

**Prospective Zimbabwean “A” Level mathematics teachers’ knowledge of
the concept of a function**



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University of the Western Cape

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A thesis submitted in Partial fulfillment of the Requirements for the Degree of
Doctor of Philosophiae, in the Department of Didactics, University of the
Western Cape.



Promoters:

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Declaration

I declare that *Prospective Zimbabwean “A” Level mathematics teachers’ knowledge of the concept of a function* is my own work, that it has not been submitted before for any degree or examination in any other university and that all the sources I have used or quoted have been indicated and acknowledged as complete references

Name: Maroni Runesu Nyikahadzoyi

Date: May 2006

Signed.....

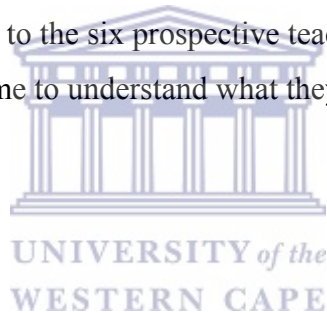


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Finally, I owe a debt of gratitude to the six prospective teachers who gave their time to participate in this study, helped me to understand what they knew and how they thought the concept should be taught.



Abstract

Prospective Zimbabwean “A” Level mathematics teachers’ knowledge of the concept of a function

By

Maroni Runesu Nyikahadzoyi

The purpose of the study was to investigate prospective ‘A’ level mathematics teachers’ knowledge of the concept of a function. The study was a case study of six prospective Zimbabwean teachers who were majoring in mathematics with the intention of completing a programme leading to certification as secondary mathematics teachers. At the time of the study the six prospective teachers were in their final year of study. Prospective teachers’ knowledge of the concept of a function was assessed through task-based interviews and reflective interviews. These interviews, which were done over a period of three months, were structured to capture the prospective teachers’ subject matter knowledge and pedagogical content knowledge for teaching the concept of a function. The interviews were also meant to capture the prospective teachers’ underlining pedagogical reasons for their choices of the examples, representations and teaching approaches when planning to teach the concept.

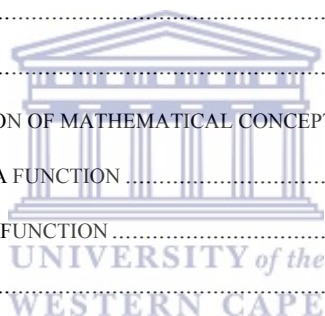
As part of the study a theoretical framework for understanding prospective teachers’ knowledge of the concept of a function was developed. The framework, which was developed, was used as an analytical tool in analyzing prospective teachers knowledge of the concept of a function.

The results of the study indicated that the prospective teachers had a process conception of a function although some of them had given a set-theoretic definition of a function in which a function is perceived as a mathematical object. They also confined the notion of a function to sets of real numbers. Functions defined on other mathematical objects (for example, the differential operator and the determinant function) were not considered as functions by five of the six prospective teachers.

Contents

DECLARATION	3
ACKNOWLEDGEMENTS	4
ABSTRACT	5
CHAPTER 1	12
THE PROBLEM AND ITS CONTEXT	12
1.1 BACKGROUND TO THE STUDY	12
1.2 RESEARCH QUESTION	18
1.3 DELIMITATIONS OF THE STUDY	19
1.4 IMPORTANCE OF THE STUDY	20
1.5 ASSUMPTIONS.....	21
1.6 THESIS OUTLINE	21
CHAPTER 2	24
PROVISIONAL THEORETICAL FRAMEWORK OF THE STUDY	24
2.1. INTRODUCTION.....	24
2.2 ROLES OF THEORETICAL FRAMEWORKS IN RESEARCH	24
2.3 CHARACTERIZATIONS OF TEACHER KNOWLEDGE OF MATHEMATICS.....	25
2.3.1 SHULMAN’S CHARACTERIZATION OF TEACHER KNOWLEDGE.....	25
2.3.2 MA’S CHARACTERIZATION OF TEACHER KNOWLEDGE OF MATHEMATICS	25
2.3.3 BALL’S CHARACTERIZATION OF TEACHER KNOWLEDGE OF MATHEMATICS	26
2.4 SUBSTANTIVE AND SYNTACTIC STRUCTURES OF PEDAGOGICAL CONTENT KNOWLEDGE	29
2.4.1 SUBSTANTIVE STRUCTURE OF PEDAGOGICAL CONTENT KNOWLEDGE	30
2.4.2 SYNTACTIC STRUCTURE OF PEDAGOGICAL CONTENT KNOWLEDGE.....	31
2.5 CONCLUSION	33
CHAPTER 3	35
LITERATURE REVIEW	35
3.1 INTRODUCTION.....	35
3.2 HISTORICAL DEVELOPMENT OF THE CONCEPT OF A FUNCTION.....	35

3.2.1 PRE-FUNCTION ERA	35
3.2.2. A FUNCTION AS A VARIABLE QUANTITY RELATED TO A CURVE.....	36
3.2.3 A FUNCTION AS AN ALGEBRAIC FORMULA	37
3.2.4 A FUNCTION AS AN ARBITRARY CORRESPONDENCE	41
3.2.5 A FUNCTION AS A SET OF ORDERED PAIRS.....	42
3.3 THE PSYCHOLOGICAL DEVELOPMENT OF THE CONCEPT OF A FUNCTION	44
3.3.1 PIAGET’S THREE FORMS OF ABSTRACTION	44
3.3.1.1 EMPIRICAL ABSTRACTION.....	44
3.3.1.2 PSEUDO-EMPIRICAL ABSTRACTION	44
3.3.1.3 REFLECTIVE ABSTRACTION.....	45
3.4 DUBINSKY’S APOS THEORY OF CONCEPT DEVELOPMENT	45
3.4.1 INTERIORIZATION	46
3.4.2 REVERSAL	46
3.4.3 CONDENSATION.....	47
3.4.4. REIFICATION.....	47
3.4.5 GENERALIZATION	47
3.5 DEFINITION-PROOF CONSTRUCTION OF MATHEMATICAL CONCEPTS	49
3.6 ALTERNATIVE CONCEPTIONS OF A FUNCTION	49
3.6.1 PRE-ACTION CONCEPTION OF A FUNCTION	50
3.6.2 ACTION CONCEPTION.....	50
3.6.3 PRE-PROCESS CONCEPTION	50
3.6.4 PROCESS OR OPERATIONAL CONCEPTION	50
3.6.5 PSEUDO-STRUCTURAL CONCEPTION	51
3.6.6 OBJECT OR STRUCTURAL CONCEPTION.....	51
3.6.7 PROCEPT CONCEPTION	52
3.6.8 PROPERTY-ORIENTED VIEW OF A FUNCTION.....	52
3.6.9 POINT-WISE CONCEPTION OF A FUNCTION.....	53
3.6.10 COVARIANCE VIEW OF A FUNCTION	54
3.7 THE PROCEPTUAL DIVIDE.....	54
3.8 REPRESENTATIONS OF THE CONCEPT OF A FUNCTION	57
3.9 THE CONCEPT OF A FUNCTION AND ITS ASSOCIATED COGNITIVE OBSTACLES.....	59
3.9.1 INHERENT COMPLEXITIES OF REIFYING THE CONCEPT OF A FUNCTION	59
3.9.1.1 PROBLEM OF SEMANTIC CONCESSIONS	59
3.9.1.2 REIFICATION-INTERIORIZATION VICIOUS CIRCLE.....	60



