsilon_X(U(Gm))$ for any $X \in \mathcal{C}$. It is not hard to see that $\bar{\mathcal{U}}^{F, \eta}$ and $\bar{\mathcal{U}}^{G, \varepsilon}$ are quasi-uniform structures on \mathcal{C} . Moreover,

Proposition 5.4.6. *The following holds true.*

- (1) $\bar{\mathcal{U}}^{F, \eta} \leq \mathcal{U}^{F, \eta}$;
- (2) $\mathcal{U}^{G, \varepsilon} \leq \bar{\mathcal{U}}^{G, \varepsilon}$;
- (3) If $\eta_X \in \mathcal{E}$, then $\bar{\mathcal{U}}^{F, \eta} = \mathcal{U}^{F, \eta}$.

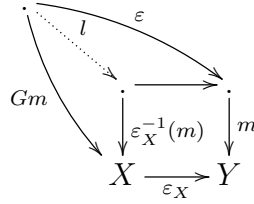
Proof. (1) Since for any $X \in \mathcal{C}$, the diagram commutes

$$\begin{array}{ccc} \cdot & \xrightarrow{e} & \cdot \\ \eta \downarrow & & \downarrow \eta_X(m) \\ \cdot & \xrightarrow{Fm} & \cdot \end{array}$$

by the diagonalization property of the $(\mathcal{E}, \mathcal{M})$ -factorization, there is t such that $Fm \circ t = \eta_X \circ e$, that is, $\eta_X(m) \leq Fm$. Thus for any $\bar{U}^{F, \eta} \in \bar{\mathcal{U}}_X^{F, \eta}$, there is $U^{F, \eta} \in \mathcal{U}_X^{F, \eta}$ such that

$$U^{F,\eta}(m) \leq \bar{U}^{F,\eta}(m).$$

(2) The pullback



commutes so that $\varepsilon_X^{-1}(m) \circ l = Gm \Leftrightarrow Gm \leq \varepsilon_X^{-1}(m)$. Thus for any $U^{G,\varepsilon} \in \mathcal{U}_X^{G,\varepsilon}$, there is $\bar{U}^{G,\varepsilon} \in \bar{\mathcal{U}}_X^{G,\varepsilon}$ such that $\bar{U}^{G,\varepsilon}(m) \leq U^{G,\varepsilon}(m)$.

□

It is well known that a full subcategory of \mathcal{C} is reflective if and only if the inclusion functor $I : \mathcal{A} \hookrightarrow \mathcal{C}$ has a left adjoint F , called a reflector. Since the reflector can also be viewed as a pointed endofunctor of \mathcal{C} , we have the following.

Proposition 5.4.7. *Let \mathcal{A} be a reflective subcategory of \mathcal{C} and \mathcal{B} be a base of a quasi-uniformity on \mathcal{C} . Then*

- (1) For any $X \in \mathcal{C}$, $\mathcal{B}_X^{\mathcal{A}} \leq \mathcal{B}_X^{\eta}$.
- (2) If \mathcal{A} is \mathcal{E} -reflective, then $\mathcal{B}_X^{\mathcal{A}} = \mathcal{B}_X^{\eta}$.

5.5 Examples

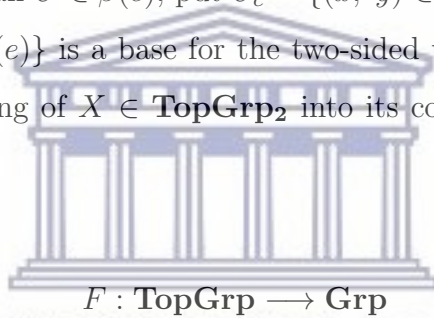
1. Let \mathbf{QUnif}_o be the category of T_o quasi-uniform spaces and quasi-uniformly continuous maps with (surjective, embeddings)-factorization system. It is known that \mathbf{bQUnif}_o (see e.g [Brü99]), the category of bicomplete quasi-uniform spaces and quasi-uniformly continuous maps is an epi-reflective subcategory of \mathbf{QUnif}_o . Let (F, η) be the bicompletion reflector into \mathbf{QUnif}_o . For any $(X, \mathcal{U}) \in \mathbf{QUnif}_o$, $\eta_X : (X, \mathcal{U}) \longrightarrow (\tilde{X}, \tilde{\mathcal{U}})$ takes each $x \in X$ to its neighbourhood filter in the topology induced by the join of \mathcal{U} and its inverse. It is known that η_X is a quasi-uniform embedding. Details about this can be found in [?]. Now, $\mathcal{B}^{F,\eta} = \{U^{F,\eta} \mid \tilde{U} \in \tilde{\mathcal{U}}_{\tilde{X}}\}$ where $U^{F,\eta} = \{(x, y) \in X \times X \mid (\eta_X(x), \eta_X(y)) \in \tilde{U}\}$ is a base of for the quasi-uniform structure $\mathcal{U}^{F,\eta}$ on X . Since η_X is quasi-uniform embedding, \mathcal{U}_X is the initial

quasi-uniformity for which η_X is quasi-uniformly continuous. Thus $\mathcal{U}_X^{F,\eta} = \mathcal{U}_X$.

