IMPLEMENTING AN INTENTIONAL TEACHING MODEL TO INVESTIGATE THE ALGEBRAIC REASONING OF GRADE 9 MATHEMATICS LEARNERS.

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Declaration

I declare that IMPLEMENTING AN INTENTIONAL TEACHING MODEL TO INVESTIGATE THE ALGEBRAIC REASONING OF GRADE 9 MATHEMATICS LEARNERS is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledge by complete references.

Jade Davids   23 October 2018

Dedication

This humble piece of work is in memory of my beloved grandparents; Ronald Davids, Audrey Davids and Ethel Rakgadi Josias and my brother Matthew Davids, for their love that is transcending time and space. To my parents Vivian and Evrille Davids, and siblings Nicole and Avron Davids, for their prayers, support and the sacrifices that they made for me. I also thank my fiancé Thabo Mohlala, for his encouragement, patience and understanding.

Acknowledgment

First and foremost I would like to thank God Almighty for giving me the strength, knowledge, ability and opportunity to undertake, persevere and complete this research study. Without his blessings, this achievement would not have been possible.

In my journey towards this degree, my supervisor and inspiration, Professor Monde Mbekwa has guided me throughout this process. I would like to express my sincere gratitude for his patience, encouragement and professional guidance. Without his able guidance, this thesis would not have been possible and I shall eternally be grateful for his assistance.

My sincere gratitude goes to my colleague; Jodi Gordon who took time out from her busy schedule to proofread and edit my work on numerous occasions. I would also like to thank all learners, my school, the Western Cape Education Department that participated and allowed me to proceed with this project, without you I would not have been able to complete it.

My acknowledgment would be incomplete without thanking the biggest source of my strength, my family. The prayers of my parents Vivian and Evrille

Davids, the love and care of my siblings Avron and Nicole Davids and the support from my fiancé Thabo Mohlala, have all made a tremendous contribution in helping me reach this stage in my life. I thank them for helping me in difficult moments where I felt like giving up and for always talking to me to follow my dream of getting this degree.

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Abstract

This research has employed an intentional teaching model to investigate the algebraic reasoning of grade 9 learners from a low socio-economic background. It has also sought to study how learners engage with algebra to make generalizations and to scrutinize any misconceptions deriving from the experience. They looked for patterns, paid attention to aspects of the patterns that are important and then generalized from familiar to unfamiliar situations. Algebraic reasoning underpins all mathematical thinking including arithmetic because it allows us to explore the structure of mathematics. This study is based on the Curriculum and Assessment Policy Statement which states that learners are expected to investigate patterns to establish the relationships between variables, as well as represent and analyse the change of patterns. The study also had a huge emphasis on algebra. According to Mphuthi & Machaba (2016):

"Algebraic expressions form part of the senior phase CAPS curriculum in South Africa. A substantial amount of time is allocated to this section on evaluating expressions and simplifications of algebraic expressions in grade 7-9."

The study is premised on a qualitative research paradigm and a design-based research methodology for data collection. A set of tasks based on algebraic patterns and generalizations was given to an opportunistic sample of 20 grade 9 learners in a school in Delft, a low socio-economic suburb about 30 kilometres from Cape Town in the Western Cape Province of South Africa. Three weeks after completing the tasks, learners were interviewed to identify their reasoning and how they felt about the tasks. The results of the study show that the majority of the learners struggled with tasks especially when asked what the rules they could derive from the patterns. Learners did

not seem to understand what they were doing because they were unable to articulate the given tasks in words and did not have knowledge of concepts like the perimeter.

Keywords: algebra; algebraic reasoning; patterns; generalisation; intentional teaching; design-based research

Chapter 1:

Research Question and Overview

1.1.   Introduction

The purpose of this study was to investigate the algebraic reasoning of grade 9 mathematics learners. Algebra is one of the most important strands in mathematics. This study sought to investigate how learners engaged with algebra and derive their generalisation of algebraic patterns. Teachers tend to focus on teaching algebra through routine practice as well as an application of rules, procedures, and techniques. Learning procedures and rules should be linked to a deeper understanding of their meaning and a flexible choice of solution methods (Kieran, 2004). According to the Curriculum and Assessment Policy Statement (CAPS) of the Department of Basic Education, students are expected to have opportunities to develop an understanding of patterns and functions, represent and analyse mathematical situations, develop mathematical models, and analyse change. Lannin (2003) says that generalizing numeric solutions can create strong connections between the mathematical content strands of number operations and algebra. This builds on what learners already know (prior knowledge) and can help to develop a deeper understanding of algebra.

1.2.   Background Information

The demand for algebra at more levels of education is increasing as stated by Egodawatte (2011). Mathematical reasoning developed through algebra is necessary through life and affects the decisions we make.

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There are many learners who have misconceptions and make errors when doing basic algebra. Learners do not see how the different strands of mathematics are interlinked. They do mathematics as different sections and not parts of a whole and this is concerning. Egodawatte (2011) states: "One should taste the whole sandwich in order to get a real sense of its ingredients."

An intentional teaching model was chosen because teacher need to use intentional teaching strategies to help children learn skills, habits or information they cannot discover on their own. Teachers intentionally play roles in guiding children's experiences and children have significant and active roles in planning and organizing learning experiences.

I chose grade 9 learners because at the end of grade 9 learners have to decide whether are not they are going to continue with Mathematics for the rest of their schooling or if they are going to opt for Mathematical Literacy or Technical Mathematics. Grade 9 learners seem to struggle with algebraic expressions and how to reason algebraically. They fail to factorise when given the following questions:

1. $9x^2 - 4$
2. $x^2 - 9x + 20$

They struggle to identify that the first question is the difference of squares and the second one they always mix up the signs.

The gap in the literature is that a lot of research was done on algebraic reasoning but none is done on how an intentional teaching model will assist in the algebraic reasoning of grade 9 learners.

The purpose of this study is to answer the following question:

- How do grade 9 learners engage with algebraic generalizations after having been exposed to an intentional teaching strategy?

1.3. Motivation

Grade 9 learners at Happy Thoughts High School do not seem to grasp algebraic reasoning and tend to make errors when solving these particular problems.

For example:

Factorise the following fully:

a) $-6k + 12k^2 - 3k^3$

b) $16y^3 - 49y$

c) $3x^2 - 12$

The following errors are common:

a) $-6k + 12k^2 - 3k^3 = -3k(2 + 4k - k^2)$

Learners identify the common factor but when they divide they do not change the signs of the terms.

$$or = 3(-2k + 12k^2 - 3k^3)$$

Learners only identify 3 as a common factor; they do not realise that the variable; k is also a common factor. Furthermore they only divide the first term with the common factor.

b) $16y^3 - 49y = y(16y^2 - 49)$

Learners only identify the common factor and not that it is the difference of squares.

$$or = (4y^2 - 7y)(4y + 7)$$

Learners identify that it's the difference of squares but fail to realise the common factor.

c) $3x^2 - 12 = 3(x^2 + 4)$

Learners identify the common factor but change the sign.

$$or = 3(x - 2)(x - 2)$$

Learners identify that it is the difference of squares but the signs are the same. Even though all of the above are small errors it has a huge impact, because it shows that learners cannot apply the basics.
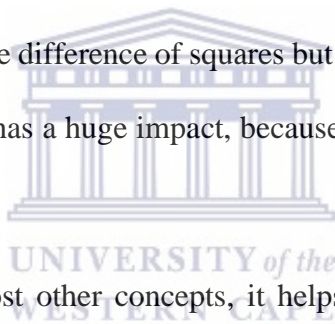
Algebra is the basis of most other concepts, it helps one develop logical thinking and problem-solving skills. The teachers fear that they will have a problem when it comes to more abstract thinking involving algebra. The purpose of this study is to figure out why this is so and to remedy the problem using generalization of patterns.

1.4. Literature Review

According to Hiebert and Carpenter (1992), error analysis in algebra is of great value in influencing instruction in a positive way. One of the main reasons for this research was to get the answer as to how learners grapple with algebra and to interview learners to gauge exactly their stepwise reasoning. The way in which algebra was introduced in South African schools reduced

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so much of the concept of algebra to the extent that it places more focus on rules for transforming and solving linear equations rather than building the foundation. Watson (2009) states that learners do not see an algebraic equation as a relationship between quantities. At Happy Thoughts High School it is seen as something that they need to calculate and get an answer to. Furthermore, it is clarified that traditional emphasis in the curriculum on finding the answer allows the learner to get by with informal and intuitive procedures in arithmetic. However, in algebra, they are required to recognize the structures that they have not been exposed to in arithmetic. Teachers, therefore, need to teach learners that it is not only about getting to the answer, but rather about understanding the procedure involved in working towards the correct answer. Watson (2009) states that the focus of algebra is on relations rather than calculations; the relation a + b = c represents three unknown quantities in an additive relationship. Teachers should also teach the properties of real numbers to learners: commutative properties, associative properties and distributive properties. I would highly recommend this to the teachers at Happy Thoughts High School as learners are unable to execute the procedures involved in algebraic equations, or problems involving algebra, due to their lack of foundation, as explained above.

Bloom's (1956) taxonomy of cognitive domain identified various levels of intellectual abilities. One of the major goals of mathematics is to foster higher intellectual functioning within the learner. Yet, learners are only aimed at achieving the end result- the answer, and not showing methods. In turn, they are not proving how one would get to the answer and this is hampering, instead of fostering, higher intellectual functioning. Children's learning are influenced by numerous factors which are interlinked with one another. Furthermore, they say that letter symbols in algebra are abstract. Learners do not see that it represents a value, they see it as a letter. In relation to Happy Thoughts High School, learners find it difficult when it comes to questions such

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as, 'work out the value of *x*' or 'check your answer by substituting a value into *x*'. They struggle because they are looking for an answer of *x* before they substitute a value for *x*. Learners see an answer as a specific number, disregarding the fact that the variable(s) in the equation could, in fact, be equivalent to a total containing a number and a variable. Watson (2009) says that it's a critical shift for learners to see a letter which represents an unknown or hidden number. Learners at Happy Thoughts also see an answer as a specific number, they do not see that a letter represents a value, and this is problematic because when they are busy checking their work they get confused due to lack of background knowledge.

Burkhardt (1988) states that the teacher will often be in a position, unusual for mathematics teachers and uncomfortable for many, to work well without knowing all the answers but this requires experience, confidence and self-awareness. This is true and it is imperative that teachers understand what they are teaching and are specialists in their field of learning and teaching; especially Mathematics teachers. At Happy Thoughts High School teachers who do not have a mathematical background are given this subject to teach. This leaves the teacher and the learners at a disadvantage because the teacher would teach according to the textbook and not really have an in-depth understanding of the rules and principles which are lacking or not explained in most textbooks. Watson (2009) states that learners rely only on remembering rules by memorising them, instead of trying to foster an understanding of the rules. This hampers their success in mathematical equations, as they tend to either forget or misapply the rule due to their lack of understanding thereof. This is true because at the school learners would try to remember the rule by using parrot techniques. When studying anything, regardless of subject, parrot style of study cannot be relied upon if one seeks to succeed. Remembering and understanding are two separate entities.