3.9.2	COMPARTMENTALIZATION OF THE DEFINITIONS AND CONCEPT IMAGES OF A FUNCTION	61
3.9.3	LACK OF FLEXIBILITY IN SWITCHING REPRESENTATIONS	62
3.9.4	NOTATIONAL COMPLEXITIES	62
3.9.5	COGNITIVE OBSTACLES ASSOCIATED WITH FORMS OF REPRESENTATION	64
3.9.5.1	COGNITIVE OBSTACLES ASSOCIATED WITH THE FUNCTION MACHINE	64
3.9.5.2	COGNITIVE OBSTACLES ASSOCIATED WITH THE ALGEBRAIC REPRESENTATIONS	65
3.9.5.3	COGNITIVE OBSTACLES ASSOCIATED WITH THE GRAPHICAL REPRESENTATIONS	65
3.9.5.4	COGNITIVE OBSTACLES ASSOCIATED WITH THE TABULAR REPRESENTATIONS	67
3.10	CONCLUSIONS	67
CHAPTER 4		68
	THEORETICAL FRAMEWORK OF THE STUDY	68
4.1	INTRODUCTION	68
4.2	COMPONENTS OF THE FRAMEWORK	68
4.2.1	SUBJECT MATTER KNOWLEDGE	69
4.2.1.1	CONCEPTIONS OF A FUNCTION	69
4.2.1.2	CONCEPT IMAGES OF A FUNCTION	71
4.2.1.3	KNOWLEDGE OF DIFFERENT REPRESENTATIONS OF THE CONCEPT	71
4.2.1.4	KNOWLEDGE OF DEFINITIONS OF A FUNCTION	72
4.2.1.5	REPertoire OF EXAMPLES OF FUNCTIONS	72
4.2.1.6	ESSENTIAL FEATURES OF A FUNCTION	73
4.2.2	PEDAGOGICAL CONTENT KNOWLEDGE	73
4.2.2.1	KNOWING DEFINITIONS OF A FUNCTION FOR TEACHING	73
4.2.2.2	KNOWING REPRESENTATIONS OF FUNCTIONS FOR TEACHING	74
4.2.2.3	USING LEARNERS' CONCEPT IMAGES OF A FUNCTION IN TEACHING	74
4.2.2.4	USING LEARNERS' CONCEPTION OF A FUNCTION TO RECAST THE TEACHING OF FUNCTIONS	75
4.2.2.5	KNOWING EXAMPLES OF FUNCTIONS FOR TEACHING	75
4.2.2.6	KNOWLEDGE OF COGNITIVE OBSTACLES ASSOCIATED WITH THE LEARNING OF FUNCTIONS	75
4.2.3	SYNTACTIC COMPONENT OF PCK	76
4.3	SOURCES OF TEACHER KNOWLEDGE FOR TEACHING THE CONCEPT OF A FUNCTION	77
4.3.1	KNOWLEDGE OF THE PSYCHOLOGICAL DEVELOPMENT OF THE CONCEPT	77
4.3.2	KNOWLEDGE OF THE HISTORICAL DEVELOPMENT OF THE CONCEPT	78
4.3.3	KNOWLEDGE OF THE NATURE OF MATHEMATICS	78
4.3.4	KNOWLEDGE OF LEARNERS' CONCEPTIONS OF A FUNCTION	79
4.3.5	KNOWLEDGE OF THE CONTEXT FOR TEACHING THE CONCEPT	79

4.4 CONCLUSIONS	80
CHAPTER 5.....	83
RESEARCH METHODOLOGY.....	83
5.1 INTRODUCTION.....	83
5.2 THE RESEARCH DESIGN.....	83
5.3 THE PARTICIPANTS	84
5.4 THE INTERVIEWS	84
5.4.1 TASK-BASED INTERVIEWS.....	85
5.4.2 INTERVIEWS BASED ON TEACHING SCENARIOS.....	86
5.4.3 REFLECTIVE INTERVIEWS	87
5.5 INTERVIEW PROCEDURES	88
5.6 DATA ANALYSIS.....	89
5.6.1 REDUCTION AND ANALYSIS BY PERSON	90
5.6.2 REDUCTION AND ANALYSIS BY TASK	90
5.6.3 ANALYSIS BY THEMES	90
5.7 RELIABILITY	93
5.7.1 CONSISTENCE OF RESPONSES.....	93
5.7.2 INTER-RATER RELIABILITY	93
5.8 VALIDITY	94
5.8.1 DESCRIPTIVE VALIDITY	94
5.8.2 INTERPRETIVE VALIDITY.....	94
5.8.3 RESPONDENT VALIDATION/ MEMBER CHECKING	95
5.8.4 ETHICAL CONSIDERATIONS.....	95
5.9 CONCLUSION	95
CHAPTER 6.....	96
DATA PRESENTATION AND DISCUSSION.....	96
6.1 INTRODUCTION.....	96
6.2 PROSPECTIVE “A” LEVEL TEACHERS’ KNOWLEDGE OF THE DEFINITION OF A FUNCTION.....	96
6.2.1 OPERABILITY OF THE DEFINITION OF A FUNCTION	99
6.2.1.1: A FUNCTION AS A RULE OF CORRESPONDENCE.....	99
6.2.1.2 A FUNCTION AS A CORRESPONDENCE.....	101
6.2.1.3 A FUNCTION AS FORMULA OR AN EQUATION.....	102
6.2.1.4 A FUNCTION AS A DEPENDENCE RELATION	103