2. The category **Unif** of uniform spaces and quasi-uniformly continuous maps is coreflective in **Qunif**. Let (G, ε) be the coreflector into **Unif**. For any $(X, \mathcal{U}) \in \mathbf{QUnif}$, $\varepsilon_X : (X, \mathcal{U} \vee \mathcal{U}^{-1}) \rightarrow (X, \mathcal{U})$ is an identity map. Since $\mathcal{U} \vee \mathcal{U}^{-1}$ is the finest quasi-uniformity on X for which ε_X is quasi-uniformly continuous, $\mathcal{U}_X^{G,\varepsilon} = \mathcal{U} \vee \mathcal{U}^{-1}$

3. Consider **TopGrp₂** the category of Hausdorff topological groups and continuous group homomorphisms with the (surjective, injective)-factorization structure. We know from [Bou66] that the category **cTopGrp₂** of complete Hausdorff topological groups (those topological groups which are complete with respect to the two-sided uniformity) is coreflective in **TopGrp₂**. Let (F, η) be the completion reflector into **TopGrp₂** and for any $X \in \mathbf{cTopGrp}$, let $\beta(e)$ be the neighbourhood filter of the identity element e . For all $U \in \beta(e)$, put $U_c = \{(x, y) \in X \times X : y \in xU \cap Ux\}$ so that $\mathcal{B}_X^c = \{U^c \mid U \in \beta(e)\}$ is a base for the two-sided uniformity \mathcal{U}^c on X . Since η_X is again an embedding of $X \in \mathbf{TopGrp}_2$ into its completion \tilde{X} , we have that $\mathcal{U}^{F,\eta} = \mathcal{U}_c$.

4. The forgetful functor



$$F : \mathbf{TopGrp} \rightarrow \mathbf{Grp}$$

is a mono-fibration. Thus by Proposition 5.3.1, every syntopogenous structure on **Grp** can be initially lifted to a syntopogenous structure on **TopGrp**.

5. Consider the functors $G : \mathbf{Qunif} \rightarrow \mathbf{Top}$ which sends every quasi-uniform space (X, \mathcal{U}) to the topological space $(X, G(\mathcal{U}))$ with $G(\mathcal{U})$, the topology induced by \mathcal{U} , obtained by taking as base of neighbourhoods at a point x the filter $\{U[x] \mid U \in \mathcal{U}\}$ where $U[x] = \{y \in X : (x, y) \in U\}$ and $F : \mathbf{Top} \rightarrow \mathbf{Qunif}$ which sends every topological space (X, \mathcal{T}) to the finest quasi-uniformity \mathcal{U} on X with $G(\mathcal{U}) = \mathcal{T}$. It is known (see e.g [DK00]) that F is left adjoint to G . For any $(X, \mathcal{T}) \in \mathbf{Top}$, the unit $\eta_X : (X, \mathcal{T}) \rightarrow (X, GF(\mathcal{T}))$ is a continuous map where $(X, GF(\mathcal{T}))$ is the set X endowed with the topology of the finest quasi-uniformity $(X, F(\mathcal{T}))$. Now $\mathcal{S}_{(X,\mathcal{U})} = \{\sqsubset_X^U \mid U \in \mathcal{U}\}$ where $A \sqsubset^U B \Leftrightarrow U(A) \subseteq B$ for any $A, B \subseteq X$ is a co-perfect syntopogenous structure on **Qunif** for any $(X, \mathcal{U}) \in \mathbf{Qunif}$. Let $(X, \mathcal{T}) \in \mathbf{Top}$, $A \sqsubset_X^\eta B \Leftrightarrow \eta_X^{-1}(GU(FA)) \subseteq B$ for any $U \in \mathcal{U}_X$. But $\eta_X^{-1}(GU(FA)) = \eta_X^{-1}(GU(A))$, $\eta_X^{-1}(GU(A))$ is a neighbourhood of A in \mathcal{T} . Thus $\mathcal{S}_X = \{\sqsubset_X^\eta \mid X \in \mathbf{Top}\}$ with

$A \sqsubset_X^\eta B \Leftrightarrow V \subseteq B$ where V is a neighbourhood of A in \mathcal{T} so that $A \sqsubset_X^\eta B \Leftrightarrow A \subseteq O \subseteq B$ for some $O \in \mathcal{T}$.

6. Let **Top** be the category of topological spaces and continuous maps with its (surjections, embeddings)-factorization structure. It is well known that **Top_o**, the category of T_o -topological spaces and continuous maps is a epi-reflective subcategory of **Top**. Define $\mathcal{S}_X = \{\sqsubset_{X_o} \mid X_o \in \mathbf{Top}_o\}$ by $A \sqsubset_{X_o} B \Leftrightarrow \overline{A} \subseteq B$ for any $X_o \in \mathbf{Top}_o$, $A, B \subseteq X_o$. Let (F, η) be the reflector into **Top_o**. For any $X \in \mathbf{Top}$, $\eta_X : X \rightarrow X/\sim$ takes each $x \in X$ to its equivalence class $[x] = \{y \in X \mid \overline{\{x\}} = \overline{\{y\}}\}$. Thus $\mathcal{S}_X = \{\sqsubset_X^{F, \eta} \mid X \in \mathbf{Top}\}$ with $A \sqsubset_X^{F, \eta} B \Leftrightarrow \eta_X^{-1}(\overline{\eta_X(A)}) \subseteq B$, $A, B \subseteq X$.



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