Remembering just gives them a whole world of equations, not an explanation of where, when and how to employ these rules effectively.

This literature which follows assisted me with my study, to answer most of my questions. In addition, what is happening at Happy Thoughts High school relates directly to the literature review. I used the interviews to deduce an answer as to why learners are misinterpreting or not understanding algebra.

They explained each step they are doing while being voice recorded while others wrote down what they have done. I believed that if we get to the root of the problem, we will be able to correct what is hampering the intellectual abilities of our learners and assist them in successfully applying correct and effective mathematical reasoning.

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Chapter 2:

Literature Review

2.1. Introduction

The goal of this chapter is to review the literature on algebraic reasoning with a particular focus on algebraic generalisation. Mulligan, Cavanagh and Keanan-Brown (2012) explain that algebraic reasoning develops through an awareness of the structural relationships of patterns and later in the structure of arithmetic. An intentional teaching model was chosen because teacher need to use intentional teaching strategies to help children learn skills, habits or information they cannot discover on their own. Grade 9 learners were chosen because at the end of grade 9 learners have to decide if they are going to continue with Mathematics or not in the FET phase of schooling (Grades 10 to 12). Grade 9 learners seem to struggle with algebraic expressions and how to reason algebraically.
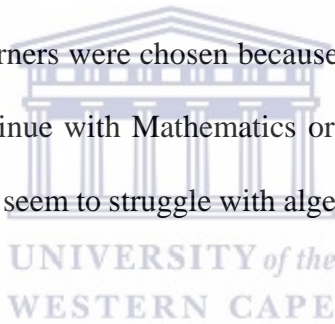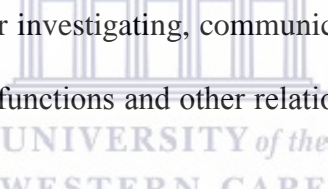
2.2. History and importance of Algebra

Algebra is one of the oldest branches of Mathematics. Osei (2006) explains that there is historical evidence that the Babylonians were experienced in its methods 4000 years ago. In 2000 B.C the Babylonians used algebraic methods to solve problems but did not use mathematical symbols. They used counting devices such as inscribed pieces of clay or metal. This type of algebra was known as rhetorical algebra and was all oral. Symbolic algebra only emerged in 1500 A.D. Symbolic algebra refers to the development of algorithms for manipulating mathematical expressions and objects. The name algebra comes from the Arabic "Al-jabr" which was the title of an algebra text written by a Persian mathematician, Al- Khwarizimi, in 820AD. It was presented as a list of rules and procedures needed to solve specific linear and quadratic equations.

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Algebra is one of the most abstract and important strands in mathematics. According to Usiskin (1988) algebra at school level is the understanding of letters (variables) and their operations. Usiskin (1988) describes four fundamental conceptions of algebra.

The first conception considers algebra as generalized arithmetic. In this conception, a variable is considered as a pattern generalizer. It is a set of integers constructed as an arithmetic progression, where learners add integers but the answers from learner to learner are different. For example, starting by 20 adding 4 or 7. The key instructions for students in this conception are "translate and generalize". According to the Curriculum Assessment Policy Statement (CAPS) of the South African Department of Basic Education (DBE):

> *"Algebra is the language for investigating, communicating most of mathematics and can be extended to the study of functions and other relationships between variables."* (CAPS, p. 10).

At school level, algebra can be described as:

- "manipulation and transformation of symbolic statements

- generalizations of laws about numbers and patterns

- the study of structures and systems abstracted from computations and relations

- rules for transforming and solving equations

- learning about variables, functions and expressing change and relationships

- modeling the mathematical structures." (Watson, 2009, p. 8).

Watson (2009) states that a good understanding of connections between numbers and relations leads to success in algebra. The second conception views algebra as a study of procedures for

solving problems. It refers to the steps you are going to follow to solve a problem or to create a pattern. According to the procedure, you shouldn't have unknowns and therefore the sum is not complete. The third conception is algebra as a relationship between quantities. If we look at the formula of the circumference of a circle ($C = 2\pi r$) represents a relationship between different quantities. The fourth conception is algebra as a study of structures, at tertiary level algebra is seen in groups, domains and fields.

Kaput (1998), has identified five forms of algebraic reasoning, which is his own conception of algebra and it is as follows:

- Algebra as generalising and formalising patterns and regularities are known as generalised arithmetic.
- Algebra as syntactically guided manipulations of symbols.
- Algebra as the study of structure and systems abstracted from computations and relations.
- Algebra as the study of functions, relations, and joint variation; and
- Algebra as modeling. (Kaput, 1998, pg. 26)
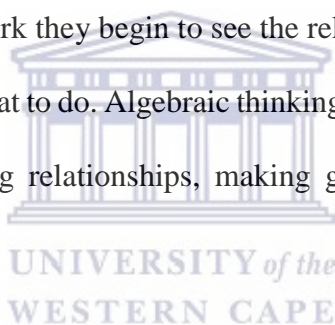
2.3. Algebraic reasoning

In this study algebraic reasoning is understood as generalised arithmetic and operating on unknown numbers or variables. It involves recognizing and analysing patterns as well as developing generalisations about these patterns.

Blanton and Kaput (2005) state that:

> Algebraic reasoning is a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways." (Blanton and Kaput, 2005, p. 99)

According to Blanton and Kaput (2003), teachers must find ways to support algebraic reasoning and create a classroom culture that values "students modelling, exploring, arguing, predicting, conjecturing, and testing their ideas, as well as practicing computational skills." When learners are interacting with the work they begin to see the relationships and connections and it is better than teachers telling them what to do. Algebraic thinking includes recognizing and analyzing patterns, studying and representing relationships, making generalizations, and analyzing how things change.

Describing algebraic reasoning and its purpose, the CAPS document states:

> The nature of algebraic reasoning is to identify and use algebraic patterns in problems are the bases of mathematical reasoning. Algebraic reasoning promotes a particular way of interpreting mathematics. It extends the mathematical reasoning of students by encouraging them to interact and engage with the generalities and relationships inherent in mathematics (Windor, 2010).

Algebraic reasoning connects the learning and teaching of arithmetic in primary school to functions and calculus in high school. Van de Walle, Karp and Bay- Williams (2011) clarify that:

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"Algebraic reasoning involves forming generalisations from experiences with numbers and computations, formalising the ideas with the use of a meaningful symbol system, and exploring the concepts of pattern and functions." (Van de Walle, Karp, Bay- Williams, 2011, p.262)

Algebraic reasoning is important because it makes learners think beyond the specific calculations they do in class, and it helps with problem solving. If it is introduced at an early stage it will help develop learners' mathematical thinking. Mulligan, Cavanagh and Keenan- Brown (2012) state that early algebraic reasoning can develop from a natural awareness of generalisations and ability to express generality. The Ontario Ministry of Education (2013) explains three actions that learners need to take to develop algebraic reasoning: offering and testing conjectures, justifying and proving, and predicting. With regard to the first action, learners need to take a guess about the problem at hand and check if it works for all the terms in the example. If not, they have to refine the conjecture. The second action, learners need to check if the conjecture will always be true, whether it will always work. The third action, by using the rule they can predict what the next or the 100th term will be. The Ontario Ministry Education (2013) state that teachers can promote algebraic reasoning in their classrooms if they plan properly. They need to think of questions that will engage learners in the actions that will develop algebraic reasoning. This contributes to the study because it goes hand in hand with the intentional teaching model that this study adopted.

2.4. Algebraic Generalisation

Generalisation is an important aspect of the learning process. The topic on algebraic generalisations was chosen because it applies to a wide variety of different situations including ones that they have not yet encountered. Generalisation is the process of defining a concept. Generalisation of patterns was chosen as it is an important part of Mathematics because it is in essence the ability to discover, replicate and justify mathematical patterns. It is interesting how learners awareness of generalisation of patterns grows. Generalisation of patterns according to Ellis (2007) is a dynamic process which engages learners in the following three activities:

- Identifying a commonality

- Extending one's reasoning beyond the range in which it is organised

- Deriving broader results from a particular scenario. Ellis (2007, p. 197)

The above-mentioned activities mentioned by Ellis one could easily pick up while learners were engaging with the tasks they had to complete.

According to the Department of Basic Education (DBE) (2011), Curriculum and Assesment Policy Statement for Mathematics grade 9, the specific aims of patterns are as follows:

- Investigate and extend numeric and geometric patterns looking for relationships between numbers, including patterns:
    - represented in physical or diagram form
    - not limited to sequences involving a constant difference or ratio
    - of learner's own creation
    - represented in tables

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- o represented algebraically
- Describe and justify the general rules for observed relationships between numbers in own words or in algebraic language". (DBE, 2011.p.3)

The section on Patterns, Functions and Algebra constitutes 35% in the grade 9 mathematics examination papers. Hence it is an important section of the grade 9 mathematics to research how learners perform in it.

Generalisation is a principle or rule that can be used as a basis of reasoning. Mason et al (1985), Mason & Lee (1996) cited in Barbosa, Palhares & Valle (2007) state that using patterns to encourage generalization is seen as a pre-algebraic activity. Warren (2006) explains that by using visualization to see how a pattern growth can be used to generate algebraic expressions. Mulligan, Cavanagh & Keanan- Brown (2012) state that early algebraic reasoning can develop from a natural awareness of generalisation and the ability to express generality. Furthermore, the role of algebraic reasoning is about promoting generalisations in mathematics at the primary and secondary years.

Lannin, Barker & Brain (2006) explain that generalising numeric patterns is viewed as a potential vehicle for shifting students from numeric to algebraic reasoning. Generalisation offers the potential to establish meaning for algebraic symbols by relating them to a scenario that learners understand. Described by Kaput (1998), generalisation involves examining different quantities and describing relationships between them. For example, if learners are given a problem they need to see patterns, make connections, check if a rule works for the whole situation and to determine the rule. According to Lannin (2006) when students are asked to generalise numeric situations in the mathematics classroom, they construct a variety of generalisations and use many different strategies. Therefore students have the ability to develop generalisations, through drawings, rules

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and counting. According to Stacy & MacGregor (2001), cited in Lannin (2006), many of the tasks used to develop clear reasoning tend to focus students on the repetitive relationship that exist in the situation, rather than promoting the use of more sophisticated strategies. English & Warren (1998) cited in Warren (2009) state that a patterning approach to algebra allows students to experience unknowns as variables, and provides students with opportunities to observe, verbalise and symbolise generalisations.