6.3 BREADTH AND DEPTH OF PROSPECTIVE “A” LEVEL MATHEMATICS TEACHERS’ UNDERSTANDING OF THE CONCEPT OF A FUNCTION.....	107
6.3.1 THE FUNCTION MACHINE AS A REPRESENTATION OF A PROCEDURE, PROCESS OR A MENTAL OBJECT	108
6.3.2 THE ALGEBRAIC EQUATION AS A REPRESENTATION OF A PROCEDURE, PROCESS OR AN OBJECT	113
6.3.3 THE VERBAL DESCRIPTION OF A FUNCTION AS A PROCEDURE, PROCESS OR A MENTAL OBJECT.	117
6.3.4 DETERMINATION OF FUNCTIONALITY	119
6.4 INFLUENCE OF PROSPECTIVE TEACHERS’ CONCEPTIONS AND IMAGES OF A FUNCTION ON THEIR PEDAGOGICAL CONTENT KNOWLEDGE.....	122
6.4.1 INFLUENCE OF TEACHERS’ CONCEPT IMAGES A FUNCTION ON CHOICES OF EXAMPLES FOR USE WITH LEARNERS.....	122
6.4.2 INFLUENCE OF TEACHERS’ CONCEPTIONS A FUNCTION ON CHOICES OF LEARNERS’ TASKS	123
6.4.3 INFLUENCE OF TEACHERS’ WAYS OF KNOWING THE CONCEPT OF A FUNCTION ON THEIR PEDAGOGICAL CONTENT KNOWLEDGE.	124
6.5 PROSPECTIVE TEACHERS’ PEDAGOGICAL REASONING ON FUNCTIONS.....	124
6.5.1 PEDAGOGICAL REASONS FOR THE CHOICE OF FUNCTIONAL REPRESENTATIONS	125
6.5.2 PEDAGOGICAL REASONS FOR THE PURSUIT OF MATHEMATICAL CONNECTIONS	128
6.5.3 PEDAGOGICAL REASONS FOR THE CHOICE OF EXAMPLES	129
6.5.4 PEDAGOGICAL REASONS FOR THE DIFFERENT LEVELS OF TEACHER AND PUPIL INVOLVEMENT	131
6.5.5 DECISIONS INFORMED BY PEDAGOGY VERSUS PEDAGOGY-FREE DECISIONS.....	133
6.6 CONCLUSIONS	134
CHAPTER 7.....	135
DISCUSSIONS, CONCLUSIONS AND RECOMMENDATIONS	135
7.1 INTRODUCTION	135
7.1.2 PROSPECTIVE TEACHERS UNDERSTANDING OF THE DEFINITION OF A FUNCTION	135
7.1.3 PROSPECTIVE TEACHERS’ BREADTH AND DEPTH OF PROSPECTIVE ‘A’ LEVEL MATHEMATICS TEACHERS’ UNDERSTANDING OF THE CONCEPT OF A FUNCTION	137
7.1.4 INFLUENCE OF PROSPECTIVE TEACHERS’ CONCEPTIONS AND IMAGES OF A FUNCTION ON THEIR PEDAGOGICAL CONTENT KNOWLEDGE.....	138
7.1.5 PROSPECTIVE TEACHERS’ WARRANTS IN THE PROCESS OF FORMULATING REPRESENTATIONS, EXAMPLES, DEFINITIONS OF A FUNCTION AS THEY PLAN TO TEACH THE CONCEPT TO AN ‘A’ LEVEL CLASS.	139
7.2 CONCLUSIONS	139
7.2.1 THE NOTION OF A FUNCTION AS A MODEL.....	141
7.3 IMPLICATIONS FOR MATHEMATICS TEACHER EDUCATION IN ZIMBABWE	141
7.3.1 USE OF TASK –BASED INTERVIEWS TO UNDERSTAND PROSPECTIVE TEACHERS’ UNDERSTANDINGS MATHEMATICS.....	141
7.3.2. THE NEED TO DEVELOP A NEW MATHEMATICS TEACHER EDUCATION PROGRAMME	142
7.4 IMPLICATIONS FOR TEACHING THE CONCEPT OF A FUNCTION IN TEACHER EDUCATION PROGRAMMES.	143
7.4.1 USE OF MULTIPLE REPRESENTATIONS	144

7.4.2 USE OF DEFINITIONS IN CONCEPT FORMATION	144
7.4.3 PROMOTING THE OPERATIONAL CONCEPTION BEFORE THE STRUCTURAL CONCEPTION.....	145
7.5 IMPLICATIONS FOR FURTHER STUDIES	145
REFERENCES.....	147
APPENDIX A:	163
INTERVIEW GUIDE: TEACHER KNOWLEDGE OF THE CONCEPT OF A FUNCTION.....	163
APPENDIX B	165
INTERVIEWS BASED ON TEACHING SCENARIOS	165
APPENDIX C.....	167
TASK-BASED INTERVIEWS	167
APPENDIX D.....	174
PROSPECTIVE TEACHERS' HYPOTHETICAL LESSON PLANS	174



CHAPTER 1

THE PROBLEM AND ITS CONTEXT

1.1 Background to the study

The Curriculum Development Unit (CDU), an organ of the Ministry of Education, coordinates curriculum development in Zimbabwe. From time to time the CDU coordinates the activities of various national panels of curriculum developers. For each subject taught in Zimbabwean schools there is a national panel of curriculum developers one of which is the Mathematics National Panel of Curriculum Developers. The Mathematics National Panel of Curriculum Developers is responsible for designing and reviewing the mathematics syllabuses for the primary and secondary school levels. It is the duty of the Mathematics subject officers at the CDU to appoint “the secretariat” and coordinates all activities. According to the CDU Guidelines (1976) members of the Mathematics National Panel of Curriculum Developers should include:

- Mathematics Education officers from Head Office in Harare and the nine provinces.
- A mathematics teacher from each of the nine provinces.
- One representative from a Mathematics Teachers’ Association.
- Representatives from the Mathematics Departments from local Universities.
- Representatives from local teachers’ colleges
- Representatives from the Zimbabwe Schools Examination Council (ZIMSEC).
- Representatives from industry and commerce.

After designing or reviewing the primary and the secondary school syllabuses members of the mathematics national panel are involved in one way or the other in implementing the mathematics curriculum. For example, mathematics lecturers in teachers colleges and universities have the responsibility of producing teachers who can implement the designed curriculum. They are also responsible for the production of resource material for use in mathematics teacher education preparation programmes and in schools. In the Zimbabwean education system, there is the Zimbabwe Schools Examination Council (ZIMSEC), which is

responsible for setting and administering the Grade Seven, 'O' and 'A' level examinations. The Zimbabwe Schools Examination Council coordinates examination item writing for the Grade Seven, 'O' and 'A' level examinations. Item writers for the Grade Seven examinations are drawn from the primary school teachers, Mathematics subject officers from ZIMSEC and Mathematics Education Officers. In the case of the 'O' and 'A' level examinations, item writers are drawn from secondary school mathematics teachers, Mathematics Education Officers from Head Office and the nine provinces and mathematics lecturers from the teachers' colleges and local universities. The examination items are based on the mathematics curriculum drawn up by the Mathematics National Panel of Curriculum Developers.

