Pricing European options: A model-free approach.

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Abstract

This paper focuses on the newly revived interest to model free approach in finance. Instead of postulating some probability measure it emerges in a form of an outer-measure. We review the behavior of a market stock price and the stochastic assumptions imposed to the stock price when deriving the *Black-Scholes* formula in the classical case. Without any stochastic assumptions we derive the *Black-Scholes* formula using a model free approach. We do this by means of protocols that describe the market/game. We prove a statement that prices a European option in continuous time.

Keywords Model free. Price path. European options. Continuous time. Protocol. Outer-measure

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

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1. Introduction

Pricing of options, commodities, securities, or even currencies has been a study given much attention in economics and finance. The *Black-Scholes* formula was introduced in 1973, in two celebrated articles, one by (Black and Scholes, 1973), the other by (Merton, 1973). This formula, has been used to price a lot of derivatives, simple and complex and has led to growth in markets for options and other derivatives, it remains the foundation for option pricing. This formula does not only price options but also gives a broad idea on how to hedge the risk involved in these markets. It can be adjusted until pricing is done by supply and demand. The formula was recognized by a Nobel Prize in 1997.

We note that a derivative is a contract whose payoff depends on the future movements of the prices of one or more commodities, securities or currencies. We will focus on European options whose payoff depend only on the price of a single security at a fixed date of maturity. A European option $\mathcal{U}$ on an underlying security S is characterised by its maturity date, say T and its payoff function, say U. Its payoff at time T is given by

$$\mathcal{U}(T) := U(S(T)). \tag{1.0.1}$$

The problem is to price $\mathcal{U}$ at a time t before time T. What price should a bank charge at time 0, say, for a contract that requires it to pay (1.0.1) at time T? The *Black-Scholes* pricing formula has been used to answer this problem.

The *Black-Scholes* formula relies on the assumption that the price of the underlying stock follows a *geometric Brownian* motion. It has been observed in (Shafer and Vovk, 2001) that this assumption limits the wildness of fluctuations in $S(t)$ and that it uses the law of large numbers on a relatively fine time scale. The appeal to the law of large numbers present challenges in real applications.

To circumvent the above mentioned difficulties, a model free approach has been suggested, see (Shafer and Vovk, 2001). This approach has received much attention and much work is being done on this, see (Takeuchi et al., 2009), (Perkowski and Prömel, 2016), (Cont, 2006), (Łochowski, 2015), (Vovk, 2008b) and (Vovk, 2015). The model free approach does not require the assumption that the market price of the underlying security follows a *geometric Brownian* motion. In particular, no a priori probabilistic assumption is needed. In a model free approach we do not start with any probability measure or some structure at outset, however, it does emerge. We look at this measure as an outer-measure. In this short project we derive the *Black-Scholes* pricing formula for European options using a model free approach.

This project is organised as follows: In Chapter 2 we look at the notations and symbols used in this project, we also look at background information, not leaving out definitions and examples where necessary. In Chapter 3 we examine the random behaviour of stock market prices and we show how the stochastic assumption is used to derive the *Black-Scholes* formula in the classical sense. In Chapter 4 we show how the *Black-Scholes* formula is derived using the model free approach. We prove the *Black-Scholes* pricing formula using the model free approach and we give a brief conclusion in Chapter 5.

2. Preliminaries

Here we give definitions that form the building blocks and understanding towards later sections and chapters. We also outline some notation used in this project.

2.1 Notations

Below we give a layout of the meanings to the notations used in this project.

$S(t)$	: Stock price
σ	: Volatility
μ	: Drift
Ω	: Sample space
$\mathcal{F}$	: σ-algebra
$\mathcal{F}_t$	: Filtration
$\mathbb{P}$	: Probability measure
$\mathbb{R}^+$	: Set of positive real numbers
$\mathcal{N}_{\mu,\sigma^2}$	: Normal distribution with mean μ and variance σ^2
LHS	: Left hand side
RHS	: Right hand side
$:=$	: LHS is equal to RHS by definition
$T > 0$	: Maturity time
$\mathbb{T} := \{n\mathrm{d}t \mid 0 \leq n \leq N\}$	: Set of time points
$W(t)$	: Wiener process
$\mathcal{I}(t)$	: Investor's capital process
GBM	: Geometric *Brownian motion*
SDE	: Stochastic differential equation
pde	: Partial differential equation
w.r.t	: with respect to

2.2 Basic concepts and results

Most of the material in this section were taken from Chapter 1 of (Shafer and Vovk, 2001), and the papers (Vovk, 2008b) and (Vovk, 2009).

We consider a game between two players, Investor and Market, over the time interval $[0, \infty)$. For each time $t \in [0, \infty)$, the value $\omega(t)$ represents the price of the financial asset at time t. First Investor chooses his trading strategy and then Market chooses a continuous function $\omega : [0, \infty) \to \mathbb{R}$ (the price path of a security).

2.2.1 Definition. (Sample space) Let Ω be the set of all continuous functions $\omega : [0, \infty) \to \mathbb{R}$.

For each $t \in [0, \infty)$, $\mathcal{F}_t$ is defined to be the smallest σ-algebra that makes all functions $\omega \mapsto \omega(s)$, $s \in [0, t]$, measurable. A process S is a family of functions $S_t : \Omega \to \mathbb{R}$, $t \in [0, \infty)$, each S_t being

$\mathcal{F}_t$-measurable. Its sample paths are the functions $t \mapsto S_t(\omega)$. An element of the σ-algebra $\mathcal{F}_\infty := \vee_t \mathcal{F}_t$ (also denoted by $\mathcal{F}$) is called an event.

2.2.2 Definition. (Filtration) A filtrtation on Ω is a family $\{\mathcal{F}_t : t \in T\}$ of sub-σ-fields of $\mathcal{F}$ with $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s < t$. A process S on $(\Omega, \mathcal{F})$ is adapted to the filtration $(\mathcal{F}_t)$ if S_t is $\mathcal{F}_t$-measurable for each t. We define

$$\mathcal{F}_\infty := \sigma\left(\bigcup_{t\in T} \mathcal{F}_t\right).$$

A filtration is said to be right-contionuous if $\mathcal{F}_t = \mathcal{F}_{t_+} := \cap_{s>t}\mathcal{F}_s$, for each $t \in T$.

2.2.3 Definition. (Stopping time) A function $\tau : \Omega \to \overline{T}$ such that $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for each time $t \in T$ is called a stopping time for the filtration. For a stopping time τ we define

$$\mathcal{F}_\tau = \{F \in \mathcal{F}_\infty : F\{\tau \leq t\} \in \mathcal{F}_t\} \quad \text{for each} \quad t \in T.$$

2.2.4 Remark. The following statements are true for stopping times:

a. If the filtration is right continuous and if $\{\tau < t\} \in \mathcal{F}_t$ for each $t \in T$ then τ is a stopping time.

b. $\mathcal{F}_\tau$ is a σ-field.

c. τ is $\mathcal{F}_\infty$-measurable.

d. An $\mathcal{F}_\infty$-measurable random variable Z is $\mathcal{F}_\tau$-measurable if and only if $Z\{\tau \leq t\}$ is $\mathcal{F}_t$-measurable for each $t \in T$.

2.2.5 Definition. For any arbitrary set $A \in \mathcal{F}$ we define

$$\mathbb{I}_A(\omega) := \begin{cases} 1 & \text{if} \quad \omega \in A, \\ 0 & \text{otherwise.} \end{cases}$$

This is called an indicator function.

If $(\mathcal{F}_t)$ is right-continuous, it is equivalent to demand that $\{\tau < t\}$ belongs to $\mathcal{F}_t$ for every t. In this case, τ is a stopping time if and only if the process $S_t = \mathbb{I}_{(0,\tau]}(t)$ is adapted. The class of sets A in $\mathcal{F}_\infty$ such that $A \cap \{\tau \leq t\} \in \mathcal{F}_t$ for all t is a σ-algebra denoted by $\mathcal{F}_\tau$. The constants, i.e $\tau(\omega) \equiv s$ for every ω are stopping times and in that case $\mathcal{F}_\tau = \mathcal{F}_s$.

Stopping times therefore appear as generalizations of constant times for which one can define a past which is consistent with the pasts of constant times. A stopping time may be thought of as the first time some physical event occurs.

2.2.6 Proposition. If E is a metric space, A is a closed subset of E and X the coordinate process on $W = C(\mathbb{R}^+, E)$ and if we set

$$D_A(\omega) = \inf\{t \geq 0; X_t(\omega) \in A\}$$

with the understanding that $\inf(\emptyset) = +\infty$, then D_A is a stopping time w.r.t the natural filtration $\mathcal{F}_t = \sigma(X_s, s \leq t)$. It is called the entry time of A.

Proof. For a metric d on E we have

$$\{D_A \leq t\} = \{\omega : \inf_{s\in\mathbb{Q}, s\leq t} d(X_s(\omega), A) = 0\}$$

and the RHS set obviously belongs to $\mathcal{F}_\tau$. $\square$

2.2.7 Proposition. If A is an open subset of E and Ω is the space of right-continuous paths from $\mathbb{R}^+$ to E, the time

$$\tau_A = \inf\{t > 0 : X_t \in A\}$$

is a stopping time w.r.t $\mathcal{F}_{t+}$. It is called the hitting time of A.