Chua & Holyes (2014) state that:

> "Generalisation has been widely acknowledged as a process involving at least one of the following activities; to examine a few particular cases to identify a commonality, to extend one's reasoning beyond those particular cases and to establish a broader result for those particular cases. " (Chua & Hoyles, 2014. p.20)

Dindyal (2007) used the following problem in research done in Australia as an example of a problem requiring generalisation from patterns:

> *Problem 1: How many small squares are there in the border of a* 5×5 *square (square drawn on a rectangular grid).*
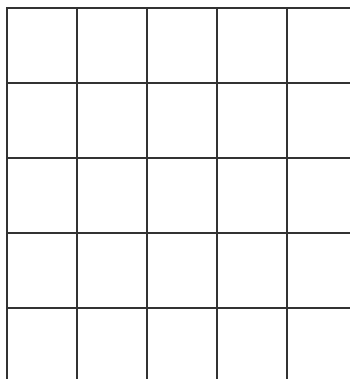
How many are there in a 6×6 square? How many are there in a 10×10 square? How many would there be in a square of $n \times n$? If there are 76 border squares in a square grid, what is the size of the grid?

Three students from school X tried different strategies, they finally came up with the correct generalization of $4(n - 1)$.

Problem 2: What is the sum of the interior angles in a triangle? From any vertex, we can divide a quadrilateral into two triangles. What is the sum of the interior angles in a quadrilateral? What is it for a pentagon, hexagon, and a decagon? What would it be for a polygon with n sides? Most of the students could get the answer, without a diagram, only one who needed help with the pentagon and hexagon, she used an additive strategy to get the interior angles of a decagon. A polygon with n sides: $(n - 2) \times 180°$.
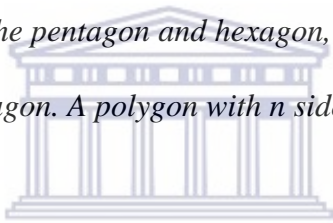
According to Dörfler (1991) cited in Zazkis & Liljedahl (2002), generalisation is both "an object and a means of thinking and communicating" (Zazkis & Liljedahl, 2002, p.63).

Generalisation contributes to the study because "algebra, and indeed all of mathematics is about generalizing patterns" (Lee, 1996, p. 103). Therefore it is essential in a study of mathematics to direct students' attention to patterns underlying a wide variety of mathematical topics; which the focus in this study is algebraic reasoning.

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2.5. Intentional Teaching

Intentional teaching was used as a model because if we do things on purpose, goal driven and with intentions, learners will follow.

Slavin (2000. p. 117) states that there is no recipe for good teaching, but the one thing that stands out most is for teachers to be intentional, "to do things on purpose". Teaching can become successful if it is geared towards a certain purpose. According to Epstein (2007. p. 2), intentional teaching embodies the knowledge, beliefs, attitudes and especially the behaviours and skills teachers employ in their work with learners. It is the intentional interaction between the teacher and the learner. To be intentional, according to Epstein (2007. p. 2) is to act purposefully, with a goal in mind and a plan to accomplish it. Intentional teachers use their knowledge, judgment, and expertise to organize learning experiences for learners.

McGravey (2010) states that through intentional acts, teachers turn children's attention to particular mathematical objects. However, some acts may alter a child's observation and others will not.

Personally, I would consider the errors they made, and explain to them where they went wrong. Furthermore, I will re-explain what they should do, step- by- step.

Morrow (2007, p. 3) defines teaching "as an activity guided by the intention to promote learning" which he later changed to "the organising of systemic learning". Intentionality refers especially to how teachers interact with learners (Epstein, 2007). According to Epstein (2007. p. 4), teaching embodies the knowledge, beliefs, attitudes and especially the behaviours and skills teachers employ in their work with learners. It is the intentional interaction between teacher and learner.

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Intentional teachers use their knowledge, judgment, and expertise to organise learning experiences for learners.

Intentional teaching contributes to study because it is the model that was used to implement algebraic reasoning of grade 9 learners. In this research I direct learners' attention was directed to the relationship between quantities so that they could move from arithmetic to algebraic thinking. Tabone (2009) states that effective teaching is the basis of all successful learning, which is why it is important to have professional and effective teachers. Pianta (2003) states that intentionality is directed, and a designed interaction between teachers and learners. Intentional teachers know the learners they work with.

In learning with intention, the learner plays a "self- initiated, goal-directed, purposive role in the learning process", Sinatra (2000) as the learner is constantly aware of what is happening, as well as how and why it is occurring. Gronlund & Steward (2011) state that teachers are intentional when they are doing things for the learners and with the learners. One way to create an intentional classroom environment is, according to Julie (2013) to make the success criteria, what learners need to know and understand, clear to the learners. Furthermore, Julie (2013) argues that opportunities are created for learners to work collaboratively and to work with different problem types.

2.6. Conclusion

Algebraic reasoning connects the learning and teaching of arithmetic in primary school to functions and calculus in high school. Intentional teaching was chosen as a model because according to Epstein (2007), it embodies the positive characteristics that teachers employ in their work with learners. Generalisation is a principle or rule that can be used as a basis of reasoning. Therefore students have the ability to develop generalisations, through drawings, rules. In this research, I wish to direct learners' attention to the relationship between quantities so that they can move from arithmetic to algebraic reasoning.

Chapter 3:

Research Design, Sampling and Data Collection

3.1. Introduction

This chapter discusses the research context, the methodological approach and the data collection process employed in this study. The chapter also touches on the reliability and validity issues, which form a critical part of research. An overview of qualitative research and document analysis is presented. The second part describes the data collection process and explains how the sample for data collection was constructed.

3.2. Research approach

Fraenkel and Wallen (1990), believe that a qualitative approach facilitates research in which the quality of relationships, activities, situations or materials may be investigated. The research paradigm for this study is qualitative. According to Golafshani (2003. p. 600), qualitative research is research that gathers findings by using real-world situations. The motivation for using qualitative research is that it uses actual data from the participants, the learners. Thus it helps one to understand the phenomenon of this study. According to Bricki and Green (2007. p. 3), "Qualitative research is about understanding some aspect of social life".
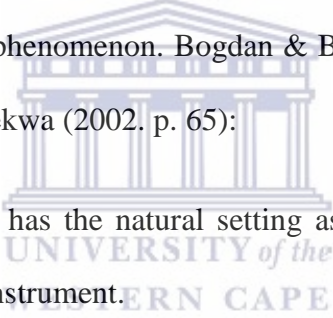
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Marshall and Rossman (1999) state that qualitative research is about uncovering the unexpected and exploring new avenues. Therefore, the research needs to be flexible so that the respondents can answer the research questions. The participants' answers and the researcher's interpretation are extremely important to the research process. Qualitative research is defined by Strauss and Corbin (1990) as any kind of research that produces findings not deduced by quantitative methods, but instead produces findings deduced from real-world settings.

Qualitative research is a scientific type of research because it seeks answers to a question; it collects evidence and produces findings that are applicable beyond the immediate boundaries of the research, which forms the bases of scientific research. Furthermore, qualitative research seeks to understand a given research problem or topic from the standpoint of the population it involves.

According to Creswell (2003), a qualitative approach is one in which the inquirer often makes knowledge claims based on experiences that are socially and historically constructed. Therefore qualitative research was used due to information having been gathered by using the data from the learners' scripts. Therefore, the research undertaken was composed of algebraic scripts being handed out in a Grade 9 Mathematics class where learners were instructed to complete the questions and describe their approaches on how they completed them. This type of research also seeks to understand a given research problem or topic from the views of the local population it involves. The Head of Department of Mathematics (HOD) who assisted the researcher first started off by teaching algebraic patterns and generalisations thereon from an intentional teaching strategy. Exercises were given and corrected. Later on, revision exercises were given on these pattern generalisations. Thirty learners were given tasks to complete, they had group discussions afterward to explain to one another their own methods, and these were recorded with cell phones. Lastly, the researcher interviewed learners on their experiences of the tasks. Marshall and

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Rossman (1999) state that qualitative research is about uncovering the unexpected and exploring new avenues. Therefore, the research needs to be flexible so that the respondents can respond to the research questions.

Qualitative research, according to Richards (2006, p. 95), is not proactive. The research paradigm is rather reactive, the expectations were explained to the learners and they had to use different methods to answer the questions, like using drawings or using the formula. The indications of form, quantity and scope must be obtained from the research question and data collected. According to Mbekwa (2002, p. 63) qualitative research is characterized by a situation in which a researcher pursues a certain phenomenon of interest, studies it, analyses it and interprets findings made on the study of the phenomenon. Bogdan & Biklen (1982. p. 27-29) summarized the following 5 points cited by Mbekwa (2002. p. 65):

1. Qualitative research has the natural setting as the direct source of data and the research is the key instrument.

2. Qualitative research is descriptive.

3. Qualitative researchers are concerned with the process rather than simply with outcomes or products.

4. Qualitative researchers tend to analyse their data inductively.

5. "Meaning" is of essential concern to the qualitative approach.

The focus of this research was on the learners' engagement with algebraic patterns in the grade 9 classroom and how they came to generalisations on the algebraic patterns.

3.3. Design- based Research

Design-based research (Brown, 1992; Collins, 1992) is a paradigm for the study of learning in context through the systemic design and study of instructional strategies and tools.

"Design-based research is a methodology designed by and for educators that seek to increase the impact, transfer, and translation of education research into improved practice. In addition, it stresses the need for theory building and the development of design principles that guide, inform, and improve both practice and research in educational contexts." (Anderson & Shattuck, 2012, p.20)

Design- based research was used as a research method which is located within a qualitative research paradigm. It is a method in which the inquirer often makes knowledge claims based on individual experiences, like interviews or analysing learners' workbooks. The researcher collects open-ended, emerging data with the primary intent of developing themes from the data (Creswell & Clark, 2007). The researcher had interviews with the participants based on their experiences of the exercises. According to Akker (2014; p. 4) design- based research typically has an explanatory and advisory aim, namely to give theoretical insights into how particular ways of teaching and learning can be promoted.

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Wang & Hannafin's (2005) definition of design-based research states:

> "Design-based research is a systemic but flexible methodology aimed to improve educational practices through iterative analysis, design, development, and implementation, based on collaboration among researchers and practitioners in the real- world settings, and leading to contextually- sensitive design principles and theories" (Wang & Hannafin, 2005, p.6).

McKenny & Reeves (2012) state that design-based research is not concerned with the movement or change in variables over time, but is concerned with how, why, and what works by means of conducting multiple iterative cycles within one larger design-based research study. Santos (2010) explains that a researcher does not solely use design-based research (DBR) to assess a certain design in order to discover how it is working, but also to generate theory. The researcher analysed the exercises to understand how learners reason when dealing with generalisations of patterns.