The scenario described above shows that the different categories of personnel: Mathematics Education Officers from the Ministry of Education, mathematics teachers, lecturers from teachers' colleges and universities and officials from the Zimbabwe Examinations Council are all involved not only in the designing of the mathematics curriculum but also in the implementation and evaluation of the mathematics curriculum. Given the scenario described above one would expect the majority of the Zimbabwean learners to excel in the public examinations. However, Moyo's (2003) analysis of Zimbabwean learners' performance in the Grade Seven, 'O' and 'A' level examinations from 1980 to 2000 shows that learners' performance across the years was always low in mathematics compared to their performance in other subjects. Worse still the analysis show that over the years the pass rate in mathematics was declining instead of improving.

Learners' poor performance in mathematics has been attributed to a number of factors some of which are: poor curricular material (Jaji and Hodzi, 1980), inappropriate mathematics teacher education preparation programmes (Ball, 2000) and teachers' inability to plan and organize rich learning experiences for the learners (Jaji, 1990). However, although evidence exists that learners' poor performance is attributed to teachers' inability to plan and organize rich learning experiences for the learners, very little is known in terms of how teacher knowledge of mathematics is organized in their mind and how they intended to use that knowledge. Given the breadth and depth of the discipline it is not feasible to study teachers' cognitions of mathematics. The focus of the current study then is to investigate prospective 'A' level mathematics teachers' knowledge of the concept of a function. The decision to

study prospective teachers knowledge of the concept of a function was based on the fact that the concept is considered as a unifying idea in the Zimbabwean mathematics curriculum.

Since independence progress has been made in Zimbabwe in localizing the mathematics syllabuses and the examination system from the University of Cambridge Local Examination Syndicate. The new mathematics syllabuses at all school levels now advocate for the use of the concept of a function as a unifying idea in mathematics teaching. Mathematics teachers are encouraged to plan pupils' activities around the fundamental mathematical ideas such as the concept of a function rather than the smaller ideas, which are supposed to be developed as parts of the fundamental ideas. The outline below shows how the concept of a function came to be adopted by the mathematics syllabus panelists as a unifying idea.

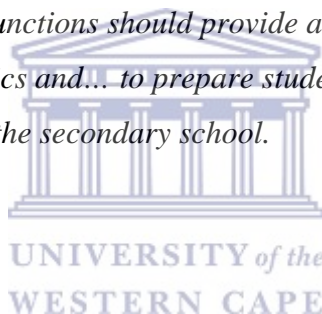
In 1996 a national panel comprising of mathematics teachers, college and university lecturers, mathematics education officers from the Ministry of Education and officials from the Zimbabwe Schools Examinations Council, was asked to come up with a new mathematics curriculum, which is relevant to the Zimbabwean situation. The national panel identified the concept of a function as one of the fundamental concept in the sense that it is basic and essential to the understanding of many areas of mathematics (Dhiwayo et al 1999).

In considering whether a mathematical concept or topic was important or not, the national mathematics syllabus panel came up with a number of criteria. The criteria involved several dimensions: the place of the topic within mathematics itself, the role of the topic outside mathematics and the relevance of the topic to the learner (Dhiwayo et al 1999).

A mathematical topic was considered important if it was fundamental to deeper mathematical study. The concept of a function was perceived as a springboard for deeper and broader mathematical content. The panel also promoted the concept of a function as a unifying theme. Before that, a symbolic, abstract approach to functions, including the vertical line test, identification of the range and the domain and the $y = f(x)$ notation was part of the traditional effort to learn about functions. However the symbolic study of functions did not give students the power to use functions to understand and describe the patterns found in the world around them (Jaji and Hodzi, 1980).

The main thrust of the new mathematics curriculum was that students should begin to see functional relationships in a variety of situations where the change of one quantity has a predictable change in another and be able to model these real-world situations. For example the students were supposed to know that equations like $C = 200x + 300$ could be the symbolic form of the relationship between talking time on the phone (denoted by x) and the charges billed (C). Finding these relationships was intimately tied to the notion of mathematics as a science of pattern and order. The panel also viewed the concept of a function important since it was considered to be useful in preparing the pupils to study other mathematical ideas such as limits, Fourier series, topology and metric spaces at University level. At a workshop held in Harare in 1996 the National mathematics syllabus panelists in conjunction with the officials from the Curriculum Development Unit resolved that:

The study of functions should begin informally in the early grades and develop in sophistication and breadth over the years of schooling. Early experiences with the concept of patterns and functions should provide a substantial base for understanding mathematics and... to prepare students for more intense study of mathematics throughout the secondary school.



(Dhiwayo et al, 1996:26)

Since then, the 'O' and 'A' Level mathematics syllabuses in Zimbabwe emphasized the study of relationships and the use of the algebraic language with which to codify them, with a focus on functions as tools for modeling simple physical phenomenon. In tertiary institutions, functions are used in a variety of ways. They are widely used in the comparison of abstract mathematical situations; for example, functions are used to tell whether two sets have the same cardinality, whether two topologies are homomorphic, or whether one group is a homomorphic image of another group. Functions are also the elements of abstract mathematical structures like vector spaces and operations on these structures such as vector addition and scalar multiplication are some of the special functions.

Although the concept of a function was part of the equation-solving approach curriculum, which was in place before the 'O' and 'A' level mathematics syllabuses were revised in 1996, it was patched in at the end of the algebra course which was designed then without providing substantial meaning or purpose (Sheehy, 1996). The call to make the concept of a

function a unifying theme in mathematics also influenced textbook publishers to revise the mathematics textbooks used in schools. One of the distinguishing features of the revised editions of the commonly used mathematics textbooks in Zimbabwean secondary schools is that the chapter on functions is now being presented earlier in the textbooks unlike in the earlier editions.

Mathematicians and mathematics educators from different parts of the world expressed similar sentiments concerning the importance of the concept of a function in the mathematics curriculum. For example, Dubinsky and Harel (1992) identified the concept of a function as the single most important concept from kindergarten to graduate school. Eisenberg (1991) proposes that having a sense for functions is one of the most important facets of mathematical thinking in that it allows students to gain insights into relationships among variables in problem solving situations. Dreyfus (1992) noted that a strong understanding of the concept of a function was a vital part of the background of any student hoping to comprehend calculus. Thompson (1994:5) wrote, “ If undergraduate mathematics curriculum does nothing else it should help students develop function sense”. Yerushalmy and Shavarts (1993) categorically stated that the concept of a function was a fundamental object of algebra, which was to be presented in a variety of representations in algebra teaching and learning from the onset. Romberg, Carpenter and Fennema (1993) claimed that attaining a deep and rich understanding of the concept of a function was crucial for success in mathematical courses at High School and College. Laughbaum, (2000.4) emphasized the importance of making the concept of a function a unifying idea by saying:

Our students see relationships in their lives, but do not know that the study of functions is the tool for analyzing and understanding them. What our students must be taught is to recognize and understand these mathematical relationships in the world they live in now, and will live in as adults. They must learn how one parameter affects the behavior of the relationship pattern. They must learn to create function relationships (in symbolic form) to model these real-world relationships.