Proof. As already observed, τ_A is a $\mathcal{F}^\circ_{t+}$-stopping time if and only if $\{\tau_A < t\} \in \mathcal{F}_t$ for each t. If A is open and $X_s(\omega) \in A$, by the right-continuity of paths, $X_t(\omega) \in A$ for every $t \in [s, s+\epsilon)$ for some $\epsilon > 0$. As a result

$$\{\tau_A < t\} = \bigcup_{s\in\mathbb{Q}, s<t} \{X_s \in A\} \in \mathcal{F}_t.$$

□

An elementary trading strategy G consists of an increasing sequence of stopping times $0 \le \tau_1 \le \tau_2 \le \cdots$ such that $\lim_{n\to\infty} \tau_n(\omega) = \infty$ for each $\omega \in \Omega$, and, of a sequence of bounded $\mathcal{F}_{\tau_n}$-measurable functions h_n, $n = 1, 2, \ldots$. To such a G and an initial capital $c \in \mathbb{R}$ corresponds the elementary capital process

$$\mathcal{K}_t^{G,c}(\omega) := c + \sum_{n=1}^{\infty} h_n(\omega) \left(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\right), \quad t \in [0, \infty), \tag{2.2.1}$$

where the value of the sum is finite for each t. The value $h_n(\omega)$ will be called Investor's bet at time τ_n, and $\mathcal{K}_t^{G,c}(\omega)$ will be called Investor's capital at time t.

A positive continuous capital process is any process S that can be represented in the form

$$S_t(\omega) := \sum_{n=1}^{\infty} \mathcal{K}_t^{G_n,c_n}(\omega), \tag{2.2.2}$$

where the elementary capital processes $\mathcal{K}_t^{G_n,c_n}(\omega)$ are required to be nonnegative, for all t and all ω, and the positive series $\sum_{n=1}^{\infty} c_n$ converges (Vovk, 2008b). Since $\mathcal{K}_0^{G_n,c_n}(\omega) = c_n$ does not depend on ω, $S_0(\omega) = \sum_{n=1}^{\infty} c_n$ also does not depend on ω and will be abbreviated to S_0. Any real-valued function on Ω is called a variable.

2.2.8 Definition. (Upper probability) We define the upper probability of a set $E \subseteq \Omega$ as the upper price of the indicator function of E, $\mathbb{I}_E$, and will be denoted as $\overline{\mathbb{P}}(E)$. That is

$$\overline{\mathbb{P}}(E) := \inf\{S_0 \mid \forall\omega \in \Omega \quad \liminf_{t\to\infty} S_t \ge \mathbb{I}_E(\omega)\}, \tag{2.2.3}$$

where S ranges over positive capital processes.

2.2.9 Remark. The definition of positive capital process corresponds to the idea where Investor divides his initial capital into a sequence of independent accounts, without him risking being bankrupt.

$E \subseteq \Omega$ is said to be null if $\overline{\mathbb{P}}(E) = 0$. A set $E \subseteq \Omega$ is almost sure if $\overline{\mathbb{P}}(E^c) = 0$, where $E^c := \Omega \setminus E$ is the complement of E. The lower probability is defined as:

$$\underline{\mathbb{P}}(E) := 1 - \overline{\mathbb{P}}(E^c). \tag{2.2.4}$$

2.2.10 Example. The upper and lower probability obey the following properties, see Chapter 1 of Shafer and Vovk (2001),

$$0 \le \underline{\mathbb{P}}(E) \le \overline{\mathbb{P}}(E) \le 1 \tag{2.2.5}$$

and

$$\underline{\mathbb{P}}(E) = 1 - \overline{\mathbb{P}}(E^c), \tag{2.2.6}$$

here E^c is the complement of $E \in \Omega$. What meaning can be related to $\overline{\mathbb{P}}(E)$ and $\underline{\mathbb{P}}(E)$? Now, let $\overline{\mathbb{P}}(E) = 0.0002$. Since $\underline{\mathbb{P}}(E)$ is between 0 and 0.0002, then is very close to zero, by (2.2.5). Then Skeptic can spend 0.0002 on $\mathbb{I}_E$, because $\mathbb{I}_E \geq 0$, this purchase does not expose him to possibile bankruptcy, and if E happens then that will result in his investment increasing drastically. But this increase is unlikely happen, which means E is also unlikely to happen. In a similar way, we can say that E is very like to happen if $\underline{\mathbb{P}}(E)$ is very close to one, which means by (2.2.5) $\overline{\mathbb{P}}(E)$ is also very close to one. By (2.2.6) if $\underline{\mathbb{P}}(E)$ is close to one, then $\overline{\mathbb{P}}(E^c)$ is close to zero and E^c is unlikely to happen which means E is likely to happen.

The upper probability is countably subadditive (see Vovk (2008a)).

2.2.11 Lemma. For any subsequence of events $E_1, E_2, \ldots$ of Ω we have

$$\overline{\mathbb{P}}\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \overline{\mathbb{P}}(E_n)$$

$\overline{\mathbb{P}}$ is an outer-measure in carathèodory's sense. A set $A \in \Omega$ is $\overline{\mathbb{P}}$-measurable if for each $E \subset \Omega$

$$\overline{\mathbb{P}}(E) = \overline{\mathbb{P}}(E \cap A) + \overline{\mathbb{P}}(E \cap A^c).$$

The following result was taken from (Vovk, 2008a).

2.2.12 Theorem. *Each event $A \in \mathcal{F}$ is $\overline{\mathbb{P}}$-measurable and the restriction of $\overline{\mathbb{P}}$ to $\mathcal{F}$ coincide with the Wiener measure W on $\Omega, \mathcal{F}$. In particular, $\overline{\mathbb{P}}(A) = \underline{\mathbb{P}}(A) = W(A)$ for each $A \in \mathcal{F}$.*

3. Behaviour of stock market prices

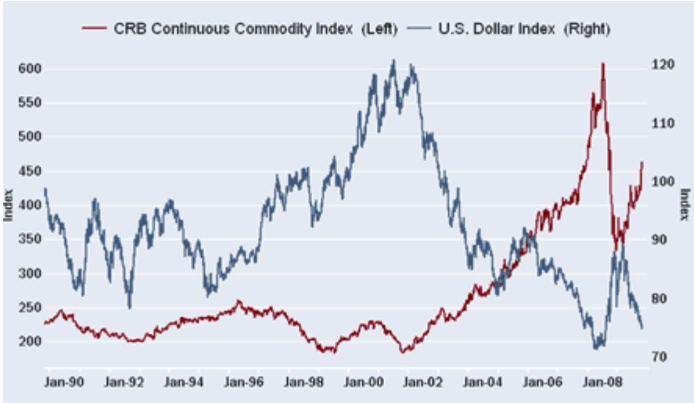

Figure 3.1: Stock market prices for oil on CRB and Dollar index from January 1990 to January 2008 (J. Michael Steele).

Louis Bachelier on his doctoral dissertation '*Theory of Speculation*', published in 1900 (Bachelier, 2011), noticed that stock market prices has an irregular and random character of changes. Having a look at Figure 3.1, we can see the irregular behaviour of stock market prices. This behaviour is now called *Brownian motion*. However, the *Brownian motion* or *Bachelier's model* as is also known had a short-coming in allowing prices to be negative. The price of a corparate share cannot be negative. But this can be overcomed by assuming that the logarithm of the share price follows a *Brownian motion*. That is, the share price itself follows a *geometric Brownian motion*.

Now that we mentioned a *Brownian motion* and *geometric Brownian motion*, let us give their definitions and examples.

3.0.1 Definition. (Linear *Brownian motion*) A real-valued stochastic process $\{B(t) : t \geq 0\}$ is called a (linear) *Brownian motion* with start x if the following holds:

- $B(0) = x$;
- the process has independent increments, i.e. for all times $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ the increments $B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), \ldots, B(t_2) - B(t_1)$ are independent random variables;
- for all $t \geq 0$ and $s > 0$, the increments $B(t+s) - B(t)$ are normally distributed with mean zero and variance s;
- almost surely, the function $t \mapsto B(t)$ is continuous.

3.0.2 Remark. We say that $\{B(t) : t \geq 0\}$ is a standard *Brownian motion* if $x = 0$. In this project we refer to the standard *Brownian motion* simply as the *Brownian motion*.

3.0.3 Definition. (Wiener process) A real-valued continuous function W on $[0, \infty)$ is called a Wiener process if the following holds:

- $W(0) = 0$,
- $W(t)$ is Gaussian with mean zero and variance t, for each $t > 0$, and

- if the intervals $[t_1, t_2]$ and $[s_1, s_2]$ do not overlap, then the random variables $W(t_2) - W(t_1)$ and $W(s_2) - W(s_1)$ are independent.

Alternatively, given a Wiener process $W(t)$, we can define any process S of the form

$$S(t) = \mu t + \sigma W(t), \tag{3.0.1}$$

where $\mu \in \mathbb{R}$ and $\sigma \geq 0$ are constants, as a *Brownian motion*.

Equation (3.0.1) implies that for any positive real number $\mathrm{d}t$, we may write

$$\mathrm{d}S(t) = \mu \mathrm{d}t + \sigma \mathrm{d}W(t). \tag{3.0.2}$$

We get a wider class of stochastic processes when the drift and volatility are allowed to depend on S and t in (3.0.2), that is,

$$\mathrm{d}S(t) = \mu(S(t), t)\mathrm{d}t + \sigma(S(t), t)\mathrm{d}W(t). \tag{3.0.3}$$

All stochastic processes satisfying stochastic differential equations of the form of (3.0.3) are called diffusion processes. This is because of the relationship with the heat equation and because the Wiener process W diffuses the probabilities for the position of the path as time goes on. A diffusion process S that satisfies (3.0.3) has the Markov property: the probabilities of the next state of S depend only on the current state, $S(t)$. This can be generalized, the drift and volatility depend on the whole preceding path of S rather than just the current value $S(t)$. This gives a wider class of processes known as Itô processes.