Furthermore, Gravemeijer & Cobb (2006) is confirming that design-based research is an iterative sequence of tightly integrated cycles of design and analysis, which is an important factor to the process of testing, improving, and understanding.
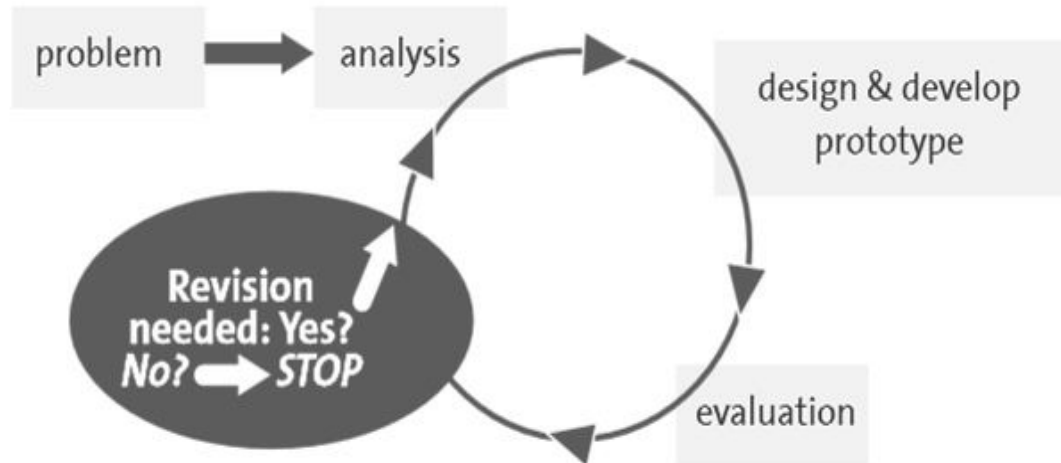
Figure 3.1: Iterations of systemic design cycles (Gravemeijer & Cobb, 2006, p.20)

This process is intended to ultimately improve the quality and functionality of a design. This relates to teaching in the sense that if one way of explanation does not work, teachers should try to rephrase and come up with new ideas to improve their teaching to accommodate all learners' educational needs. The purpose of this study was to investigate how learners reason algebraically, and intentional teaching was used as a model to address this. The tasks were designed based on generalisations of patterns. In figure 1, the design cycle is presented. The cycle shows the factors of the systemic design cycle quite clearly. The **problem** at hand was**:** what processes are involved in algebraic reasoning by grade 9 learners? The **analysis** took place when the researcher gave learners tasks on generalisation to complete. They had to discuss in the groups. The researcher took the learners' answers and checked what method was used to answer the questions. This was further achieved by checking for similarities and differences in both the methods they used and their answers. **Design and develop prototype:** the researcher prepared questions and told the learners that they could use any method they had knowledge to answer the questions.

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Evaluation: Before presenting the participants with the questions, the researcher revisited generalisation of patterns so that the learners were reminded of the topic. Thereafter the researcher collected data and analysed it. Data was collected by means of exercises that learners had to complete followed by interviews. The methods learners used to answer questions was observed, the researcher also looked for similarities and differences in the way that learners answered the questions. The Head of Department of Mathematics explained the task to them, what was expected of them to do. The learners did not fully comprehend the task and therefore the researcher had to take a different method of explanation. Learners were given the tasks (figure 3) to do and it was used to analyse the different method on how learners answered the questions and their understanding. As seen in the above diagram, design based research embodies a process of testing, improving and understanding which is one of the best ways to learn. Thus, when linking it to intentional teaching one has a winning recipe destined to succeed if employed correctly. Bakker (2004) states that:
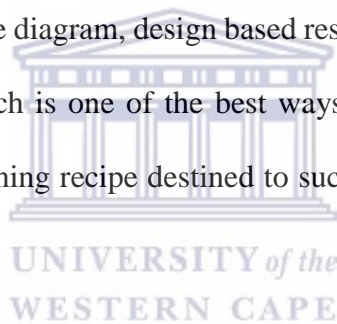
> "In design-based research are intertwined with the following saying that is also in social and cultural traditions: If you want to understand something you have to change it, and if you want to change it you have to understand it." (Bakker, 2004, p. 37)

3.4. Research participants and Sampling

The research was conducted at Happy Thoughts High School, a technical high school, in Delft. Delft is an economically deprived residential area on the outskirts of Cape Town, South Africa. It is situated next to the Cape Town International Airport.

A Technical High School is designed to bring academic and technical schooling to the learners. At a Technical High School learners are not only equipped with skills for certain trades (Electrical, Mechanical and Civil engineering and Architecture) but also professional disciplines. This school was chosen because it is the school where the researcher teaches. It has different feeder schools so learners with different backgrounds enter the school in grade 8. The research was conducted with 20 learners in the researcher's class. Hence the study took an opportunistic sampling strategy because the target group was chosen by the researcher. Instead of randomly selecting a sample, the researcher collected her data from her own class. Opportunity sampling is when a sample is readily available. In the context of this study, the sample was opportunistic as it was based on a class which was taught by the researcher.

The sample consisted of 12 boys and 8 girls, between the ages of 15 and 16, in grade 9. Their levels of understanding vary, there were 10 learners who were top achievers, 6 learners who were average, and 4 who were below average.

I gathered the information in two ways. The first was an exercise, secondly, it was tasked for learners to do and thirdly an interview, this took place over approximately three days. I have 30 learners in the grade 9 class. After receiving permission from the participants, all of them participated but only 20 allowed me to use their work as part of the research. Ten learners felt intimidated because they only wanted to give the right answers, they chose not to participate because out of shame and fear. Hence the sample selected for this study is 20 grade 9 learners. I have marked their scripts Learner A- Learner T, in order to have the ability to explain learners work with reference to specific scripts while keeping the participants anonymous. I designed an exercise for the learners on algebraic patterns.

Even though learners were split into groups they first completed the exercise individually and afterward explained what they have done compared with the others in the group. Qualitative research was used due to information having been gathered by using the data from the learners' scripts- this took place in the classroom. Furthermore, these scripts were collected, analysed and then presented.

The learners were divided into groups of 3. There were 30 learners who formed 10 groups. Only 20 learners gave consent for their scripts to be used by handing them in and the other 10 did not hand in their scripts. All of them participated from the beginning. The learners were allowed to sit with their friends so that they weren't afraid to share ideas and could communicate freely. The topic of the lesson was revision of patterns. When it was done in the first term learners understood what to do but when it was in a different context or when they had to get the general term they struggled. The Head of Department of Mathematics (HOD) introduced the lesson by revisiting patterns, the definition of patterns (things arranged by following a rule). The Head of Department of Mathematics (HOD) used 5 different examples to explain how to extend the pattern and how to get the rule. He started with example one and showed them different ways to get the next pattern: trial and error, drawing and counting. The learners had to do example two and we would all discuss it after a few minutes. He continued with this up until example 5. The initial lesson conducted for this class was based on generalisation of patterns. The HOD was intentional in his method by following principles of intentional teaching. He clearly articulated the objectives of the lesson which learners had to reiterate, and gave them examples to work through both individually and in groups.

Epstein (2007) states that:

> "Intentional teachers act with specific outcomes or goals in mind for children's development and learning." (Epstein, 2007, p. 1)

The lessons and exercises given to learners took two days to complete.

The following are the examples that were given:

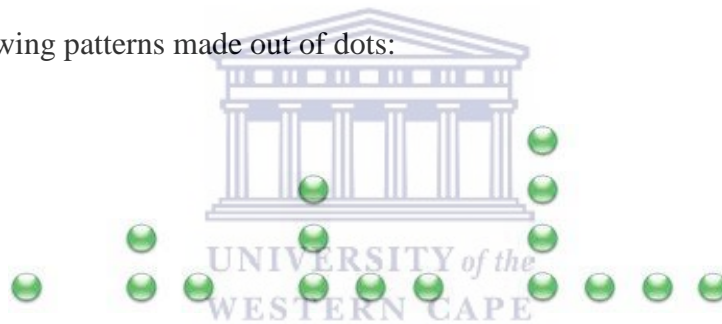**Example 1**

Consider the following patterns made out of dots:

Figure 1   Figure 2   Figure 3   Figure 4

a) Complete the table below:

Figure	1	2	3	4	5	10	100
Number of dots used	1	3	5	7			

b) How many dots would be used in figure p (Determine a formula for the number of dots)?

c) Is there a figure that uses 146 dots? Explain your answer.

Example 2

Matchsticks are arranged as shown in the following figures:
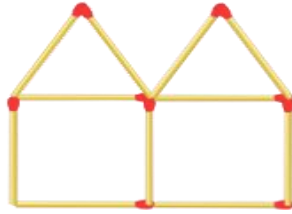

Figure 1   Figure 2   Figure 3
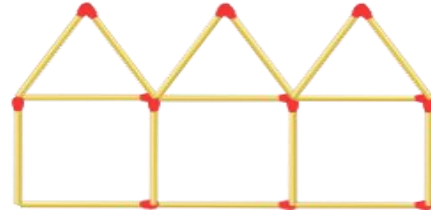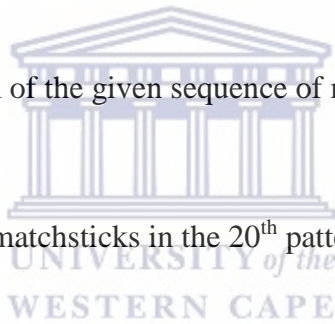
a) Determine the number of matchsticks in the next figure (figure 4) if the pattern is continued.

b) Write down a general term of the given sequence of matchsticks in the form: $T_n = $ _____

c) Determine the number of matchsticks in the 20th pattern

Example 3

The following pattern is given:

$$5; \ 2; \ -1; \ -4 \ ....$$

a) Complete the table below show all your work:

Term position	1	2	3	4	5	6	15
Term value	5	2	-1	-4			

https://etd.uwc.ac.za

b) Write down the rule or general term for the pattern above.

c) If the term value is -100; determine the term position.
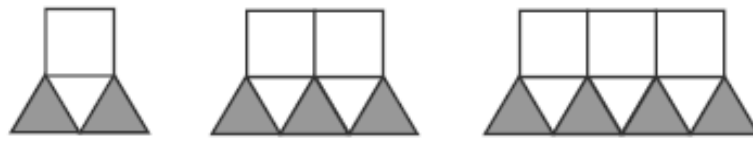
Example 4

The following pattern is given:

Figure 1   Figure 2   Figure 3

a) Complete the table below:

Number of white squares	Number of grey triangles
1	2
2	3
3	4
20	
N	

b) The number of grey triangles are _____ the number of white triangles.

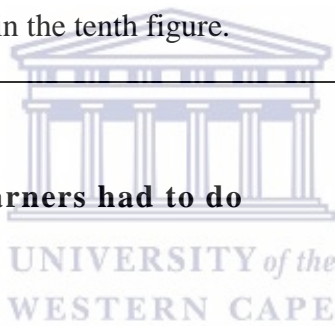c) Determine the general term for the number of grey triangles.