The need to make the concept of a function one of the central themes in the mathematics curriculum was also echoed by different associations of mathematicians and reform movements. For example, the American Mathematical Association of Two-Year Colleges (AMATYC) (1995) in its Standards document recommended that the concept of a function be one of the central themes of the Standards for content. In 1967 the Cambridge Conference

on the Learning of Mathematics in Schools strongly recommended that the concept of a function be part of the curriculum since it was viewed as useful in organizing the material to be taught (Ponte, 1984). In the United States, as early as 1959, the Commission on Mathematics of the College Entrance Examination Board published a report in which they described a nine-point programme for school mathematics reform in light of the New Mathematics movement. The fourth point called for a judicious use of unifying ideas, one of which was the concept of a function. The Commission described the modern definition of the concept of a function as unifying all the previous ideas of a function. It stated:

We have presented several different points of view from which function (and relations) may be considered. We may describe them as sets of ordered pairs, sets of points, tables, correspondences, or as mappings. We may emphasize the rule or we may concentrate attention on the sets... the modern point of view of function is not contradictory to any of them, but unifies them all.

(Cited in Tall, 1992:23)

The National Council of Teachers of Mathematics (NCTM) (1989) emphasized the importance of the concept of a function as the unifying principle in the development of secondary mathematics curriculum. This is elaborated in the NCTM Curriculum Evaluation Standards for School Mathematics (1989,154).

The concept of a function is an important unifying idea in mathematics. Functions, which are special correspondences between the elements of two sets, are common throughout the curriculum. In arithmetic, functions appear as the usual operations on numbers where a pair of numbers corresponds to a single number, such as the sum of the pair; in algebra, functions are relationships between variables that represent numbers; in geometry, functions relate sets of points to their images under motions such as flips, slides and turns and in probability, they relate events to their likelihood. The concept of a function is also important because it is a mathematical relationship of many input-output situations found in the real world, including those that recently have arisen as a result of technological advances. An obvious example is the $\sqrt{2}$ key on the calculator

The above statement reflects the various roles of the concept of a function across different fields of mathematics such as algebra, geometry, probability and arithmetic. It is clear that

the sentiment that the concept of a function is both central and essential in today's mathematics curriculum is not just peculiar to Zimbabwe but to many countries. As the concept of a function is taking a new meaning in the field of mathematics and the field of mathematics continues to take new places in society, there is a perceived need to define and present it differently to students.

Despite widespread agreement that the mathematics curriculum should be centred around the concept of a function, the complex process of developing a conceptual understanding of functions continues to be difficult for students to master. Students are often unsuccessful at establishing the correct connections between various functional representations (Jaji, 1990). The qualitative interpretation of functional graphs is especially problematic. Sfard's research (1989) offers one possible explanation for the difficulties students encounter in terms of developing a sound understanding of the concept of a function. Sfard's findings suggest that students acquire a procedural or process conception of functions first, enabling them to understand the concept of a function on a rudimentary level, but have difficulty making transition or 'reifying' to a structural or object conception of function. Students who can reify functions perceive them as abstract notions whose representations may be algebraic, graphical or numeric. The knowledge of these different representation systems and the linkages between them can provide students with a rich conceptual understanding of functions (Kaput, 1989). However, there is a feeling that mathematics teachers should be held responsible for students' inability to master the fundamental mathematical ideas such as the concept of a function, since teachers' conceptions of mathematical ideas have a significant influence on what happens in classrooms (Amit, 1991). Given the importance of functions in mathematics, it is crucial to explore the nature of prospective "A" level mathematics teachers' knowledge of the concept of a function. Also, since the process of learning is influenced by the teacher, it is important to understand how prospective teachers explain what a function is to students, what they emphasize and what they do not, and the ways they choose to help students understand.

1.2 Research Question

The broad research question for the study was: What is the nature of prospective 'A' level mathematics teachers' knowledge for teaching the concept of a function? The research

question was broken down into sub-questions centred on subject matter knowledge and pedagogical content knowledge

- What are the prospective teachers' understandings of the definition of a function?
- What is the breadth and depth of prospective 'A' level mathematics teachers' understanding of the concept of a function?
- What is the influence of prospective teachers' conceptions and images of a function on their pedagogical content knowledge?
- What are the prospective teachers' warrants in the process of formulating representations, examples, definitions of a function as they plan to teach the concept to an 'A' level class?

1.3 Delimitations of the study

The current study focused on prospective teachers' subject matter knowledge of the concept of a function and their interpretations of some hypothetical students' mathematical thinking about the concept of a function. The prospective teachers' knowledge of the concept of a function was assessed by introducing the concept to prospective teachers through scenarios where a class of 'A' level students proposes alternative solutions to some tasks involving the concept. The prospective teachers were then invited to engage in the mathematics by examining how students might have arrived at their different answers and considering how they might have reacted to the students' responses. In this way subject matter knowledge and pedagogical content knowledge for teaching the concept of a function were brought together in mutual interaction. In short, the study explored what Ernest (1989) refers to as the enacted mental models for teaching the concept of a function. The decision not to study prospective teachers' knowledge of the concept of a function while in their actual teaching practice was deliberate since institutional and organizational factors in schools act as constraints.

Zimbabwean school administrators and school boards, preoccupied with test scores, put pressure on teachers to emphasize basic skills, computation and memorization of facts. Time is segmented into forty-minute periods, content must be covered, and pupils must be prepared not only for the external examinations, but also for the next level. Moreover the large class sizes make experimenting with pedagogy risky for teachers who are to maintain order and routine. Teachers often do not have the time to plan and organize deeper experiences for pupils, nor can they afford the luxury of engaging learners in investigative

work. Teachers are pressured to make sure that pupils master the required content. It is for the above reasons why prospective teachers' espoused knowledge for teaching the concept of a function was not investigated in practice.

1.4 Importance of the study

The success of a teacher education programme is measured in terms of the resulting curricular changes taking place. In some cases it is change manifested in teachers' awareness; in others, it is change demonstrated in their work with students. Carlson et al (2002) suggest that teacher education programmes should address at least two main issues: first, teachers' perceptions (belief, and attitudes), and second, teachers' skills that are needed for the day-to-day classroom activities. The current study is likely to provide the much needed information by mathematics educators at Masvingo State University as to the nature of prospective teachers' cognitions of the concept of a function, how they intend to teach the concept and the underlying reasons for their pedagogical decisions they might make as they teach the concept to 'A' level mathematics learners.