An American astrophysicist M. F. Maury Osborne published the first detailed study of the hypothesis that $S(t)$ follows a *geometric Brownian motion*. This led to the *geometric Brownian motion* also being called Osborne's log-Gaussian model, see (Shafer and Vovk, 2001) page 201

If $\ln S(t)$ follows a *Brownian motion*, then we have

$$\mathrm{d} \ln S(t) = \mu \mathrm{d}t + \sigma \mathrm{d}W(t). \tag{3.0.4}$$

It follows that $\frac{\mathrm{d}S(t)}{S(t)}$ satisfy a SDE of the form

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu \mathrm{d}t + \sigma \mathrm{d}W(t). \tag{3.0.5}$$

The Equation (3.0.5) can be written as

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t), \tag{3.0.6}$$

whose solution is given by

$$S(t) = S(0)e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}. \tag{3.0.7}$$

3.0.4 Example. Let $S(t)$ be the price of FMC stock at time t from the present. Assume that $S(t)$ is a GBM (*geometric Brownian motion*) with drift $\mu = -0.05$ and volatility $\sigma = 0.4$. If the current price of FMC stock is $\$2.50$, what is the probability that the price will be at least $\$2.60$ one year from now.

Since $S(t)$ is a GBM, $\ln(S(t))$ is a regular *Brownian motion* with drift $\mu = -0.05$ and volatility $\sigma = 0.4$. We want to know the probability that $\ln(S(1)) \geq \ln(2.60)$ given that $\ln(S(0)) \geq \ln(2.50)$. This means

$$\ln(S(1)) - \ln(S(0)) \geq \ln(2.60) - \ln(2.50) = \ln\left(\frac{2.60}{2.50}\right) = \ln(1.04) \approx 0.0392.$$

In this case $Z = (\ln(S(1)) - \ln(S(0)) - \mu t)/\sigma$ is a standard normal random variable. So

$$\begin{aligned}\mathbb{P}(\ln(S(1)) - \ln(S(0)) > 0.0392) &= \mathbb{P}(\ln(S(1)) - \ln(S(0)) + 0.05 > 0.0392 + 0.05)\\ &= \mathbb{P}(Z > 0.223)\\ &= 1 - \mathbb{P}(Z \leq 0.293)\\ &= 1 - \Phi(0.293)\\ &= 1 - 0.5884\\ &= 0.4116\end{aligned}$$

where $\Phi(Z) = \mathbb{P}(Z \leq z)$ is the distribution function of standard normal random variable. Hence, the probability that the price will be at least $\$2.60$ one year from now is 41%.

In the following sections we follow the work of (Shafer and Vovk, 2001), also the definitions were taken from the above mentioned source.

3.1 The variation spectrum

We use nonstandard analysis, because it allows us to derive into continuous time the fundamental idea in which two players alternate moves. Our time interval include all real numbers between 0 and T inclusive. But we divide this interval into an infinitely large number N of steps of equal infinitesimal length $\mathrm{d}t := \frac{T}{N}$. The set of the time points is given by

$$\mathbb{T} := \{n\mathrm{d}t \mid 0 \leq n \leq N\}.$$

The infinitely large N is the number of rounds of play between the players (Investor and Market), where by the infinitesimal number $\mathrm{d}t$ is the time taken to complete each round.

Given a function f on $\mathbb{T}$, we set

$$\mathrm{d}f(t) := f(t + \mathrm{d}t) - f(t)$$

whenever $t \in \mathbb{T} \setminus \{T\}$. We call f a continuous function if $\sup_{t\in\mathbb{T}\setminus\{T\}} |\mathrm{d}f(t)|$ is infinitesimal.

3.1.1 Definition. Given a continuous function f on $\mathbb{T}$, and a real number $p \in \mathbb{R}^+$, we define

$$\mathrm{var}_f(p) := \sum_{t\in\mathbb{T}\setminus\{T\}} |\mathrm{d}f(t)|^p ,$$

the p-variation of f.

$\mathrm{vex}f$ is said to be the variation exponent of f.

3.1.2 Lemma. Suppose f is a continuous function on $\mathbb{T}$. Then there exists a unique real number (we sometimes include $-\infty$ and ∞) $\mathrm{vex}f \in [1, \infty]$ such that

(a) $\mathrm{var}_f(p)$ is infinitely large when $1 \leq p < \mathrm{vex}f$, and

(b) $\mathrm{var}_f(p)$ is infinitesimal when $p > \mathrm{vex}f$.

Proof. If f is a constant function, then $\mathrm{var}_f(p) = 0$ for all $p \in [1, \infty)$. This means that $\mathrm{vex}f = 1$. Suppose that f is not a constant function. Then it suffices to consider $p_1, p_2 \in \mathbb{R}$ satisfying $1 \leq p_1 < p_2$ and to show that the ratio

$$\sum_t |\ \mathrm{d}f(t)\ |^{p_2} \Big/ \sum_t |\ \mathrm{d}f(t)\ |^{p_1}, \tag{3.1.1}$$

where $t \in \mathbb{T} \setminus \{T\}$, is infinitesimal. To do this, we need to show that the ratio (3.1.1) is less than ϵ, for an arbitrary $\epsilon \in \mathbb{R}^+$. Suppose that $\epsilon_1 > 0$ and is so small that $\epsilon_1^{p_2-p_1} < \epsilon$. Now, since f is continuous, $\mid \mathrm{d}f(t) \mid \leq \epsilon_1 \quad \forall t$. So we have

$$\begin{aligned}\sum_t \mid \mathrm{d}f(t) \mid^{p_2} &= \sum_t \mid \mathrm{d}f(t) \mid^{p_2-p_1} \mid \mathrm{d}f(t) \mid^{p_1} \\ &\leq \epsilon_1^{p_2-p_1} \sum_t \mid \mathrm{d}f(t) \mid^{p_1} \\ &< \epsilon \sum_t \mid \mathrm{d}f(t) \mid^{p_1},\end{aligned}$$

dividing both sides by the positive sum on the RHS of the inequality we obtain our ratio

$$\sum_t \mid \mathrm{d}f(t) \mid^{p_2} \Big/ \sum_t \mid \mathrm{d}f(t) \mid^{p_1} < \epsilon.$$

□

In many games between Investor and Market, the ability of Investor to hedge the sale of an option depends on Market choosing his moves so that $\mathrm{var}_N(2+\epsilon)$ for some small postive ϵ.

$\frac{1}{\mathrm{vex} f}$ the Hölder exponent, denoted by $H(f)$. f is said to have a bounded variation if $\mathrm{vex} f = 1$. $\mathrm{vex} f = 1$ when f is bounded and monotonic (entirely nonincreasing or nondecreasing function), then $\mathrm{var}_f(1) = \sum_t \mid \mathrm{d}f(t) \mid = \mid f(T) - f(0) \mid$, which is not infinitesimal but finite. One can reach the same conclusion when f is bounded and $[0,1]$ can be divided into finite number of intervals where f is monotonic. This justifies that ordinary well-behaved functions have variation exponent 1. We call f stochastic when $H(f) = \frac{1}{2}$, substochastic when $H(f) < \frac{1}{2}$ and superstochastic when $H(f) > \frac{1}{2}$. A diffusion process is expected to be stochastic, a substochastic process is less irregular, while a superstochastic process is more irregular, compared to a diffusion process.

3.2 The relative variation spectrum

3.2.1 Definition. A function f on $\mathbb{T}$ is said to be positive if $f(t) > 0 \quad \forall t \in \mathbb{T}$. We call a positive function f relatively continuous if $\sup_{t \in \mathbb{T} \setminus \{T\}} \mathrm{d}f(t)/f(t)$ is infinitesimal, and we set

$$\mathrm{var}_f^{\mathrm{rel}}(p) := \sum_t \left| \frac{\mathrm{d}f(t)}{f(t)} \right|^p,$$

for a relatively continuous f. We call $\mathrm{var}_f^{\mathrm{rel}}(p)$ the relative p-variation of f and $\mathrm{var}_f^{\mathrm{rel}}$ the relative variation spectrum.

3.2.2 Lemma. If f is a relatively continuous function on $\mathbb{T}$, then there exists a unique real number $\mathrm{vex}^{\mathrm{rel}} f \in [1, \infty]$ such that

(a) $\mathrm{var}_f^{\mathrm{rel}}(p)$ is infinitely large for $1 \leq p < \mathrm{vex}^{\mathrm{rel}} f$, and

(b) $\mathrm{var}_f^{\mathrm{rel}}(p)$ is infinitesimal for $p > \mathrm{vex}^{\mathrm{rel}} f$.

The proof to this result is similar to that of Lemma 3.1.2. We call $\mathrm{vex}^{\mathrm{rel}} f$ the relative variation exponent of f.

A function f on $\mathbb{T}$ is said to be strictly positive if there exists a real number $\epsilon > 0$ such that $f(t) > \epsilon \quad \forall t \in \mathbb{T}$. It is bounded if there exists a finite real number C such that $\sup_t \mid f(t) \mid < C$.

3.2.3 Lemma. If a strictly positive and bounded function f is relatively continuous, then $\mathrm{vex}^{\mathrm{rel}} f$ coincides with its absolute counterpart: $\mathrm{vex}^{\mathrm{rel}} f = \mathrm{vex} f$.

3.2.4 Proposition. The path $W : [0, T] \to \mathbb{R}$ of a standard Wiener process satisfies $\mathrm{vex} W = 2$ almost surely. Moreover, $\mathrm{var}_W(2) \approx T$ almost surely.