Figure 3.2: The examples learners had to do

Each learner was given the worksheet (the tasks) to complete. The worksheet was prepared by the professors who are the supervisors of this research. The Head of Department (HOD) of Mathematics at my school assisted in the classroom so that there was no bias while I observed the learners and made my notes. Learners were split into groups, but the HOD let them do the tasks individually and afterward, they recorded their discussions on each question, they discussed their different ways of answering each question. He wanted them to work on their own first so that there was no chance of them being able to get answers from each other.

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3.5. Credibility and Trustworthiness

Credibility is the how confident the qualitative researcher is in the truth of the research study's findings. In this research it was very important to be truthful and that the participants could rely upon the researcher. The study's findings are credible, transferable, confirmable, and dependable. Trustworthiness is all about establishing transferability; conformability, dependability.

Transferability is how the qualitative researcher demonstrates that the research study's findings are applicable to other contexts. In my research I have used qualitative research design, I gave the learners an exercise and interviewed them in groups. The importance was how the learners answered the questions and the observations. According to Merriam (2002), validity was concerned with the extent to which the findings of one study could be applied to other situations. The aforementioned means that if someone else was to use this method, they would yield the same results.

Conformability is the degree of neutrality in the research study's findings. In other words, this means that the findings are based on participants' responses and not any potential bias or personal motivations of the researcher. This involves making sure that researcher bias does not skew the interpretation of what the research participants said to fit a certain narrative. I asked a colleague to implement the research to the learners, to help with the data collection and also to ensure that I was not biased in the process or to see if any information was misinterpreted. I would make sure that the information was very clear to them, why the research was being done, what I was studying, how the information would be collected and used.

Finally, dependability is the extent that the study could be repeated by other researchers and that the findings would be consistent.

Trustworthiness is crucial when it comes to qualitative research because it would show how reliable the researcher is. It refers to the consistency and stability of the "learners" and the researcher to collect the data.

In this research, the researcher was the class teacher but had the Head of the Department for Mathematics helped with the research to improve reliability. The participants were learners from the researches class but only 20 learners gave permission to be part of the researcher which was acceptable because participating was voluntary. Cell phones were used as audio recorders to collect the data, it was placed on the tables but only three groups recording was clear enough to use. These learners switched off the recording while they were busy and only switched it on when everybody was done and then they discussed what they did.

In order to begin this part of my research, the HOD of Mathematics explained what the research was about to the participants. He continued to explain the topic, as they had done it during the previous term. While he was busy with the explanations I observed the lesson. This was done in order to eliminate any chance of bias. Learners were given the instructions; participation was voluntary so they could decide to not hand in their answers at any time. The exercise was used to grasp how learners answered the questions, what methods they used, how they were thinking.

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3.6. Ethical Considerations

Ethics are the moral principles that govern a person's behaviour and how to conduct an activity.

I worked with learners at school, so it was important to pay close attention to ethical issues. Firstly, I sought informed consent; from the Western Cape Education Department (WCED), the school and the parents. This was achieved by sending letters to parents explaining the research and the anticipated learner involvement. The letter was available in three different languages: Afrikaans, English and IsiXhosa so that each parent would understand what the research was to be about. The parents were provided with contact details and they were encouraged to phone or to meet to discuss any concern they had. Secondly, written permission from teachers, the principal and the respective district (Metro-North Education Department) was required for the class that was part of the research. Finally, I explained to the learners how they would be involved in the research. This explanation was also available to them in writing in Afrikaans, English and isiXhosa.

The second issue involved anonymity. The learners were to be recorded via journal notes, voice recorders and interviews. No names were to be used in subsequent publications of the data including the Master's thesis and nobody would be able to identify who said what. The names of learners and the school were fictionalised. I guaranteed that all information would be highly confidential.

The third issue concerned voluntary participation. Each learner was asked if they wished to participate, they also had the option to freely withdraw from participating. I started with 30 learners but only 20 learners handed in their papers and the others did not want to be part of the

research anymore. I explained to them that no names will be used and that there is no right and wrong answer. They still did not want to be part of the research. It was important that the research was conducted in accordance with research ethics.

3.7. Conclusion

This chapter explained the methodology of the study, which described the sample and how data were collected. Qualitative research allows for the analysis of being gained from the natural settings of where learners were completing exercise based on generalisation of patterns. The documents that were analysed is from a Grade 9 class of 2015. The issues of validity and reliability and ethical consideration were completely covered in this study. The following chapter will look at the findings of the research.
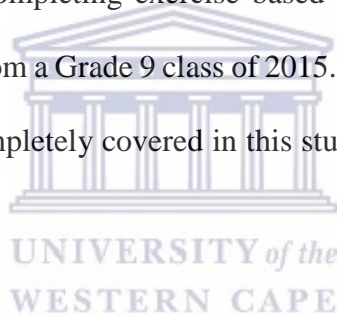
Chapter 4:

Data Collection, Analysis and Findings of this Study

4.1. Introduction

The previous chapter involved a discussion of the research design that includes the research method and the way the data was collected and analysed. This chapter discusses the findings arising from the study.

4.2. Analyses of Worksheets

The research was conducted in the researcher's class. The HOD was the implementer, he conducted the lessons and stayed for the whole process of data collection. He used intentional teaching as his teaching method because two days before he explained the examples to them step-by- step. Worksheets were given to the learners and they worked on them for about three days. The HOD went through the tasks with them and the researcher observed the process. These documents were collected to be analysed.

These documents were used to identify the similarities and differences in how learners answered the questions. This analysis is presented in discussion form, based on each question.

The researcher observed the learners work individually with the tasks and group discussions they had afterward. Only three group recordings were used because the other recordings were of poor sound quality. Learners discussed each question explaining to one another how they got to a particular answer.

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The following were tasks given to the learners to do after completed the initial exemplary exercises:

Task 1

(a)   Complete the table below. Show all your work in the space below the table

Term Number	1	2	3	4	5	10
Term Value	6	9	12			

(b)   Write a rule or formula to find the term value for any term number when you know the

term values of the first three consecutive terms.
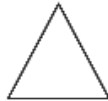
(c)   The term value is 72. What is the term number? (Explain your reasoning below).

Task 2

Tandeka arranges triangular tiles. The sides of the triangle are 1cm each. She arranges the tiles in rows and finds the perimeter of each arrangement.
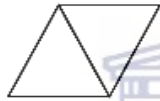
1 Tile

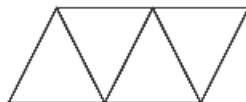Perimeter: 3 cm

2 Tiles

Perimeter: 4 cm

3 Tiles

4 Tiles

3.1   What is the perimeter if 4 tiles are arranged in the same pattern?

3.2   What is the perimeter if 10 tiles is arranged in the same pattern? Write down how you

         found the perimeter for the 10 tiles.

3.3   Write a rule/formula for finding the perimeter, P, if n is the number of triangular tiles in

         the pattern. Explain how the parts of your formula are related to the pattern in the

         picture above.

3.4   Jonathan said: *When square tiles with sides of 1 cm each are used then the rule is*

                                $P = 2n + 2.$

         Do you agree with Jonathan? Write a reason for your answer.
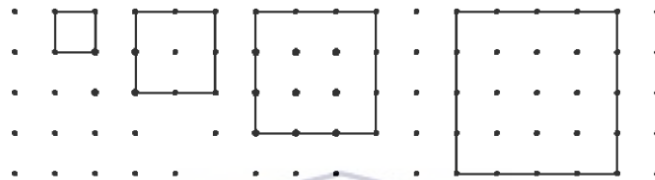
3.5   Write a rule for finding the perimeter if an arrangement is made with 5-sided tiles if the

          sides are equal and the length of a side is 1 cm.

3.6   What do think the rule is when 10-sided tiles are used if the sides are equal and the

         length of a side is 1 cm. Write a reason why you think your rule is correct.
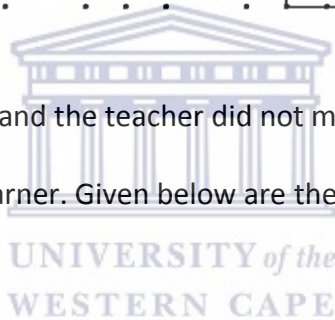
Task 3

A grade 8 class was given the following task.

Squares of different sizes are drawn and the dots inside each square are counted. Write a rule or formula to find the number of dots inside the square when you know the length of the side of the square.

The learners handed in their work and the teacher did not mark their work. She wrote comments for the work of each learner. Given below are the answers of three learners.

Learner A

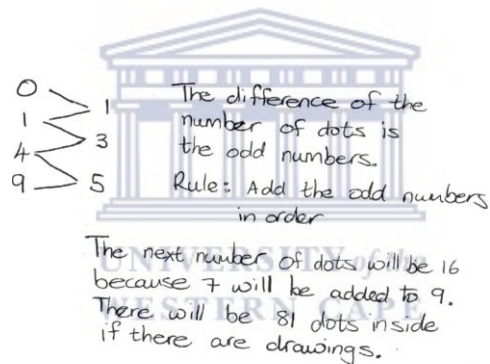The number of dots are square numbers. Even 0 is a square number. To get 0,1 is subtracted from the length of the side. This gives 0 and the square number for 0 is 0

The next number of dots = 2−1 = 1,
The square number of 1 is 1.
The rule is
Number of dots = (length of side − 1)²
                                                squared.

Learner B

SL = length of side of square

D = number of dots. The number of dots are all square numbers. The rule is D = $(SL)^2$ if the SL is always the one before the next one.

So for SL = 2 then you square SL before the 2, which is 1 then D = $(1)^2$ = 1 for SL. If 1 then D = $(1)^2$ = 1 for SL. It works the same for all the other SL.

Learner C

The difference of the number of dots is the odd numbers.

Rule: Add the odd numbers in order

The next number of dots will be 16 because 7 will be added to 9. There will be 81 dots inside if there are drawings.

Teacher comments.

Comment 1

You worked correctly by finding the difference between the number of dots. Your rule works but if will be difficult to find the number of dots inside a very big square, say one when you have 50 squares. Your rule does not connect the number of dots to the length of the side of the square. Try to find the connection.

Comment 2

The way you organized your answer by getting the number of dots in more squares is good. Your rule is correct for the example that you give. When making the rule it should work for all the information that is given and you rule does not work to get the number of dots (0) when the length of the side is 1. Check with other learners in the class how you can change your rule so that it also works for this case.

Comment 3

You obtained a correct rule. It is always wise to show that you have checked your rule with all the drawings and also try to create other drawings of which shows to show that your rule still works.

Which comment corresponds to which learner's answer?

Which comment do you agree with most? Give reasons for your answer.

Figure 4.1: The tasks the learners had to complete

The learners first worked individually and afterwards had discussions. The participants were 20 learners from the researchers' class. The following is based on the learners' individual work. When it came to the group discussions some only gave the answers, but from the submitted scripts one could see what was done when looking at their calculations.