Although pedagogical knowledge or the knowledge of teaching has long been recognized in Zimbabwe as an important component of an educators' preparation for the classroom, the current certification tests in teacher education programmes do not emphasize assessment of pedagogical content knowledge. The extent of pedagogical content knowledge assessed in current teacher education certification tests is limited to testing candidates' knowledge of ways of assessing students' understanding of mathematics, the interpretation of those assessment results and the use of assessment data to positively impact on student achievement. The current certification assessments do not focus on pedagogy to a level appropriate enough to assess the pedagogical content knowledge required by the new teaching standards. They focus on what teachers do and ignore what they think and why.

Researchers are now beginning to alter their views of the teacher to encompass a more active, cognizing agent whose thought and decisions influence all aspects of classroom instruction and learning (Fennema, et .al, 1989). This change in the conceptualization of the teacher should coincide with a move from assessing teachers' knowledge in quantitative terms to the more recent qualitative attempts to describe the subtleties of teachers' conceptions of the subject areas (Ball, 1991). It is hoped that the findings of the current study

will hopefully provide both formative and summative information on the prospective teachers' conceptualization of the concept of a function. In other words, the findings will act as the basis for quality assurance in relation to newly qualified teachers' teaching skills and quality monitoring in relation to the mathematics teacher preparation programme at Masvingo State University.

Following the Cognitively Guided Instruction (CGI) approach to teacher education Fennema, et al, (1989), concluded that changes in teaching occur as teachers gain knowledge about children's thinking as revealed in the strategies their pupils use to solve word problems. Analogous changes are likely to occur in mathematics teacher educators once the findings of this study are available to them.

1.5 Assumptions

In carrying out the study it was assumed that, more than any other single factor, teachers influence what mathematics students learn and how well they learn it. Students' mathematical knowledge, their self-confidence and dispositions towards mathematics are shaped by the teachers' mathematical and pedagogical decisions. The Professional Teaching Standards of the National Council of Teachers of Mathematics (1991) describes four major responsibilities of the teacher in creating the mathematics classroom as a place for thinking and learning. These responsibilities include: posing worthwhile mathematical tasks, orchestrating stimulating mathematical discourses, thoughtful planning and reflecting on their teaching and creating classrooms where mathematical thinking is central. The study also assumed that teachers who would have had a strong preparatory teacher education programme could accomplish these responsibilities. In other words I assumed that the six prospective teachers have had a strong and effective mathematics teacher education programme.

1.6 Thesis outline

This report includes seven chapters and four appendices. Chapter I provided the background to the study. In the background to the study the changes in the Zimbabwean mathematics curriculum were described. One notable change in the Zimbabwean mathematics curriculum is that there has been a deliberate attempt to make the concept of a function a unifying idea in both algebra and other school mathematics courses. However the performance of learners

in mathematics has not improved over the years. It is against that background that I decided to investigate prospective Zimbabwean 'A' level mathematics teacher's knowledge of the concept of a function.

The second chapter outlines a provisional theoretical framework of the study. The provisional theoretical framework does not specifically describe features of teacher knowledge of the concept of a function but teacher knowledge of mathematics in general. However it was used as a tool in focusing the literature review in a bid to understand better what constitutes teacher knowledge of the concept of a function.

The third chapter constitutes a review of the related literature on the historical development of the concept of a function, alternative conceptions of a function, theories on the psychological development of the concept, different ways of representing functions and the cognitive obstacles associated with the learning of the concept. The above knowledge domains were reviewed since it was felt that they influence the teachers' content knowledge and pedagogical content knowledge for teaching the concept.

The fourth chapter outlines the theoretical framework that was developed in order to understand what constitutes the prospective teachers' knowledge of the concept. The theoretical framework was also used as an analytical tool for analyzing Zimbabwean prospective teachers' knowledge of the concept of a function.

The fifth chapter describes the research methodology used in the study. The chapter discusses how the task-based interviews, teaching scenarios and reflective interviews were used to collect data to answer the sub-questions raised in Chapter 1. Characteristics of the prospective secondary mathematics teachers are also described.

In chapter 6 the results of the study are outlined. The chapter characterizes the prospective teachers' understandings of the definition of a function, the extent to which they can use the definitions, their understanding of the properties of a function, their ability to translate from one representational form of a function to another and their warrants for judging the worth of products and processes of pedagogical reasoning that underline the choice of representations and examples when teaching mathematical concepts.

The final chapter gives a summary of the study. The major conclusions of the study are also highlighted. Lastly recommendations for further study and the implications of the findings for teaching the concept of a function are highlighted.



CHAPTER 2

PROVISIONAL THEORETICAL FRAMEWORK OF THE STUDY

2.1. Introduction

In this chapter a provisional theoretical framework of teacher knowledge for teaching mathematics is described with the intention of generating an analytical framework for classifying and describing prospective Zimbabwean teachers knowledge of the concept of a function

2.2 Roles of theoretical frameworks in research

Newman (1991) defines a theoretical framework as an orientation or a way of looking at a social phenomenon or construct. Theoretical frameworks are principally used in research to guide empirical inquiry as they provide a structure for an inclusive explanation of an empirical phenomenon, its scope and how we should look at and think about the phenomenon or a construct. The purpose of a theoretical framework then is to provide an orderly scheme for classification and description of a construct or social phenomenon. When faced with a subject of research, one can immediately identify its crucial aspects or variables by using the theoretical framework.

Theoretical frameworks do not only direct a researcher to the important questions but can also be used as analytical tools to make sense of research data. When exploring teacher knowledge of the concept of a function, an appropriate theoretical framework should describe in detail the various components of what constitutes teacher knowledge for teaching the concept. However, in the absence of an appropriate theoretical framework, there was a need to develop one. This was done by making a survey of recent literature on teachers' subject matter knowledge of mathematics.

2.3 Characterizations of teacher knowledge of mathematics

In order to develop a theoretical framework for studying prospective ‘A’ level teachers’ knowledge of the concept of a function, it is essential to first understand how teachers’ knowledge of mathematics is characterized by Shulman (1986), Ma (1999) and Ball (2000).

2.3.1 Shulman’s characterization of teacher knowledge

Shulman (1986) argued that in addition to knowledge of subject matter, teachers’ knowledge should include what he called pedagogical content knowledge, which consists of the ways of representing and formulating the subject that makes it comprehensible to others. Pedagogical content knowledge (PCK), like other domains of knowledge, has a body of accumulated ideas that have been constructed and used – in this case, pedagogical representations of the subject matter. The idea of pedagogical content knowledge (PCK) substantially improved mathematics teacher educators’ understanding of the knowledge required for teaching. The concept implies that not only must teachers know content deeply, know it conceptually and know the connections among ideas, but must know the representations for and the common student difficulties with particular ideas.