Proof. Let $\epsilon > 0$ be arbitrarily small. For each $N = 1, 2, \ldots$, we have

$$\mathbb{P}\left(\left(W\left((n+1)\frac{T}{N}\right) - W\left(n\frac{T}{N}\right)\right)^2 \geq T + \epsilon\right) \leq e^{-Nc(\epsilon)}, \tag{3.2.1}$$

$$\mathbb{P}\left(\left(W\left((n+1)\frac{T}{N}\right) - W\left(n\frac{T}{N}\right)\right)^2 \leq T + \epsilon\right) \leq e^{-Nc(\epsilon)}, \tag{3.2.2}$$

where $c(\epsilon)$ is a positive constant. These inequalities follow from

$$\mathbb{E}\left(W\left((n+1)\frac{T}{N}\right) - W\left(n\frac{T}{N}\right)\right)^2 = \frac{T}{N}$$

and the standard large-deviation results, where $\mathbb{P}$ and $\mathbb{E}$ denote the usual probability and expectation, respectively. Combining (3.2.1) and (3.2.2) with the Borel-Cantelli lemma (Chung and Erdos, 1952) we obtain that, with probability 1,

$$\left|\sum_{n=0}^{N-1}\left(W\left((n+1)\frac{T}{N}\right) - W\left(n\frac{T}{N}\right)\right)^2 - T\right| \geq \epsilon$$

only for finitely many N, hence, $|\mathrm{var}_W(2) - T| < \epsilon$ almost surely. Since ϵ is arbirarily small, then $\mathrm{var}_W(2) \approx T$ almost surely. $\square$

3.2.5 Proposition. The path $S : [0, T] \to \mathbb{R}$ of the diffusion process governed by (3.0.6) satisfies $\mathrm{vex} S = 2$ almost surely, where $T > 0$.

Proof. Since the solution to the diffusion process (3.0.6) is given by (3.0.7) for a standard Wiener process W, we have

$$\begin{aligned}
\mathrm{d}S(t) &= S(t)e^{(\mu - \frac{1}{2}\sigma^2)\mathrm{d}t + \sigma \mathrm{d}W(t)} - S(t) \\
&= S(t)e^{\theta(t)\left((\mu - \frac{1}{2}\sigma^2)\mathrm{d}t + \sigma \mathrm{d}W(t)\right)}\left((\mu - \frac{1}{2}\sigma^2)\mathrm{d}t + \sigma \mathrm{d}W(t)\right) \\
&\asymp \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}W(t),
\end{aligned}$$

where $\theta(t) \in (0, 1)$ and $a(t) \asymp b(t)$ means that $|a(t)| \leq c|b(t)|$ and $|b(t)| \leq c|a(t)|$ for some constant

$c > 0$ (may be dependent on the path $S(t)$). Hence

$$\begin{aligned}
\mathrm{var}_S(2) &\asymp \sum_{t\in\mathbb{T}\setminus\{T\}} \left((\mu - \frac{1}{2}\sigma^2)\mathrm{d}t + \sigma \mathrm{d}W(t)\right)^2 \\
&= \left(\mu - \frac{1}{2}\sigma^2\right)^2 \sum_{t\in\mathbb{T}\setminus\{T\}} (\mathrm{d}t)^2 + 2(\mu - \frac{1}{2}\sigma^2)\sigma \sum_{t\in\mathbb{T}\setminus\{T\}} \mathrm{d}t\mathrm{d}W(t) + \sigma^2 \sum_{t\in\mathbb{T}\setminus\{T\}} (\mathrm{d}W(t))^2 \\
&\approx \sigma^2 \mathrm{var}_W(2).
\end{aligned}$$

For a standard Wiener process W, $\mathrm{var}_W(2) \approx T$ almost surely (see proposition 3.2.4). This shows that $\mathrm{var}_S(2)$ is neither infinitesimal nor infinitely large, almost surely. □

3.2.6 Remark. The propositions 3.2.4 and 3.2.5 together shows that the diffusion process of the form (3.0.6) satiisfies $\mathrm{vex}S = 2$ almost surely. Investor can multiply his capital significantly unless Market makes $\mathrm{vex}S = 2$, we are going to prove this claim in the next chapter.

3.3 *Black-Scholes* formula

In derivative pricing, it is important to consider factors that influence the price an investor would be willing to pay, including the investor's attitude towards risk and the prospects of the underlying security. The formula derived by Black and Scholes consists of only time to maturity T, the option's payoff function U, the current price $S(t)$ and the volatility of the price.

In discussion with Merton, it came up that the formula requires very little economic theory for its justification.

For a game between Investor and Market in which Investor is allowed to continuously adjust the amount of stock S, and the following protocol is from (Shafer and Vovk, 2001).

THE BLACK-SCHOLES PROTOCOL

Parameters: $T > 0$ and $N \in \mathbb{N}$; $\mathrm{d}t := T/N$

Players: Investor, Market

Protocol:

$$\begin{aligned}
&\text{Market announces } S(0) > 0. \\
&\mathcal{I}(0) := 0 \\
&\text{FOR } t = 0, \mathrm{d}t, 2\mathrm{d}t, \ldots, T - \mathrm{d}t: \\
&\qquad \text{Investor announces } \delta(t) \in \mathbb{R}. \\
&\qquad \text{Market announces } \mathrm{d}S(t) \in \mathbb{R}. \\
&\qquad S(t + \mathrm{d}t) := S(t) + \mathrm{d}S(t). \\
&\qquad \mathcal{I}(t + \mathrm{d}t) := \mathcal{I}(t) + \delta(t)\mathrm{d}S(t).
\end{aligned} \tag{3.3.1}$$

$\mathcal{I}$ is Investor's capital process. The move $\delta(t)$ by Investor is the number of shares of the stock over the time t to $t + \mathrm{d}t$, and again over t to $t + \mathrm{d}t$, Market's move $\mathrm{d}S(t)$ is the change in the stock price, then $\delta(t)\mathrm{d}S(t)$ is Investor's gain or loss.

The *Black-Scholes* formula for pricing options relies on the assumption that the market price $S(t)$ follows a diffusion process/*geometric Brownian motion*. This assumption is used in two essential ways:

Taming the Market: The diffusion model of the form (3.0.3) limits the fluctuations in $S(t)$. The $\sqrt{\mathrm{d}t}$ effect: the change in $S(t)$ over $\mathrm{d}t$ has the magnitude of the order $(\mathrm{d}t)^{1/2}$. This is the wildness of the fluctuations of $S(t)$, however, one can imagine even wilder fluctuations.

Averaging Market changes: Equation (3.0.3) allows the use of the law of large numbers on a fine time scale. It says that comparable changes in $S(t)$ are independent over some nonoverlapping time intervals.

Model free approach does not require the diffusion model (3.0.3) for any of these purposes.

To the *Black-Scholes* protocol given above, Black and Scholes added the assumption that $S(t)$ follows a *geometric Brownian motion*. This means that Market's moves must obey the stochastic differential equation,

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t). \tag{3.3.2}$$

To derive the *Black-Scholes* formula using model free approach we do not need the full force of this assumption. In fact, the assumption boils down to the three following assumptions:

1. **The $\sqrt{\mathrm{d}t}$ effect**. That is, the $\mathrm{d}S(t)$ have order of magnitude $(\mathrm{d}t)^{1/2}$, from the stochastic differential Equation (3.3.2) and the fact that $\mathrm{d}W(t)$ is an increment of a *Wiener* process we have this order of magnitude. In Equation (3.3.2), we can neglect the term, $\mu S(t)\mathrm{d}t$, because $\mathrm{d}t$ is much smaller than $(\mathrm{d}t)^{1/2}$.

2. **Standard deviation proportional to price**. Squaring both sides of Equation (3.3.2)

$$\begin{aligned}(\mathrm{d}S(t))^2 &= \mu^2 S^2(t)(\mathrm{d}t)^2 + 2\mu\sigma S^2(t)\mathrm{d}t\mathrm{d}W(t) + \sigma^2 S^2(t)(\mathrm{d}W(t))2 \\ &= S^2(t)\left(\mu^2(\mathrm{d}t)^2 + 2\mu\sigma\mathrm{d}t\mathrm{d}W(t) + \sigma^2(\mathrm{d}W(t))^2\right).\end{aligned} \tag{3.3.3}$$

   Since $\mathrm{d}W(t)$ is of order $(\mathrm{d}t)^{1/2}$, then $(\mathrm{d}W(t))^2$ is of order $\mathrm{d}t$ and dominates the other terms, and to approximate $(\mathrm{d}S(t))^2$ in Equation (3.3.3) we can drop the $(\mathrm{d}W(t))^2$ term and replace it by its expected value $\mathrm{d}t$.

3. **Authorization to use the law of large numbers**. $(\mathrm{d}W(t))^2$ has mean $\mathrm{d}t$ and variance $2(\mathrm{d}t)^2$, since $\mathrm{d}W(t) \sim N(0, \mathrm{d}t)$. That is,

$$(\mathrm{d}W(t))^2 = \mathrm{d}t + z \tag{3.3.4}$$

   where z has mean zero and variance $2(\mathrm{d}t)^2$.

   Summing Equation (3.3.4) over all N increments

$$\mathrm{d}W(0), \mathrm{d}W(\mathrm{d}t), \mathrm{d}W(2\mathrm{d}t), \ldots, \mathrm{d}W(T-\mathrm{d}t)$$

   we obtain

$$\sum_{n=0}^{N-1}(\mathrm{d}W(n\mathrm{d}t))^2 = T + \sum_{n=0}^{N-1} z_n$$

   for $\mathrm{d}t$ sufficiently small, we expect the z_n terms to average to zero.