Task 1 a) was relatively easy for learners to do. The learners used different methods to complete the table. Ten learners wrote what was added and used trial and error to get to the general term for 1b. Six learners out of the group of 20 learners only completed the table without showing their calculations.

Figure 4.2: Learner A's method to complete the table.

Task 1

(a)   Complete the table below. Show all your work in the space below the table

Term Number	1	2	3	4	5	10 r3+3
Term Value	6	9	12	15	18	33

$Tn = 3n + 3$

$1 \times 3 + 3 = 6$
$2 \times 3 + 3 = 9$
$3 \times 3 + 3 = 12$
$4 \times 3 + 3 = 15$
$5 \times 3 + 3 = 18$
$10 \times 3 + 3 = 33$

Figure 4.3: Learner I's method to complete the table.

Various calculation strategies were adopted by learners in order to complete the tables. Learner A, for example, used the functional notation formula $T(n) = 3n + 3$ to complete the table but did not indicate how the formula was derived as indicated in figure 21. The same pertained to other learners. In one other case learners subtracted successive terms to find the common difference in the first three terms given and used this to find succeeding terms as indicated in figure 22.

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Task 1

(a)   Complete the table below. Show all your work in the space below the table

Term Number	1	2	3	4	5	10
Term Value	6	9	12	15	18	33

$9 - 6 = 3$

$12 - 9 = 3$

$15 - 12 = 3$

$T_5 = 15 + 3 = 18$

$T_6 = 18 + 3 = 21$

$T_7 = 21 + 3 = 24$

$T_8 = 24 + 3 = 27$

$T_9 = 27 + 3 = 30$

$T_{10} = 30 + 3 = 33$

Figure 4.4: Learner P's method to complete the table.

Learner P used the common difference and kept on adding till the 10th term.

Task 1 c) was similar to the worked examples prior to the worksheet so they answered it well. Only 4 learners out of a group of 20 leaners explained their reasoning. Different methods were used as shown below.

(c)   The term value is 72. What is the term number? (Explain your reasoning below).

$72 - 3 = 69$

$69 \div 3 = 23$

Figure 4.5: Learner E's method (Task 1c)

Without using the formula and substituting 72 in the formula, learner E's method is similar in what should be done when using the formula.

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(c) The term value is 72. What is the term number? (Explain your reasoning below).

$$72 = 3n + 3$$
$$72 - 3 = 3n$$
$$\frac{69}{3} = \frac{3n}{3}$$
$$23 = n$$

Figure 4.6: Learner D (Task 1 c)

(c) The term value is 72. What is the term number? (Explain your reasoning below).

$$Tn = 3n + 3$$
$$72 = 3n + 3$$
$$72 - 3 = 3n$$
$$\frac{69}{3} = \frac{3n}{3}$$
$$23 = n$$

Replace Tn with 72 and not n, because we are looking for the term number

Figure 4.7: Learner T (Task 1 c)

Learner D and T substituted 72 into the formula. Learner D calculated "n" by using simple algebra with explanation. Learner T only explained why 72 was substituted into Tn.

What was very important in the worksheets, was that learners had to read and understand what was asked of them before attempting to answer the questions. While observing the learners kept on asking if their answers were correct. The HOD told them that the method they use is what is important and not their answers. For task 2 some prior knowledge was required, they had to

know what perimeter was. I observed that question 3.6 was the question that got the learners thinking because they engaged in their groups on methods to solve this question.
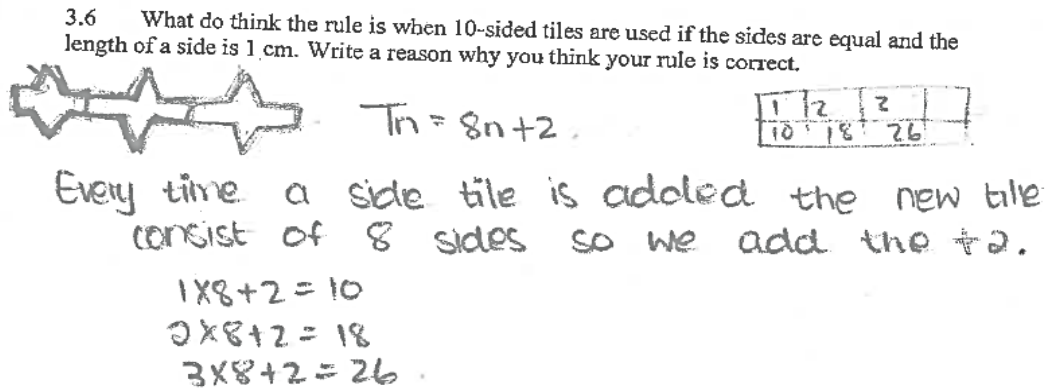
3.6   What do think the rule is when 10-sided tiles are used if the sides are equal and the length of a side is 1 cm. Write a reason why you think your rule is correct.

$Tn = 8n + 2$

1	2	2
10	18	26

Every time a side tile is added the new tile consist of 8 sides so we add the +2.

$1 \times 8 + 2 = 10$
$2 \times 8 + 2 = 18$
$3 \times 8 + 2 = 26$

Figure 4.8: Learner I (Task 2 - 3.6)

3.6   What do think the rule is when 10-sided tiles are used if the sides are equal and the length of a side is 1 cm. Write a reason why you think your rule is correct.

$Tn = 8n + 2$

Using my picture the 1st one will be 10, 2nd one 18 and third one 26

1	2	3
10	18	26

$\rightarrow Tn = 8n + 2$

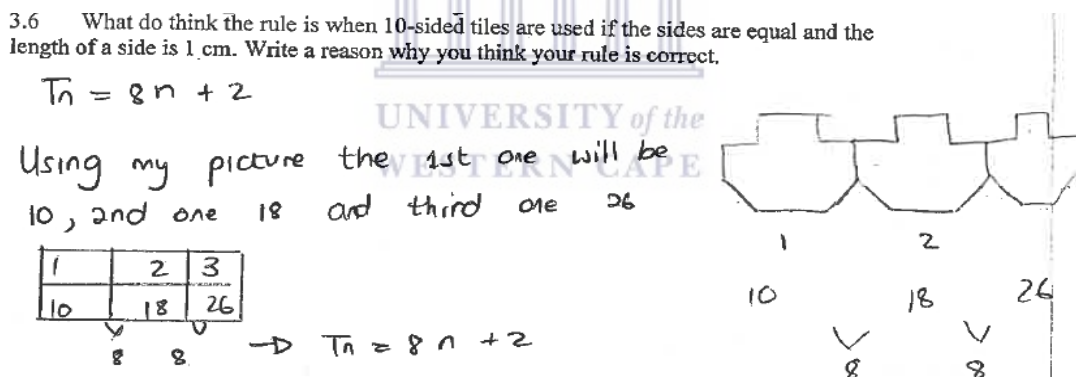| 1	2		
10	18	26	

8   8

Figure 4.9: Learner P (Task 2 – 3.6)

Learner I and P used drawings, a table and calculations to explain the formula that was derived. Learner I and P has the same formula, but their ways of reasoning are different.

It was interesting to see the different drawings the learners came up with and their way of explaining their methods used.

For Task 3 an example of a question was given to which three learners' answers and 3 teacher comments were given. Learners had to a) map what comment fits with what answer and b) identify which comment they agree with the most. The learners in the class did not understand what to do at first. The HOD had to intervene and explain to them what they had to do. Four learners out the group did not answer task 3 at all. 16 learners only answered part A of the question but not which one they agreed with most.
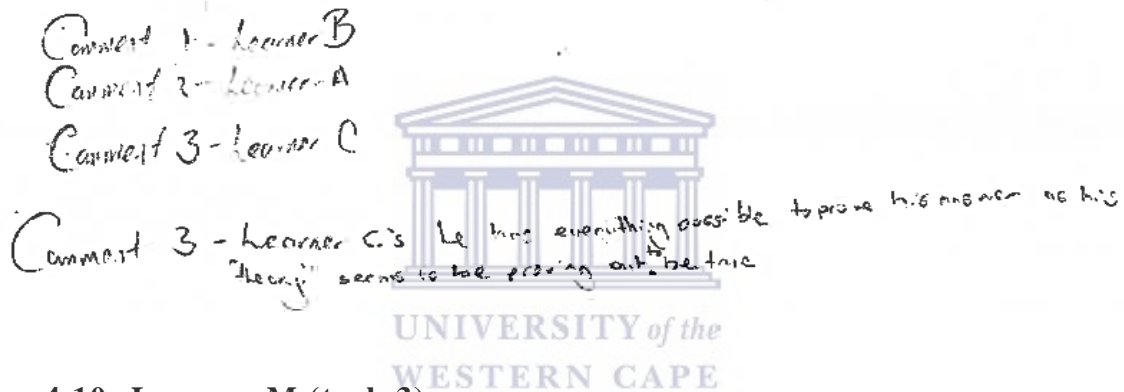

Figure 4.10: Learner M (task 3)

Learner M's answer for the second question was faint; he wrote the following:

"Comment 3- Learner C's has everything possible to prove his answer as his "theory" seems to be turning out to be true.

Figure 4.11: Learner R (task 3)

Learner R agrees mostly with comment two because "to solve problems together, you can correct each other in case one is wrong or has made a mistake"

All the learners had different reasoning to which comment fits with which learner. 8 out of 20 learners first started reasoning the question themselves before looking at the learners' answers.

The second was a group interview to collect in-depth information. I only interviewed three groups; the first group consisted of 5 learners, the second group consisted of 3 learners and the third group of two learners.

TASK 1

1. What method was used to get the next term?

2. How did you come up with your rule?

Figure 4.12: Questions based on Task 1

Question 1: All of the participants responded that they added 3 to get the next term. They did not go into depth in this explanation. One put it this way: "I saw that 3 was added the whole time. So I used the formula to check if substituting the term in the formula will give me the correct answer". 5 out of the 8 participants used the formula in question 1 and rewrote it in question 2. They also mentioned that they used the general formula; for the nth term of a linear sequence: $T_n = a + (n-1)d$.

TASK 2

3. What method did you use to get the perimeter of the 4 tiles?

4. Do you agree with Jonathan's statement: when square tiles with sides of 1cm each are used then the rule is $P = 2n + 2$?

5. When you calculated the rule for 10 tiles what method did you use?

6. What do you notice if you look at the answers of 3.3, 3.5, and 3.6?

Figure 4.13: Questions based on Task 2

Question 3. All of them stated that they continued drawing and counted the perimeter.

Question 4: Most of them said they agreed with Jonathan, the only one said she doesn't agree, but she couldn't remember why. One said that when he substituted 1 into the rule $P = 2n + 2$ he got 4 which is equal to the perimeter of the tile.