That concept of PCK makes clear that knowledge of mathematics for teaching encompasses more than what is taught and learned in conventional mathematics courses. Included here is knowledge of what is typically difficult for students, representations that are most useful for teaching a special idea or procedure and ways of developing a particular idea, for example the ordering of decimals, the advantages and disadvantages of particular metaphors or analogies and how they might distort the subject matter. For example, both ‘take away’ and ‘borrowing’ create problems for students’ understanding of subtraction. Ball (2000) remarked that such problems couldn’t be discerned generically because they require a careful mapping of the metaphors against core aspects of the concept being learned and against how learners interpret the metaphor.

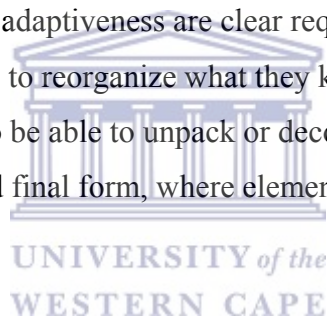
2.3.2 Ma’s characterization of teacher knowledge of mathematics

Ma (1999) described what she called profound understanding of fundamental mathematics (PUFM) in terms of the breadth, depth and thoroughness of the knowledge teachers need. Depth, according to Ma, refers to the ability to connect ideas to the large and powerful ideas

of the domain, whereas ‘breadth’ has to do with connections among ideas of similar conceptual power. Thoroughness is essential in order to weave ideas into a coherent whole. In addition to the premium she placed on connections, Ma also emphasized flexibility as held in a multiplicity of representations and approaches.

Drawing on Bruner’s (1960) ideas about the structure of a discipline, Ma stresses the importance of teachers knowing and attending to the simple but powerful basic concepts and principles of mathematics and developing basic attitudes such as, to seek to justify claims, to seek consistency in an idea across contexts and to know how as well as why. How such PUFM is used in practice is both dynamic and situated in contexts. Ma argues that teachers’ knowledge of mathematics for teaching must be like an experienced taxi driver’s knowledge of a city, whereby one can get to significant places in a wide variety of ways, flexibly and adaptively.

According to Ma, flexibility and adaptiveness are clear requirements for teaching. She argues that teachers must be able to reorganize what they know in response to a particular context. To do this, one needs to be able to unpack or deconstruct his/her own mathematical knowledge into less polished and final form, where elemental components are accessible and visible.



2.3.3 Ball’s characterization of teacher knowledge of mathematics

Ball (2000) contested the assumption that mathematically proficient people can solve mathematically implicated problems that arise in the course of teaching by pointing out that content knowledge needed for teaching mathematics is different from mathematical content knowledge held by mathematicians. One distinguishing feature of teacher knowledge of mathematics she highlighted is that it is in a decompressed form. In contrast, a powerful characteristic of mathematics is its capacity to compress information into abstract and highly usable forms. When ideas are represented in compressed symbolic form, their structure becomes evident and new ideas and actions are possible because of the simplification afforded by the compression and abstraction. Mathematicians are said to rely on this compression in their work (Ball and Bass, 2000). Ball pointed out that most personal knowledge of mathematics, which is desirably and usefully compressed, could be inadequate for teaching. Although mathematics is a discipline in which compression is central, its

polished, compressed forms can obscure one's ability to discern how learners are thinking at the roots of that knowledge. As Freudenthal (1983:469) states:

I have observed not only with other people but also with myself ... that sources of insight can be clogged by automatisms. One finally masters an activity so perfectly that the question of how and why students don't understand them is not asked anymore, cannot be asked anymore and is not even understood anymore as a meaningful and relevant question.

Knowing mathematics flexibly in and for teaching requires a transcendence of tacit understanding that characterizes much personal knowledge (Polanyi, 1958). Because teachers must be able to work with content for students in its growing, not finished state, they must be able to do something perverse: work backward from mature and compressed understanding of the content and unpack its constituent elements (Cohen, 2002). For example, they must be able to hear students' ideas and to hypothesize about their origin, status and direction. In order to ascertain the opportunities for learning embedded in the examples and work that they assign, teachers must be able to decompress a mathematical task, considering its diverse possible trajectories of enactment and engagement.

Ball (2000) remarked that teaching mathematics entails work with microscopic elements of mathematical knowledge, elements invisible for someone with mature mathematical fluency. She illustrated the importance of decompressed knowledge by saying that speculating on why a six year-old might write "1001" for –"one hundred and one"- and not reading it as a mistaken count – "one thousand and one"- requires the capacity to appreciate the elegance of the compressed notation system that adults use readily for numbers but that is not automatic for learners. After all, Roman numerals follow precisely the same structure as the young child's inclination, each element with its own notation, e.g., CI for "one hundred and one" – without the place value core of the Arabic system.

According to Ball, the hallmark of expert mathematics teachers' knowledge is attained when the teacher is able to see and hear someone else's perspective, to make sense of a student's apparent error or appreciate a student's unconventionally expressed insight. To do this the teacher has to unpack one's own highly compressed understandings. She goes further to say that even producing a comprehensive explanation depends on this capacity to unpack one's own knowledge, since an explanation works if its logical steps are small enough to make

sense for particular learners or class based on what they currently know or do not know (Ball, 2000).

According to Ball (2000) teacher knowledge of mathematics also includes knowledge of mathematical definitions and criteria that will enable a teacher to select definitions that are pedagogically appropriate. To Ball (2000) the most important criterion for a good definition is whether or not a definition is usable or operable by pupils at a particular level. For her, a definition of a mathematical object is useless, no matter how mathematically refined or elegant, if it includes terms that are beyond the prospective user's knowledge. In the case of the concept of a function the highly compressed modern set theoretic definition was developed by Bourbaki for the study of Analysis and hence it may not be appropriate for students who are studying Calculus courses.

Ball pointed that definitions must be based on elements that are themselves already defined and understood, hence defining a function in terms of a domain and range is meaningless to a learner who is not familiar with the terms domain and range. What is needed is being able to understand and work with definitions in the classroom, with pupils, treating them in a way that respects the role definitions play in doing and knowing mathematics at different levels of the mathematics curriculum. Knowing how definitions function and what they are supposed to do, together with knowing a well acceptable definition in a discipline, would equip a teacher for the task of developing a definition that has mathematical integrity which is acceptable to pupils.