3.4 The classical derivation of the *Black-Scholes* formula

We want to find the price of a European option $\mathcal{U}(t)$ at time t that pays $U(S(T))$ at maturity time T. We first suppose that there is such a price and let it depend only on time, t, and on the current price of the stock, $S(t)$, such that $\mathcal{U}(t) = \overline{U}(S(t), t)$. To find $\overline{U}$, we use Taylor's series expasion on $\overline{U}$ and we only consider terms of order $\mathrm{d}t$ and smaller, we get

$$\mathrm{d}\overline{U}(S(t), t) \approx \frac{\partial \overline{U}}{\partial s}\mathrm{d}S(t) + \frac{\partial \overline{U}}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}(\mathrm{d}S(t))^2. \tag{3.4.1}$$

The term in $\mathrm{d}S(t)$ is of order $(\mathrm{d}t)^{1/2}$. The term in $\mathrm{d}t$ and the term in $(\mathrm{d}S(t))^2$ are each of order $\mathrm{d}t$. These $\mathrm{d}t$ terms have coefficients who are always positive, hence their cumulative effect is nonnegligible. The $\mathrm{d}S(t)$ are much larger, but their overall effect may be comparable to that of the $\mathrm{d}t$ terms, this is because they oscillate between negative and positve values and their coefficient varies slowly. We substitute the RHS of (3.3.3) for $(\mathrm{d}S(t))^2$ in (3.4.1) we obtain

$$\mathrm{d}\overline{U}(S(t), t) \approx \frac{\partial \overline{U}}{\partial s}\mathrm{d}S(t) + \frac{\partial \overline{U}}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}S^2(t)(\mu^2(\mathrm{d}t)^2 + 2\mu\sigma\mathrm{d}t\mathrm{d}W(t) + \sigma^2(\mathrm{d}W(t))^2), \tag{3.4.2}$$

retaining terms of order $(\mathrm{d}t)^{1/2}$ and $\mathrm{d}t$ only, we get

$$\mathrm{d}\overline{U}(S(t), t) \approx \frac{\partial \overline{U}}{\partial s}\mathrm{d}S(t) + \frac{\partial \overline{U}}{\partial t}\mathrm{d}t + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 \overline{U}}{\partial s^2}(\mathrm{d}W(t))^2. \tag{3.4.3}$$

The term in $(\mathrm{d}W(t))^2$ varies slowly because of its coefficients, we can replace $(\mathrm{d}W(t))^2$ with $\mathrm{d}t$ in (3.4.3)

$$\mathrm{d}\overline{U}(S(t), t) \approx \frac{\partial \overline{U}}{\partial s}\mathrm{d}S(t) + \left(\frac{\partial \overline{U}}{\partial t} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 \overline{U}}{\partial s^2}\right)\mathrm{d}t. \tag{3.4.4}$$

Here we used the law of large numbers. But it is only valid if the coefficient $S^2(t)\frac{\partial^2 \overline{U}}{\partial s^2}$ holds steady during enough $\mathrm{d}t$ for the $(\mathrm{d}W(t))^2$ to average out. We now look at *Black-Scholes* protocol. According to (3.3.1),

$$\mathrm{d}\mathcal{I}(t) = \delta(t)\mathrm{d}S(t),$$

where $\delta(t)$ is the amount of stock Investor holds from t to $t + \mathrm{d}t$. If we compare this with (3.4.4), we notice that our goal is achieved by setting

$$\delta(t) := \frac{\partial \overline{U}}{\partial s}(S(t), t), \tag{3.4.5}$$

provided we have

$$\frac{\partial \overline{U}}{\partial t}(S(t), t) + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 \overline{U}}{\partial s^2}(S(t), t) = 0 \tag{3.4.6}$$

for all t, any value $S(t)$ takes. The problem is then reduced to a mathematical. We need to find a function $\overline{U}(s, t)$, for $0 < t < T$ and $0 < s < \infty$, that satisfies the pde

$$\frac{\partial \overline{U}}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \overline{U}}{\partial s^2} = 0 \tag{3.4.7}$$

and the final condition $\overline{U}(s, t) \to U(s)$ as $t \to T$. Notice that this is a heat equation. This pde has a solution,

$$\overline{U}(s, t) = \int_{-\infty}^{\infty} U(se^z)\mathcal{N}_{-\sigma^2(T-t)/2, \sigma^2(T-t)}(\mathrm{d}z). \tag{3.4.8}$$

So an approximate price at time t for the European option $\mathcal{U}$ with maturity T and payoff $U(S(T))$ is given by

$$\mathcal{U}(t) = \int_{-\infty}^{\infty} U(S(t)e^{z})\mathcal{N}_{-\sigma^2(T-t)/2,\sigma^2(T-t)}(\mathrm{d}z), \tag{3.4.9}$$

where $S(t)$ is the price of the stock at time t, $D(t) = \sigma^2(T-t)$ is the remaining volatility, $\mathcal{N}_{\mu,\phi}$ is the normal distribution with mean μ and variance ϕ. This is the *Black-Scholes* formula for an arbitrary European option.

3.4.1 Remark. The volatility parameter, σ, plays a role in the derivation while the drift parameter, μ, does not appear in the *Black-Scholes* formula. The use of law of large numbers enables us to move from Equation (3.4.3) to Equation (3.4.4).

Alternatively, we can derive the *Black-Scholes* formula as follows:

We again make use a Taylor's series expansion, and taking terms of order $\mathrm{d}t$ and smaller, we get

$$\mathrm{d}\overline{U}(S(t),t) \approx \frac{\partial \overline{U}}{\partial s}\mathrm{d}S(t) + \frac{\partial \overline{U}}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}(\mathrm{d}S(t))^2. \tag{3.4.10}$$

The RHS of the approximation is the increment in the capital process of an investor who holds shares of two securities over the period of time t to $t+\mathrm{d}t$:

- $\frac{\partial \overline{U}}{\partial s}$ shares of S, and
- $-\sigma^2\frac{\partial \overline{U}}{\partial t}$ shares of a security $\mathcal{D}$ whose price per share at t is $\sigma^2(T-t)$, and which pays a continuous dividend per share amounting, over the period from $t+\mathrm{d}t$, to

$$-\frac{\sigma^2}{\frac{\partial \overline{U}}{\partial t}}\frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}(S(t))^2. \tag{3.4.11}$$

The second term on the RHS of (3.4.10) is the capital gain from holding the $-\sigma^{-2}\frac{\partial \overline{U}}{\partial t}$ shares of $\mathcal{D}$, and the third term is the total dividend.

The *Black-Scholes* equation tells us to choose the function $\overline{U}$ so that the dividend per share, (3.4.11), reduces to $\left(\frac{\mathrm{d}S(t)}{S(t)}\right)^2$, and the increment in the capital process, (3.4.10), becomes

$$\mathrm{d}\overline{U}(S(t),t) \approx \frac{\partial \overline{U}}{\partial s}\mathrm{d}S(t) + \frac{\partial \overline{U}}{\partial t}\mathrm{d}t - \frac{\partial \overline{U}}{\partial t}\frac{(\mathrm{d}S(t))^2}{\sigma^2 S^2(t)}. \tag{3.4.12}$$

Now the stochasticity assumption comes into play, that is, $S(t)$ follows a *geometric Brownian motion*. This assumption tells us that $(\mathrm{d}S(t))^2 \approx \sigma^2 S^2(t)\mathrm{d}t$, using the rules that $(\mathrm{d}t)^2 = \mathrm{d}t\mathrm{d}W(t) = 0$ and $(\mathrm{d}W(t))^2 = \mathrm{d}t$, so that (3.4.12) reduces to

$$\mathrm{d}\overline{U}(S(t),t) \approx \frac{\partial \overline{U}}{\partial s}\mathrm{d}S(t) + \frac{\partial \overline{U}}{\partial t}\mathrm{d}t - \frac{\partial \overline{U}}{\partial t}\mathrm{d}t,$$

it is easier to interpret in the following form

$$\mathrm{d}\overline{U}(S(t),t) \approx \frac{\partial \overline{U}}{\partial s}\mathrm{d}S(t) - \sigma^{-2}\frac{\partial \overline{U}}{\partial t}(-\sigma^2\mathrm{d}t) - \sigma^{-2}\frac{\partial \overline{U}}{\partial t}(\sigma^2\mathrm{d}t).$$

The capital gain on each share of $\mathcal{D}$, $-\sigma^2 \mathrm{d}t$, is cancelled by the dividend, $\sigma^2 \mathrm{d}t$. So there is no point in holding any number of shares of $\mathcal{D}$. The increment in the capital process is

$$\mathrm{d}\overline{U}(S(t), t) \approx \frac{\partial \overline{U}}{\partial s} \mathrm{d}S(t),$$

which we achieve by holding $\frac{\partial \overline{U}}{\partial s}$ shares of S. This shows how the stochasticity assumption is being eliminated. If the market does really price $\mathcal{D}$ like we just explained, then we can do without the stochastic assumption.

4. Option pricing formula in model free approach

4.1 The model free derivation of the *Black-Scholes* formula

We now focus on the model free version of the *Black-Scholes* formula. We require that Market prices both security S which pays no dividends and a derivative security $\mathcal{D}$ that pays dividends $(\mathrm{d}S(t)/S(t))^2$, and the constraints on how wild the price changes can be are adopted as constraints on how Market make his moves. Let us describe how these give a *Black-Scholes* formula.

Assume that between time 0 and T, Investor trades in two securities:

(1) a security S which pays no dividends, and

(2) a security $\mathcal{D}$, each share of which pays the dividend $(\mathrm{d}S(t)/S(t))^2$.

This gives the following protocol (the protocol is taken from Shafer and Vovk (2001)):

THE NEW BLACK-SCHOLES PROTOCOL

Parameters: $T > 0$ and $N \in \mathbb{N}$; $\mathrm{d}t := T/N$

Players: Investor, Market

Protocol:

Market announces $S_0 > 0$ and $D_0 > 0$.
$\mathcal{I}(0) := 0$
FOR $t = 0, \mathrm{d}t, 2\mathrm{d}t, \ldots, T - \mathrm{d}t$:
  Investor announces $\delta(t) \in \mathbb{R}$ and $\lambda(t) \in \mathbb{R}$.
  Market announces $\mathrm{d}S(t) \in \mathbb{R}$ and $\mathrm{d}D(t) \in \mathbb{R}$.
  $S(t + \mathrm{d}t) := S(t) + \mathrm{d}S(t)$.
  $D(t + \mathrm{d}t) := D(t) + \mathrm{d}D(t)$.
  $$\mathcal{I}(t + \mathrm{d}t) := \mathcal{I}(t) + \delta(t)\mathrm{d}S(t) + \lambda(t)(\mathrm{d}D(t) + (\mathrm{d}S(t)/S(t))^2). \tag{4.1.1}$$

Additional Constraints on Market:

(1) $D(t) > 0$ for $0 < t < T$ and $D(T) = 0$,

(2) $S(t) \geq 0$ for all t, and

(3) the wildness of Market's moves is constrained.