Question 5: All of them said they used drawings to get to the rule of the 10 tiles and they used Jonathan's rule as a base to work on.

Question 6: 1 learner said that she noticed that all the answers had a plus 2 (added 2). Another learner said that the answers are kind of the same. While someone else said they are halves of each other. 3.5 is 10 cm which is half of 3.6.

TASK 3
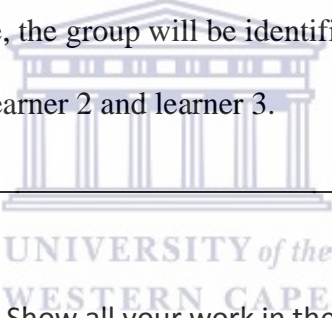
7. What method did you use to answer the questions?

Figure 4.14: A question based on Task 3

Question 7: Most of them first tried to figure out what the pattern was. The one group agreed that it was square numbers. They further looked at the learners' answers and then mapped it with the teacher comments. Two of them couldn't answer because they didn't do that question.

In both interviews, there are parts that blurred out but in those cases, I either used the other group's answers or someone else's answer if more than one participant in the group responded.

The following represents the discussion of each task and its sub-questions:

Only three groups' discussions were used, the others weren't clear because learners spoke too softly. Seemingly the device that was used did not record the discussion well. Each group consisted of 3 learners. In each case, the group will be identified and the learner who answered in the group; for example, learner 1, learner 2 and learner 3.

Task 1

   a) Complete the table below. Show all your work in the space below the table

Term Number	1	2	3	4	5	10
Term Value	6	9	12			

Figure 4.15: Task 1 a

Task 1 a) Out of the 3 groups; 4 learners explained that the formula was used: $T_n = a + (n-1)d$. Also that 3 was substituted as the difference. In group 1, learner 3 explained that she substituted -3, learner 2 explained that it cannot be -3 because the pattern is increasing. In group 2 learner 1 said that the rule is 3n +3 and the answer was 3, 6, 9, 15, 18 and 33, no discussion was

given on how the formula was derived. Learner 3 did not agree with the last term that is equal to 33 and stated that it is 21.

b) Write a rule or formula to find the term value for any term number when you know the term values of the first three consecutive terms.

Figure 4.16: Task 1 b

Task 1 b) All participants of the three groups used their answers in a) to determine the answer in b. In group 1, learner 1 explained that he substituted 3 for the value of "a" and not "d".

Learner 2 explained that 3 is the difference and should be substituted by the difference and that 6 is the first value, so it should be substituted by a. In group 2; all the learners just rewrote the general term because they calculated it in the previous questions.
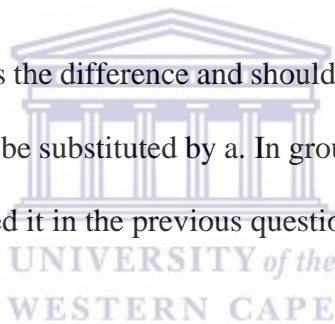
c) The term value is 72. What is the term number? (Explain your reasoning below).

Figure 4.17: Task 1 c

Task 1c) Most learners from the 3 groups used the formula derived in b to get the term number. Learner 1 in group 1 and learner 3 in group 2 explained that they cannot substitute 72 by n, because they need to get n. In group 2 learner 2 explained that 72 was substituted for Tn : $72 = 3n + 3$, furthermore that the $+ 3$ must be taken to the other side. This indicated incorrect wording used, instead of saying that the inverse operation should have been done to remove the $+ 3$, and

got 23 as the term. In group 3 learner 1 explained that the 72 was not substituted; 23 was substituted to get 72 and therefore the number was 23.

Task 2

Tandeka arranges triangular tiles. The sides of the triangle are 1cm each. She arranges the tiles in rows and finds the perimeter of each arrangement.

1 Tile
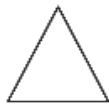
Perimeter: 3 cm

2 Tiles

Perimeter: 4 cm

3 Tiles

4 Tiles

2.1   What is the perimeter if 4 tiles are arranged in the same pattern?

Figure 4.18: Task 2- 2.1

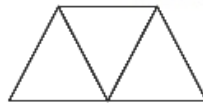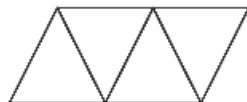Task 2; 2.1) All the learners from the 3 groups said that that they counted the number of lines on the outside of the tiles, the perimeter of 6 was derived. In group 2; learner 1 said that she got 6cm because she counted each triangles perimeter.

In group 2; learner 2 said that: "every time you add one triangle to the previous arrangement and when you counted the sides you get 6". He got 9 as the perimeter and explained that he added the lines on the inside of the triangle as well. Learner 3 of the same group told learner 2 that he misunderstood what perimeter was because it is the length around the shape, and didn't see it as the length around the "new shape".

2.2	What is the perimeter if 10 tiles is arranged in the same pattern? Write down how you found the perimeter for the 10 tiles.

Figure 4.19: Task 2 – 2.2.

Task 2, 2.2) Learner 2; in group 1 and learner 1; in group 2 explained that a drawing was used to get the perimeter of 10 tiles, the perimeter was counted and it was 12. None of the three groups used a different method to solve the question.

2.3	Write a rule/formula for finding the perimeter, P, if n is the number of triangular tiles in the pattern. Explain how the parts of your formula are related to the pattern in the picture above.

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Figure 4.20: Task 2 – 2.3

Task 2, 2.3) In group 1; learner 1 said that they just kept on drawing to get the perimeter, no formula was derived.

2.4   Jonathan said: *When square tiles with sides of 1 cm each are used then the rule is* *P = 2n + 2.* Do you agree with Jonathan? Write a reason for your answer.

Figure 4.21: Task 2 – 2.4

Task 2, 2.4) In group 1; learner 2 explained that 4n +2 would be a better option because a square has 4 sides. Learner 2 did not agree with Jonathan. In group 2; learner 3 agreed with Jonathan because when he drew squares and counted it, he discovered that the rule is true.

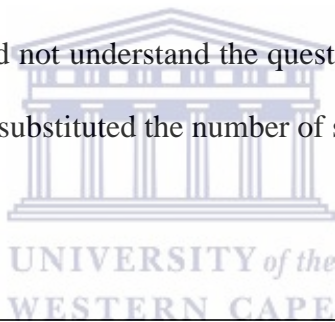| |
|---|
| 2.5   Write a rule for finding the perimeter if an arrangement is made with 5-sided tiles if

the sides are equal and the length of a side is 1 cm. |

Figure 4.22: Task 2 – 2.5

Task 2, 2.5) In group 1; learner 1 said the rule is 4n because it is a 5 sided tile. In group 2; learner 1 said that she did not understand the question. Learner 3 from the same group said that a pentagon has 5 sides, and used drawings to get to his answer. He got his difference as 3, therefore the term is $T_n = 3n + 2$.

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| 2.6 | What do think the rule is when 10-sided tiles are used if the sides are equal and the |
| | length of a side is 1 cm. Write a reason why you think your rule is correct. |

Figure 4.23: Task 2 – 2.6

Task 2, 2.6) In group 1; learner 2 said that they used the previous pattern, so it is 10n. In group 2; learner 1 and learner 2 did not understand the question. In group 2; learner 3 explained that the rule is 6n + 4 and when he substituted the number of sides, the rule worked.

Task 3

A grade 8 class was given the following task.

Squares of different sizes are drawn and the dots inside each square are counted. Write a rule or formula to find the number of dots inside the square when you know the length of the side of the square.

The learners handed in their work and the teacher did not mark their work. She wrote comments for the work of each learner. Given below are the answers of three learners.

Learner A

The number of dots are square numbers. Even 0 is a square number. To get 0, 1 is subtracted from the length of the side. This gives 0 and the square number for 0 is 0.

The next number of dots = 2 - 1 = 1. The square number of 1 is 1.

The rule is

Number of dots = (length of side - 1) squared.

Learner B

SL	1	2	3	4	5	6
D	0	1	4	9	16	25

SL = length of side of square
D = number of dots. The number of dots are all square numbers. The rule is
D = (SL)² if the SL is always the one before the next one.
So for SL = 2 then you square SL before the 2, which is 1 then D = (1)² = 1 for SL.
If 1 then D = (1)² = 1 for SL. If It works the same for all the other SL.
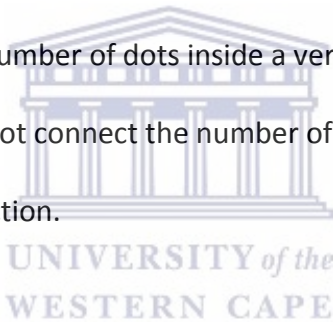
Learner C

The difference of the number of dots is the odd numbers.

Rule: Add the odd numbers in order

The next number of dots will be 16 because 7 will be added to 9. There will be 81 dots inside if there are drawings.

Teacher comments.

Comment 1

You worked correctly by finding the difference between the number of dots. Your rule works but if will be difficult to find the number of dots inside a very big square, say one when you have 50 squares. Your rule does not connect the number of dots to the length of the side of the square. Try to find the connection.

Comment 2

The way you organized your answer by getting the number of dots in more squares is good. Your rule is correct for the example that you give. When making the rule it should work for all the information that is given and you rule does not work to get the number of dots (0) when the length of the side is 1. Check with other learners in the class how you can change your rule so that it also works for this case.

Comment 3

You obtained a correct rule. It is always wise to show that you have checked your rule with all

the drawings and also try to create other drawings of which shows to show that your rule still

works.

Which comment corresponds to which learner's answer?

Which comment do you agree with most? Give reasons for your answer.

Figure 4.24: Task 3

Task 3: Group 1; learner 2 said that $p \times l$, which means that it is the perimeter multiplied
by the length. Learner 1 and 2 stated the following: Comment 1 corresponds with learner A,
Comment 3 corresponds to learner C. Comment 2 corresponds to the learner B. Furthermore they
agree with learner C because when they have done it you get the same answers and it makes the
most sense.

In group 2 learner 1 and 3 said the rule is $T_n = n^2$. Learner 2 did not agree with the others
in the group. He used $T_n = (n + 1)^2$, and asked the others to check if their rule is correct. Group
2 had the same mapping as group 1. In group 2 learner 1 agreed mostly with comment 2 because
of another way of looking at the answer that was questioned. Learner 2 agreed with comment 3
because when he substituted the term value he obtained the same answers as in the information.
Learner 3 said that maybe he understood the question wrong because he got that Comment 3 fits
mostly with learner 3, the reasoning was when he added the odd numbers he got the same answers
as learner 3; in task 3.

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In most cases, learners just gave the answer without explanation. There were discussions mostly in task 2.

4.3. Students' responses to the exercises

To support the task given to the learners and to fill the gaps, interviews were used. The information collected from the interviews was presented in a narrative form that includes the description and analysis of data.

The purpose of the interviews was to find out how the learners felt about the task and the questions that were asked.