Once $\mathcal{D}$ pays its last dividend, at time T, it becomes worthless: $D(T) = 0$. So Market is required to make his $\mathrm{d}D(t)$ add to $-D(0)$. We assume zero interest rate and Investor starts with zero capital.

We consider a European option $\mathcal{U}$ with maturity T and a payoff function U. Assume that the price of $\mathcal{U}$ before T is given in terms of the current prices of S and D by

$$\mathcal{U}(t) = \overline{U}(S(t), D(t)),$$

with the function $\overline{U}(s, D)$ satisfying the initial condition

$$\overline{U}(s, 0) = U(s). \tag{4.1.2}$$

We approximate the increment in the option's price, $\mathcal{U}$, from t to $t + \mathrm{d}t$ by means of a Taylor's series expansion:

$$\mathrm{d}\overline{U}(S(t), D(t)) \approx \frac{\partial \overline{U}}{\partial s}\mathrm{d}S(t) + \frac{\partial \overline{U}}{\partial D}\mathrm{d}D(t) + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}(\mathrm{d}S(t))^2. \tag{4.1.3}$$

We assume that the rules of the game constrain Market's moves $\mathrm{d}S(t)$ and $\mathrm{d}D(t)$ such that terms of higher order are negligible in the Taylor's expansion. Putting together Equation (4.1.1) in the protocol and Equation (4.1.3), shows that we need

$$\delta(t) = \frac{\partial \overline{U}}{\partial s}, \quad \lambda(t) = \frac{\partial \overline{U}}{\partial D}, \quad \text{and} \quad \frac{\lambda(t)}{S^2(t)} = \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial s^2}.$$

Putting together the two equations involving $\lambda(t)$ above shows that we require $\overline{U}$ to satisfy the pde

$$-\frac{\partial \overline{U}}{\partial D} + \frac{1}{2}s^2\frac{\partial^2 \overline{U}}{\partial s^2} = 0, \tag{4.1.4}$$

for all s and all $D > 0$. This is the *Black-Scholes* equation, for the market where both S and D are traded. Making use of the initial condition (4.1.2), the solution to the pde is

$$\overline{U}(s, D) = \int_{-\infty}^{\infty} U(se^z)\mathcal{N}_{-D/2,D}(\mathrm{d}z). \tag{4.1.5}$$

The *Blackk-Scholes* formula for this market is given by (4.1.5). The price for the European option $\mathcal{U}$ in this market is

$$\mathcal{U}(t) = \int_{-\infty}^{\infty} U(S(t)e^z)\mathcal{N}_{-D(t)/2,D(t)}(\mathrm{d}z). \tag{4.1.6}$$

To hedge this price, one needs to hold a continuously changing portfolio, containing $\frac{\partial \overline{U}}{\partial s}(S(t), D(t))$ shares of S and $\frac{\partial \overline{U}}{\partial D}(S(t), D(t))$ shares of $\mathcal{D}$ at time t. If $S(t)$ follows a GBM, then the derivative $\mathcal{D}$ is unnecessary. In this case, the dividends of the derivative $\mathcal{D}$ are independent nonnegative random variables with expected value $\sigma^2\mathrm{d}t + (\mu\mathrm{d}t)^2 \approx \sigma^2\mathrm{d}t$. The remaining dividends at time t will add to almost excatly $\sigma^2(T - t)$, this is by the law of large numbers, and which gives the classical case.

In the following sections we follow closely the work of Shafer and Vovk (2001), Chapter 10 and Chapter 11.

4.2 Bachelier pricing in continuous time

For simplicity and better understanding, we begin with the Bachelier game, even though it allows negative market stock prices. In the Bachelier game, Investor trades in two securities: a security S, which pays no dividends, and a security $\mathcal{D}$, which pays regular dividend equal to the square of the most recent change in the price of S. Market sets prices for S and $\mathcal{D}$ at each time step; prices $S_0, \ldots, S_N$ for S and $D_0, \ldots, D_N$ for $\mathcal{D}$. At point n, the dividend paid by $\mathcal{D}$ is $(\Delta S_n)^2$, where $\Delta S_n := S_n - S_{n-1}$. But N is infinitely large, and the sequences $S_0, \ldots, S_N$ and $D_0, \ldots, D_N$ define functions S and D, respectively, on the time interval $\mathbb{T}$:

$$S(n\mathrm{d}t) := S_n \quad \text{and} \quad D(n\mathrm{d}t) := D_n$$

for $n = 0, \ldots, N$.

The following protocol is taken from Shafer and Vovk (2001)

BACHELIER'S PROTOCOL WITH CONSTRAINED VARIATION

Parameters: $N, \mathcal{I}_0, \delta \in (0,1)$

Players: Investor, Market

Protocol:

Market announces $S_0 \in \mathbb{R}$ and $D_0 > 0$.
FOR $n = 1, \ldots, N$:
Investor announces $M_n \in \mathbb{R}$ and $V_n \in \mathbb{R}$.
Market announces $S_n \in \mathbb{R}$ and $D_n \geq 0$.

$$\mathcal{I}_n := \mathcal{I}_{n-1} + M_n \Delta S_n + V_n((\Delta S_n)^2 + \Delta D_n). \tag{4.2.1}$$

Additional Constraints on Market: Market must set $D_N = 0, D_n > 0$ for $n < N$ and must make $S_0, \ldots, S_N$ and $D_0, \ldots, D_N$ satisfy the following condition

$$\inf_{\epsilon \in (0,1)} \max(\mathrm{var}_S(2+\epsilon), \mathrm{var}_D(2-\epsilon)) < \delta \quad \text{for a small} \quad \delta > 0.$$

We can write (4.2.1) as follows

$$\mathcal{I}_{n+1} = \mathcal{I}_n + M_{n+1}(S_{n+1} - S_n) + V_{n+1}((S_{n+1} - S_n)^2 + (D_{n+1} - D_n))$$

or

$$\mathrm{d}\mathcal{I}_n = M_{n+1}\mathrm{d}S_n + V_{n+1}((\mathrm{d}S_n)^2 + \mathrm{d}D_n) \tag{4.2.2}$$

for $n = 0, \ldots, N-1$. Investor chooses moves M_n and V_n in the situation

$$S_0 D_0 \cdots S_{n-1} D_{n-1}. \tag{4.2.3}$$

So a strategy for Investor is a pair of functions, $\mathcal{M}$ and $\mathcal{V}$, each of which maps each situation of the form (4.2.3) to a real number. If Market's moves are given by S and D, and Investor decides to use the strategy $(\mathcal{M}, \mathcal{V})$, (4.2.2) becomes

$$\mathrm{d}\mathcal{I}_n = \mathcal{M}(S_0 D_0 \cdots S_n D_n)\mathrm{d}S_n + \mathcal{V}(S_0 D_0 \cdots S_n D_n)((\mathrm{d}S_n)^2 + \mathrm{d}D_n). \tag{4.2.4}$$

If Investor follows the strategy $(\mathcal{M}, \mathcal{V})$ and Market plays (S, D), then Investor's total change in capital is denoted by $\mathcal{I}^{\mathcal{M},\mathcal{V}}(S, D)$. Summing the increments (4.2.4) we obtain

$$\mathcal{I}^{\mathcal{M},\mathcal{V}}(S, D) = \sum_{n=0}^{N-1} \left(\mathcal{M}(S_0 D_0 \cdots S_n D_n)\mathrm{d}S_n + \mathcal{V}(S_0 D_0 \cdots S_n D_n)((\mathrm{d}S_n)^2 + \mathrm{d}D_n) \right). \tag{4.2.5}$$

Investor's final capital will be $\alpha + \mathcal{I}^{\mathcal{M},\mathcal{V}}(S, D)$ at time T for some initial capital α at time 0. If his decision on how many shares of each security to hold over the time period depends on the current prices of S and $\mathcal{D}$, then $\mathcal{M}$ and $\mathcal{V}$ are just functions of two variables and (4.2.5) reduces to

$$\mathcal{I}^{\mathcal{M},\mathcal{V}}(S, D) = \sum_{n=0}^{N-1} \left(\mathcal{M}(S_n, D_n)\mathrm{d}S_n + \mathcal{V}(S_n, D_n)((\mathrm{d}S_n)^2 + \mathrm{d}D_n) \right).$$

This can also be written as

$$\mathcal{I}^{\mathcal{M},\mathcal{V}}(S, D) = \sum_{t \in \mathbb{T} \setminus \{T\}} \mathcal{M}(S(t), D(t))\mathrm{d}S(t) + \sum_{t \in \mathbb{T} \setminus \{T\}} \mathcal{V}(S(t), D(t)) \left((\mathrm{d}S(t))^2 + \mathrm{d}D(t) \right).$$

4.3 *Black-Scholes* formula in continuous time in model free

Assume now that over some period of time, say $0 < n \leq N$, Investor trades in two securities:

(1) a security S that pays no dividends, and

(2) a security $\mathcal{D}$, each share of which pays the dividend $(\Delta S_n/S_{n-1})^2$.

This produces the following protocol:

THE BLACK-SCHOLES PROTOCOL IN CONTINUOUS TIME (Shafer and Vovk, 2001)

Players: Investor, Market

Protocol:

$\mathcal{I}_0 := 0$

Market announces $S_0 > 0$ and $D_0 > 0$.

FOR $n = 1, \dots, N$:

Investor announces $M_n \in \mathbb{R}$ and $V_n \in \mathbb{R}$.

Market announces $S_n > 0$ and $D_n \geq 0$.