The interview was done with the same learners who agreed on the voice recordings and a few other learners from the class. Group 1 had three additional learners; for group 2 only 1 learner came to the interview but had 3 additional learners with. The interviews were only done with two groups, which was a total of 9 learners.

TASK 1

1. What method was used to get the next term?

2. How did you come up with your rule?

Figure 4.25: Interview questions based on task 1.

Task 1,1. In group 1, interviewee 1 said that they were just adding 3 to get the next term. Interviewee 2 stated that the formula was used; $T_n = a + (n-1)d$.

In group 2; interviewee 1 said that they used the arithmetic formula: $T_n = a + (n-1)d$.

Task 1.2, In group 1; interviewee 3 explained that the difference was 3. Learner 2 said that the formula was used and the value of "a" and "d" was substituted.

In group 2, interviewee 2 said that he could see that there was a constant difference. He played around with the numbers to see what works. He lastly substituted the x- values to check if it works.

TASK 2
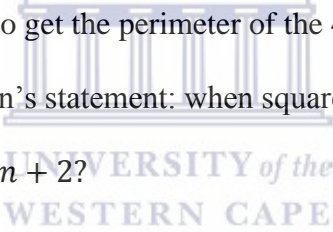
3. What method did you use to get the perimeter of the 4 tiles?

4. Do you agree with Jonathan's statement: when square tiles with sides of 1cm each are used then the rule is $P = 2n + 2$?

5. When you calculated the rule for 10 tiles what method did you use?

6. What do you notice if you look at the answers of 2.3, 2.5, and 2.6?

Figure 4.26: Interview questions based on task 2.

Task 2: 3, In group 1, interviewee 4 said that they used drawings and got the answer.

In group 2, interviewee 3 said that 1 tile consists of 3 tiles, therefore the perimeter of a tile is 3. He drew the four tiles and counted it and got to 6cm.

Task 2; 4, In group 1; interviewee 1 just said yes that they agree with Jonathan. In group 2; interviewee 3 said that he agrees because the perimeter of a square is 4 cm and 2 was added to the previous pattern. When you substitute the term number you get the term value.

Task 2. 5, In group 1; interviewee 2 said that they used Jonathan's rule to determine the rule of 10 tiles. In group 1; interviewee 1 said that she used drawings to derive the method.

In group 2; interviewee 2 said because it is 10 sided tiles, so you can substitute it into the formula. Furthermore, he said that he saw a pattern between the answers, that it was for the three tiles n +2 and for the square 2n +2, so he went to calculate all the shapes and got to a 10sided tile: 8n +2.

Task 2, 6. In group 1; interviewee 4 explained that 2 was added in each pattern. In group 2; interviewee 1 said the answers are the same, and the number of n is constantly going up. He added that the term number is always added with 2. In group 2; interviewee 2 said that he noticed that it is halves of each other. 2.5 answer is half of the answer of 2.3.

TASK 3

7. What method did you use to answer the questions?

**Figure 4.27: Interview questions based on task 3.**

Task 3, 7, In group 1, interviewee 5 explained that they used square numbers. Interviewee 3 said that they just mapped which comment fit with a learners' calculations.

In group 2; interviewee 2 said that he first tried to do the answer on his own and got $T_n = (n + 1)^2$. Furthermore, she said that some of the questions were confusing at first. Interviewee 1 said that it was a bit challenging and would probably get 50% for this task. Some parts of the recordings weren't clear because the learners' spoke too soft.

4.5. Conclusion

This chapter discussed the analysis of data and the findings of the study on how learners reasoned algebraically. Some learners only had the answer without a supporting reason. About 3 learners did not complete task 3 and it was picked up in the recordings that they either did not have time to do it or did not know what to do.

The next chapter of this study presents the interpretation and discussion of the findings with some recommendations and a conclusion.
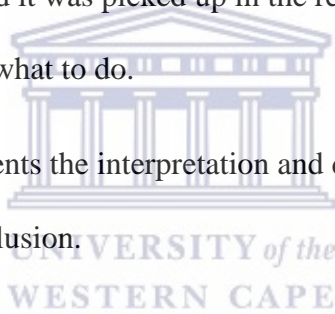
Chapter 5:

Interpretation of Findings, Summary and Reflections

5.1. Introduction

The aim of this study was to investigate the algebraic reasoning of grade 9 learners in an intentional teaching context. In essence, this research investigated how intentional teaching could be implemented to assist how learners' algebraic reasoning. The significance of this research lies in its contribution to a better understanding of how learners reason algebraically. This chapter includes discussions based on the tasks given to learners to do. It also makes a recommendation for teaching and concludes the thesis.
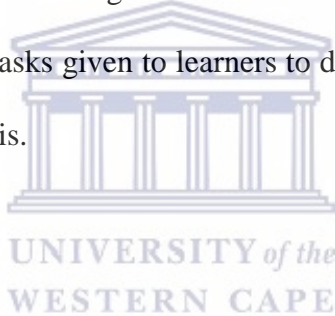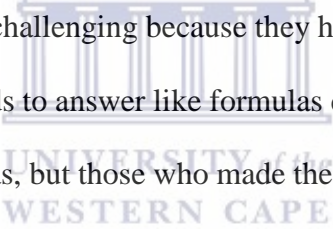
5.2. Interpretation of data

The research was conducted in the researcher's class with 30 learners, only 20 gave consent for their scripts to be used, therefore analysis was done with only 20 scripts. Learners were given a task to complete and the following is the interpretation of the task.

Task 1 was a knowledge question, learners found it easy to answer but wanted to know if their answers were right. Learners are more interested in whether their answer is right and less interested in the process of getting to the answer.

Despite directly accessing learners' mathematical thinking and reasoning behind their actions it is true that we can have access to their thinking through other methods such as when they explain their work and through interviews. In question 1 it was already picked up that learners'

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algebraic thinking are not sophisticated because they just answered the questions without giving reasons. One could argue that they did not understand fully what they were doing or that they could explain what they were doing. A few learners made mistakes by substituting incorrectly, for example substituting 3 for the value of "a" and not "d". They did not know the difference between the first term and the constant difference. During the group discussion, one of the members in the group explained the difference between "a" and "d". The confusion also came when they had to find the value of n. Just because the question asked: "determine the term number"; in the group, it was explained that 72 should not be substituted by n, because they need to calculate n. Lack of practice played a role here because learners mostly get confused with the term number and the term value.

Task 2 learners found a bit challenging because they had to read and figure out what to do and they could use different methods to answer like formulas drawings and tables. There was also a confusion with what perimeter was, but those who made the error stated that they did not have a clear understanding of what perimeter was. Most learners just left out questions and when they were discussing they couldn't participate and did not ask for assistance.

The last two questions in task 2 were a bit of a problem for learners to answer because learners had to use what was given and had to write down the rule. Eleven learners out of the group only answered the questions without explanation, 6 learners used drawings to answer their questions and 3 used tables. The 11 that only answered the questions made mistakes and didn't get to the right general term.

During group discussions, in the 1 group, 1 learner used a drawing and the other used a table whilst the last one only had the answer. The two with the illustrations explained to the one who only wrote down the answer that it should have been a 5 sided tile, so therefore the first term

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was 5. Even though they added 4 tiles only, the outside ones should be counted because it is the perimeter. Knowing what perimeter meant played a big part in task 2.

For task 3 there was different answers and different reasoning to why they chose those combinations. This was a good question because all of them who were interviewed explained why they believed that the comment should go with a certain calculation.

The researcher first wanted each learner to work on their own because mostly with group work there is always one who leans on others. When it came to the group discussions and learners had differences, one or two in the group would suggest to make a drawing, use a table or to use a formula to check if the answers were right. This was a comfort to many because when they were uncertain they could use the drawings or calculations. In one group a few learners just left out the answer when they did not understand. During group discussions, they did not participate and copied down answers from the others. This is what happens in most classes I have been involved in. When learners do not understand they do not tell the teacher or ask for assistance. Consequently, when they get home they are lost and do not try to do their homework and hence during corrections and feedback they write down answers without understanding the work.

The researcher also discovered that as teachers we assume that there are certain things learners should know when they are in grade 9. For example, we assume that learners should know by this stage the meaning of perimeter and forget that we need to reinforce everything for learners to understand.

Mason (1996, p.75) states that school algebra instruction has been continuously criticized for "rushing from words to single letter symbols". This is true because learners fail to make the connection between words and letters (variables). Several researchers have suggested that

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generalisation of pattern as a preferred introduction to algebra. When the term algebra is used it includes two distinct concepts: algebraic thinking and algebraic symbolism. There is a lack of agreement among researchers as to the relationship between the two. Some view the algebraic symbols as a necessary component of algebraic thinking, while others consider them as an outcome or as a communication tool. Further, different perspectives are argued on the relationship between algebraic reasoning and generalization. Kieran (1989, p. 165) on the other hand states that, "generalization is neither equivalent to algebraic thinking, nor does it even require algebra.

Charbonneau (1996, p. 35) considers symbolism as central to algebra, but "not the whole of algebra". He explains symbolism as a language that may condense the presentation of an argument and as a means to solve problems.

For Kaput and Blanton (2001), agrees that generalising and formalising patterns and constraints is one of the forms of the 'complex composite' of algebraic reasoning.

5.3. Limitations of the research

The two major limitations of this study was that the researcher could not present all the data because of the poor sound quality of the recordings. Some leaners spoke too softly and there was a lot of interruption from the school intercom which was beyond the researcher's control. The second major limitation was that intentional teaching was a once off engagement with the learners. This was a limited duration of implementation and despite of intentional teaching, learners could not provide adequate explanations to their solutions.

Another limitation is that even though the researcher started with 30 learners, only 20 learners gave consent for their worksheets to be used because it was voluntary. This research encountered challenges regarding personal feelings. Ten learners were reluctant to answer the tasks because they wanted to give the right answers only and felt ashamed of their answer and decided not to give consent for it to be used. Even though this was the case, it had no impact on the responses obtained from the rest of the class. For question 2.6 a few learners only gave the answer and did not provide reasoning. They also could not engage in deeper level discussions. This was an indication that they had a poor conceptual and procedural understanding. Even though recordings were made of group discussions about two recordings were unclear.

5.4.   Conclusion

As stated at the beginning of this research, one of the major issues is that learners' reasoning skills in mathematics are not as sophisticated as one would expect. Intentional teaching was used so that learners could understand what the content of the study was all about. This research revealed that some learners lacked understanding. It is important for learners to gain a correct understanding of concepts in the beginning. "The best time to learn mathematics is when it is first taught; the best way to teach mathematics is to teach it well the first time."

Learners were not aware of their errors and keep on committing the same procedural errors. They lack conceptual knowledge and therefore cannot check their solutions. Therefore it is important to use intentional teaching during the lesson.

The researcher hopes that this research has contributed to mathematics in general.

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