$\mathcal{I}_n := \mathcal{I}_{n-1} + M_n \Delta S_n + V_n((\Delta S_n/S_{n-1})^2 + \Delta D_n)$.

Additional Constraints on Market: Market must ensure that S is continuous, $\inf_n S_n$ is positive and not infinitesimal, and $\sup_n S_n$ is finite. He must also ensure that D is continuous, $D_n > 0$ for $n = 1, 2, \dots, N-1$, $D_N = 0$, $\sup_n D_n$ is finite, and $\operatorname{vex} D < 2$.

Investor's move M_n is the number of shares of the security S he holds over time n, V_n is the number of shares of the security $\mathcal{D}$ he holds over n. Market's move is to announce the change in prices S_n and D_n per share over n, and hence $M_n \Delta S_n + V_n((\Delta S_n/S_{n-1})^2 + \Delta D_n)$ is Investor's gain or loss. $\mathcal{I}$ denotes Investor's capital process. Security $\mathcal{D}$ is worthless after paying its last dividend, at time N, that is, $D_N = 0$. Investor starts with zero capital, but can borrow money to buy stock or borrow stock to sell at whatever price he pleases. We assume that the interest rate is zero, for simplicity. The following Theorem is the main result of this study. In (Shafer and Vovk, 2001) the proof is not explicitly given and here we give a detailed proof.

4.3.1 Theorem. *Let $U : (0, \infty) \to \mathbb{R}$ be Lipschitzian and bounded below. Then in the Black-Scholes protocol in continuous time, the price at time 0 (right after $S(0)$ and $D(0)$ are announced) for the European option $U(S(T))$ is*

$$\int_{\mathbb{R}} U(S(0)e^z)\mathcal{N}_{-D(0)/2, D(0)}(\mathrm{d}z) \tag{4.3.1}$$

Before we give the proof let us state two well-known inequalities. For nonnegative sequences X_n and Y_n and $p, q \in \mathbb{R}^+$, Hölder's inequality, says that if $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{n=1}^{N} X_n Y_n \leq \left(\sum_{n=1}^{N} X_n^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{N} Y_n^q\right)^{\frac{1}{q}}, \tag{4.3.2}$$

and Jensen's inequality, says that if $p \leq q$, then

$$\left(\sum_{n=1}^{N} X_n^p\right)^{\frac{1}{p}} \geq \left(\sum_{n=1}^{N} X_n^q\right)^{\frac{1}{q}}. \tag{4.3.3}$$

The two inequalities imply that if $\frac{1}{p}+\frac{1}{q}\geq 1$, then

$$\sum_{n=1}^{N} X_nY_n \leq \left(\sum_{n=1}^{N} X_n^{2+\epsilon}\right)^{1/(2+\epsilon)} \left(\sum_{n=1}^{N} Y_n^{2-\epsilon}\right)^{1/(2-\epsilon)}. \tag{4.3.4}$$

Proof. We assume that vexS ≤ 2 and that U is smooth with derivatives $U^{(1)}$ - $U^{(4)}$ all bounded. We want to find a strategy $(\mathcal{M},\mathcal{V})$ such that

$$\int_{\mathbb{R}} U(S(0)e^{z})\mathcal{N}_{-D(0)/2,D(0)}(\mathrm{d}z)+\mathcal{I}^{\mathcal{M},\mathcal{V}} \approx U(S(T)). \tag{4.3.5}$$

For any $D\geq 0$ and $s>0$ we set

$$\overline{U}(s,D) := \int_{\mathbb{R}} U(se^{z})\mathcal{N}_{-D(t)/2,D(t)}(\mathrm{d}z), \tag{4.3.6}$$

which is continuous and satisfies the initial condition $\overline{U}(s,0)=U(s)$. It can be verified that the *Black-Scholes* pde is

$$\frac{\partial\overline{U}}{\partial D}=\frac{1}{2}s^2\frac{\partial^2\overline{U}}{\partial s^2}, \tag{4.3.7}$$

for all $s\in\mathbb{R}$ and all $D>0$. In fact (4.3.6) with the given initial condition solves (4.3.7).

Applying Taylor's formula to $\mathrm{d}\overline{U}(s,D)$ we have

$$\mathrm{d}\overline{U}=\frac{\partial\overline{U}}{\partial s}\mathrm{d}S(t)+\frac{\partial\overline{U}}{\partial D}\mathrm{d}D(t)+\frac{1}{2}\frac{\partial^2\overline{U}}{\partial s^2}(\mathrm{d}S(t))^2+\frac{\partial^2\overline{U}}{\partial s\partial D}\mathrm{d}S(t)\mathrm{d}D(t)+\frac{1}{2}\frac{\partial^2\overline{U}}{\partial D^2}(\mathrm{d}D(t))^2. \tag{4.3.8}$$

Applying the Taylor's formula to $\frac{\partial^2\overline{U}}{\partial s^2}$, we find

$$\frac{\partial^2\overline{U}}{\partial s^2}=\frac{\partial^2\overline{U}}{\partial s^2}+\frac{\partial^3\overline{U}}{\partial s^3}(\Delta S)+\frac{\partial^3\overline{U}}{\partial D\partial s^2}(\Delta D), \tag{4.3.9}$$

where ΔS and ΔD satisfy $|\Delta S|\leq|\mathrm{d}S|$ and $|\Delta D|\leq|\mathrm{d}D|$ respectively. Substituting (4.3.7) and (4.3.9) into (4.3.8) we obtain

$$\begin{aligned}\mathrm{d}\overline{U}&=\frac{\partial\overline{U}}{\partial s}\mathrm{d}S+\frac{\partial\overline{U}}{\partial D}\left(\mathrm{d}D+\left(\frac{\mathrm{d}S}{S}\right)^2\right)+\frac{1}{2}\frac{\partial^3\overline{U}}{\partial s^3}(\Delta S)(\mathrm{d}S)^2\\&+\frac{1}{2}\frac{\partial^3\overline{U}}{\partial D\partial s^2}(\Delta D)(\mathrm{d}S)^2+\frac{\partial^2\overline{U}}{\partial s\partial D}(\mathrm{d}S)(\mathrm{d}D)+\frac{1}{2}\frac{\partial^2\overline{U}}{\partial D^2}(\mathrm{d}D)^2\end{aligned} \tag{4.3.10}$$

We take Investor's strategy as

$$\left(\frac{\partial\overline{U}(S,D)}{\partial s},\frac{\partial\overline{U}(S,D)}{\partial D}\right) \tag{4.3.11}$$

where the first argument is the share of S and the second argument is the share of $\mathcal{D}$. Integrating from 0 to T, we obtain

$$\begin{aligned}&|\left(\overline{U}(S(T),D(T))-\overline{U}(S(0),D(0))\right)-\left(\mathcal{I}(S(T),D(T))\right)|\\&\leq\frac{1}{2}\sup\left|\frac{\partial^3\overline{U}}{\partial s^3}\right|\mathsf{var}_s(3)+\frac{1}{2}\sup\left|\frac{\partial^3\overline{U}}{\partial D\partial s^2}\right|\sum_t|dD|\,|dS|^2\\&+\sup\left|\frac{\partial^2\overline{U}}{\partial D\partial s}\right|\sum_t|dD|\,|dS|+\frac{1}{2}\sup\left|\frac{\partial^2\overline{U}}{\partial D^2}\right|\mathsf{var}_D(2),\end{aligned} \tag{4.3.12}$$

all suprema taken over the convex hull of $\{(S(t), D(t))|0 < t < T\}$. Now we prepare to bound the suprema in (4.3.12). An application of (4.3.7) enables us to rewrite the partial derivatives as follows

$$\begin{aligned}
\frac{\partial^3 \overline{U}}{\partial D \partial s^2} &= \frac{1}{2}\frac{\partial^2}{\partial s^2}\left(s^2 \frac{\partial^2 \overline{U}}{\partial s^2}\right) = \frac{\partial^2 \overline{U}}{\partial s^2} + 2s\frac{\partial^3 \overline{U}}{\partial s^3} + \frac{1}{2}s^2\frac{\partial^4 \overline{U}}{\partial s^4}, \\
\frac{\partial^2 \overline{U}}{\partial D \partial s} &= \frac{1}{2}\frac{\partial}{\partial s}\left(s^2 \frac{\partial^2 \overline{U}}{\partial s^2}\right) = s\frac{\partial^2 \overline{U}}{\partial s^2} + \frac{1}{2}s^2\frac{\partial^3 \overline{U}}{\partial s^3}, \\
\frac{\partial^2 \overline{U}}{\partial D^2} &= \frac{1}{2}\frac{\partial}{\partial D}\left(s^2 \frac{\partial^2 \overline{U}}{\partial s^2}\right) = \frac{1}{2}s^2\frac{\partial^3 \overline{U}}{\partial D \partial s^2}.
\end{aligned} \tag{4.3.13}$$

To bound our partial derivatives we first note that for $x \in \mathbb{R}$ and a standard Gaussian variable ψ, we have $\mathbb{E}\left(e^{x\psi}\right) = e^{x^2/2}$. Therefore, applying Leibniz differential rule, we get

$$\begin{aligned}
\left|\frac{\partial^n \overline{U}}{\partial s^n}\right| &= \left|\int_{\mathbb{R}} U^{(n)}(se^z)e^{nz}\mathcal{N}_{-D/2,D}(\mathrm{d}z)\right| \le \left|\left|U^{(n)}\right|\right| \int_{\mathbb{R}} e^{nz}\mathcal{N}_{-D/2,D}(\mathrm{d}z) \\
&= \left|\left|U^{(n)}\right|\right| \mathbb{E}\left(e^{n(-D/2+\psi\sqrt{D})}\right) = \left|\left|U^{(n)}\right|\right| e^{-nD/2}e^{n^2D/2} = \left|\left|U^{(n)}\right|\right| e^{n(n-1)D/2}.
\end{aligned} \tag{4.3.14}$$

From (4.3.13) and (4.3.14) we obtain

$$\left|\frac{\partial^3 \overline{U}}{\partial s^3}\right| \le c_3 e^{3C}, \tag{4.3.15}$$

$$\left|\frac{\partial^3 \overline{U}}{\partial D \partial s^2}\right| \le \left|\frac{\partial^2 \overline{U}}{\partial s^2}\right| + 2C\left|\frac{\partial^3 \overline{U}}{\partial s^3}\right| + \frac{1}{2}C^2\left|\frac{\partial^4 \overline{U}}{\partial s^4}\right| \le c_2e^C + 2Cc_3e^{3C} + \frac{1}{2}C^2c_4e^{6C}, \tag{4.3.16}$$

$$\left|\frac{\partial^2 \overline{U}}{\partial D \partial s}\right| \le C\left|\frac{\partial^2 \overline{U}}{\partial s^2}\right| + \frac{1}{2}C^2\left|\frac{\partial^3 \overline{U}}{\partial s^3}\right| \le Cc_2e^C + \frac{1}{2}C^2c_3e^{3C}, \tag{4.3.17}$$

$$\left|\frac{\partial^2 \overline{U}}{\partial D^2}\right| \le \frac{1}{2}C^2\left|\frac{\partial^3 \overline{U}}{\partial D \partial s^2}\right| \le \frac{1}{2}C^2c_2e^C + C^3c_3e^{3C} + \frac{1}{4}C^4c_4e^{6C}, \tag{4.3.18}$$

where c_2, c_3, c_4 and C are constants. This completes the proof for the case of smooth U with bounded derivatives $U^{(1)}$-$U^{(4)}$.

Now we remove the restriction on $||U^{(2)}||$-$||U^{(4)}||$. We introduce a new function V by

$$V(s) := \int_{\mathbb{R}} U(s+z)\mathcal{N}_{0,\sigma^2}(\mathrm{d}z); \tag{4.3.19}$$

where $c_3 = ||V^{(3)}||$ and $c_4 = ||V^{(4)}||$. Let us check that $U(s)$ is close to $V(s)$, we use the fact that U is Lipschitzian with coefficient c:

$$\begin{aligned}
|V(s) - U(s)| &= \left|\int_{\mathbb{R}} U(s+z) - U(s)\mathcal{N}_{0,\sigma^2}(\mathrm{d}z)\right| \le \int_{\mathbb{R}} |U(s+z) - U(s)|\mathcal{N}_{0,\sigma^2}(\mathrm{d}z) \\
&\le c\int_{\mathbb{R}} |z|\mathcal{N}_{0,\sigma^2}(\mathrm{d}z) = c\sigma\int_{\mathbb{R}} |z|\mathcal{N}_{0,1}(\mathrm{d}z) = c\sigma\sqrt{2/\pi},
\end{aligned} \tag{4.3.20}$$

where the last equality follows from $\int_0^\infty y^{2n+1}e^{-y^2/2}\mathrm{d}y = 2^n\int_0^\infty x^n e^{-x}\mathrm{d}x = 2^n\Gamma(n+1) = 2^n n!$. We also get

$$\begin{aligned}
\left|\int_{\mathbb{R}} V(S(0)+z) - U(S(0)+z)\mathcal{N}_{0,D(0)}(\mathrm{d}z)\right| &\le \int_{\mathbb{R}} |V(S(0)+z) - U(S(0)+z)|\mathcal{N}_{0,D(0)}(\mathrm{d}z) \\
&\le c\sigma\sqrt{2/\pi}.
\end{aligned} \tag{4.3.21}$$

Now we find the upper bounds for all derivatives $V^{(n)}$. For $n = 0, 1, \ldots$, we have

$$V^{(n)} = \frac{1}{\sigma^{n+1}\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(x-s)^2/(2\sigma^2)} H_n\left(\frac{x-s}{\sigma}\right) U(x)\mathrm{d}x, \tag{4.3.22}$$

where H_n are Hermite's polynomials (proof of (4.3.22) is by induction on n, see (Shafer and Vovk, 2001)). Assuming, $U(s) = 0$, without loss of generality, for $n = 1, 2, \ldots$, we get

$$\begin{aligned}\left|V^{(n)}(s)\right| &\leq \frac{1}{\sigma^{n+1}\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(x-s)^2/(2\sigma^2)} \left|H_n\left(\frac{x-s}{\sigma}\right)\right| c\,|x-s|\,\mathrm{d}x \\ &= \frac{c}{\sigma^{n-1}\sqrt{2\pi}} \int_{\mathbb{R}} e^{-y^2/2} \left|H_n(y)\right| |y|\,\mathrm{d}y. \end{aligned} \tag{4.3.23}$$

We get the following bounds:

$$\left\|V^{(3)}\right\| \leq \frac{c}{\sigma^2\sqrt{2\pi}} \int_{\mathbb{R}} e^{-y^2/2} \left|y^3 - 3y\right| |y|\mathrm{d}y \leq \frac{c}{\sigma^2\sqrt{2\pi}} \int_{\mathbb{R}} e^{-y^2/2}(y^4 + 3y^2)\mathrm{d}y = \frac{6c}{\sigma^2}, \tag{4.3.24}$$

and

$$\begin{aligned}\left\|V^{(4)}\right\| &\leq \frac{c}{\sigma^3\sqrt{2\pi}} \int_{\mathbb{R}} e^{-y^2/2} \left|y^4 - 6y^2 + 3\right| |y|\mathrm{d}y \leq \frac{2c}{\sigma^3\sqrt{2\pi}} \int_{\mathbb{R}} e^{-y^2/2}(y^5 + 6y^3 + 3y)\mathrm{d}y \\ &= \frac{2c}{\sigma^3\sqrt{2\pi}}(8 + 12 + 3) = \frac{46c}{\sigma^3\sqrt{2\pi}}. \end{aligned} \tag{4.3.25}$$

Equation (4.3.20) still holds for the new function, but (4.3.21) becomes

$$\begin{aligned}\left|\overline{V}(S(0), D(0)) - \overline{U}(S(0), D(0))\right| &= \left|\int_{\mathbb{R}} V(S(0)e^z) - U(S(0)e^z)\mathcal{N}_{-D(0)/2, D(0)}(\mathrm{d}z)\right| \\ &\leq \int_{\mathbb{R}} |V(S(0)e^z) - U(S(0)e^z)|\mathcal{N}_{-D(0)/2, D(0)}(\mathrm{d}z) \\ &\leq c\sigma\sqrt{2/\pi} \end{aligned} \tag{4.3.26}$$

Now taking a small $\sigma > 0$. Combining (4.3.20), (4.3.26) and the boundedness of $||V^{(2)}||$-$||V^{(4)}||$ shows that our goal (4.3.5) can be obtained for any Lipschitzian U. The last thing to do is to remove the assumption $\mathrm{vex}S \leq 2$. Since U is bounded from below, we can assume that Investor's capital never drops below some known constant (if it does that would mean Market violated some of his obligations, then Investor can choose zero moves). But spending an arbitrary small $\epsilon > 0$ on $\mathcal{D}$ will make sure that Investor's dividends

$$\sum_t \left(\frac{\mathrm{d}S(t)}{S(t)}\right) \geq \frac{\mathrm{var}_S(2)}{(\inf_t S(t))^2}$$

from holding $\mathcal{D}$ will be infinitely large when $\mathrm{vex}S > 2$ and will compensate any losses incurred by his main hedging strategy. □

Investor can multiply his capital significantly unless Market makes $\mathrm{vex}S = 2$ (part of this result is incorperated in the proof Theorem 4.3.1). If $\mathrm{vex}S > 2$, one can become infinitely rich by buying the security $\mathcal{D}$, and if $\mathrm{vex}S < 2$, one can become infinitely rich by selling $\mathcal{D}$.

4.3.2 Proposition. For any arbitrarily small $\epsilon > 0$ there exixts a strategy which, starting from ϵ at the moment when $S(0)$ and $D(0)$ are announced, never goes to debt and earns more than 1 if:

- S is continuous,
- $\sup_{0\le t\le T} S(t) < \infty$,
- $\inf_{0\le t\le T} S(t)$ is positive and non-infinitesimal, and
- $\mathrm{vex} S \neq 2$.

Proof. Assume that the conditions are satisfied and we show how to get rich when $\mathrm{vex} S < 2$ or $\mathrm{vex} S > 2$. We first show how to get rich when $\mathrm{vex} S > 2$. Since $\mathrm{vex} S > 2$, then $\mathrm{var}_S(2)$ is infinitely large, while $D(0)$ is finite. Buying $\$\epsilon$ worth of $\mathcal{D}$, we will get an infinitely large amount,

$$\epsilon \sum_t \left(\frac{\mathrm{d}S(t)}{S(t)}\right) \geq \epsilon \left(\sup_t S(t)\right)^{-2} \mathrm{var}_S(2),$$

in dividends, however, ϵ is arbitrarily small. Now assume that $\mathrm{vex} S < 2$. In this case, $\mathrm{var}_S(2)$ is infinitesimal. Selling $\$1$ worth of $\mathcal{D}$, we will get at least 1 when vex$S$ is different from 2. $\square$

5. Conclusion

In this paper we considered a market that prices two derivative securities, stock S and the security $\mathcal{D}$. We employed protocols for the market that prices stock S which pays no dividends and security $\mathcal{D}$ which pay dividend $(\Delta S_n/S_{n-1})^2$ each time step. No stochastic assumptions or probability structures are imposed on our securities for these protocols. Our aim was to price European options using model free approach. Through Theorem 4.3.1, we achieved our goal. $U(S(T))$ is a fair price for the European option with maturity T, this result is similar to the classical case, the difference is that we don't start with any probability structure at outset, but it emerges and the remaining volatility $\sigma^2(T-t)$ is equivalent to the price of the security $\mathcal{D}$ at time 0.

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