Ball (2000) also emphasized that teaching mathematics requires a special sort of sensitivity to the need for precision. Precision requires that the language and ideas be meticulously specified so that mathematical problem solving is not necessarily impeded by ambiguities of meaning and interpretation. The need for precision is relative to the context and use. For example, a rigorous and precise definition of a function ' f ' as a rule that to each element $x \in X$ assigns a unique element $y \in Y$ where X , the set of input values, is called the domain of f , the set of possible output values of Y is called the codomain of f and the set of actual output values is called the range of f - would not be precise for a student who can't distinguish the difference between possible and actual values or the difference between the range and the codomain. Knowing what definitions are supposed to do, and how to select

and/or construct definitions that are appropriately and usefully precise for pupils at a certain level demands a flexible and serious understanding of mathematical language and what it means for a definition to be precise.

Another important aspect of subject matter knowledge of mathematics required for teaching identified by Ball (2000) is knowledge of its connectedness, both across mathematical domains at a given level and across time as mathematical ideas develop and extend.

Teaching requires teachers to help pupils make connections across mathematical domains, helping students build links and coherence in their knowledge. For example, teachers should know how the concept of a function is related to the concept of a sequence. Besides, teachers must know how to present ideas/concepts using multiple representations, that is, numeric, graphic and symbolic representations.

The kind of teacher knowledge of mathematics described by Ball is not something a mathematician would necessarily have by virtue of having studied advanced mathematics, neither would it be familiar to a high school biology teacher by virtue of his teaching experience. This kind of knowledge is quite clearly mathematical although formulated around the need to make ideas accessible to others. Unlike Shulman who was working on the assumption that someone who has the pedagogical content knowledge for a given discipline is able to use that knowledge in teaching, Ball contested that possessing a body of pedagogical content knowledge (PCK) may not always equip teachers with the flexibility needed to manage the complexities of practice. In her view, pedagogical content knowledge as perceived by Shulman was only a component of PCK.

2.4 Substantive and syntactic structures of Pedagogical content knowledge

Ball (2000) argued that an extensive repertoire of pedagogical representations, as conceived by Shulman (1986), was not sufficient for teaching since there is no repertoire of representations that could possibly suit all the possible teaching contexts. Ball (2000) proceeded to extend Shulman's notion of pedagogical content knowledge by considering pedagogical content knowledge as having both the substance (i.e., representations, definitions and examples) and the syntax, i.e., the standards that guide the pedagogical reasoning entailed in representing subject matter in teaching.

2.4.1 Substantive structure of pedagogical content knowledge

According to Schwab (1978) substantive knowledge of a discipline consists of the key facts, concepts, principles and explanatory frameworks. If pedagogical content knowledge is considered as a discipline, then the various representations, metaphors, definitions, examples and explanatory frameworks used in the teaching of a school subject would constitute the substantive component of pedagogical content knowledge. One of the most important issues that arise in mathematics education is the fact that ways need to be found to promote understanding in mathematics among learners. Carpenter, et al (1996) hinted that multiple representations of concepts could be utilized to help students develop deeper, more flexible understanding. They defined understanding as the way certain information can be represented and structured. Moreover, they affirm that mathematics is understood if its mental representation is part of a network of representations.

More recently the National Council of Teachers of Mathematics (NCTM, 2000:67) has included the use of representations as one of the new standards in mathematics teaching and learning. The representation standards state that:

Instructional programs from prekindergarten through grade 12 should enable students to create and use representations to organize, record, and communicate mathematical ideas; select, apply and translate among mathematical representations to solve problem; and to use representations to model and interpret physical, social and mathematical phenomena.

Kaput (1992) says that the use of more than one representation or notation system helps to illustrate a better picture of a mathematical concept or idea and that it provides diverse concretizations of a concept, carefully emphasize and suppress aspects of complex concepts, and promote the cognitive linking of representations. Moreover, Dorfler (1993) expressed his motives for using external representations in mathematics. He argued that first, representations are an inherent part of mathematics; second, representations are multiple concretizations of a concept; third, representations are used locally to mitigate certain difficulties; and last, representations are intended to make mathematics more attractive and interesting. In summary, it has been shown that the use of multiple representations is a useful tool for promoting better understanding of the key concepts in the mathematics curricula.

Knowledge of the different ways of representing a mathematical idea is what Ball would refer to as the substantive structure of pedagogical content knowledge.

2.4.2 Syntactic structure of pedagogical content knowledge

According to Ball (2000) the syntactic structure of PCK included the warrants for PCK and the criteria of those warrants. The warrants for PCK are the considerations that are used to appraise, evaluate and modify representations, definitions and teaching approaches as well as guide their generations. She proposed a framework of pedagogical content knowledge, which included, besides the substance of PCK, the syntax of PCK, i.e., the warrants for judging the worth of products and processes of pedagogical reasoning that underline representing mathematics to students. The warrants for judging the worth of products and processes of pedagogical reasoning included mathematical warrants, warrants based on learning theories, knowledge of learners and the context.

Mathematical warrants

Ball (2000) considered the substance and nature of mathematics to be a critical dimension of any pedagogical representation. Ball emphasized that representations must distil and simplify ideas without distorting them hence she argued that the first mathematical warrant should be that a mathematical representation should feature the conceptual essence of the content at hand, not just surface or procedural characteristics. The second mathematical warrant she proposed was that beyond the substance of mathematical knowledge, representation should be epistemologically appropriate, i.e. they should appropriately portray a disciplinary view of what it means to do and know mathematics. The last mathematical warrant is that teachers should ensure that a representation is likely to support the development of appreciation of and propensity towards mathematics.

Warrants based on learning theories

Pedagogical representations should help pupils learn particular topics or ideas. Ball (2000) has identified three criteria emerging from this purpose: focus, differentiation and multifacedness. The criterion of focus addresses the extent to which the form and relation of the representation call attention to the conceptual essence of the content, i.e., representations should spotlight the central components. Since mathematics is a system of ideas and ways of thinking the learners must understand the different ideas and how they fit together. Differentiation is the key factor here. The representations used should make the parts plainer

and the thinking apprehensible. For example, a good representation of a function should help learners to distinguish relations which are functions from non-functions. Since most concepts have multiple conceptual dimensions any single representation is unlikely to carry all of them. Thus, a third warrant for learning is multifacedness. It is important that over time the pedagogical representations encountered by students should be multiple and should round out the conceptual essence of particular ideas.

Warrants based on knowledge of students

Ball also alluded that for representations to be helpful to students in learning mathematics, they must be accessible to students. This accessibility is primarily a function of their comprehensibility. Use of learners' prior knowledge is central in ensuring that representations are accessible to learners.

To attract students' focus, interest is considered to be a critical criterion. Interest includes both those things about which pupils care as well as those things which might stimulate pupils' engagement in mathematics. Choosing representations that interest pupils per se or that expand their horizons of interest help to convey that mathematics is itself interesting.

Warrants based on the context for teaching

According to Ball valuable pedagogical representations must suit the contexts of teaching by being both feasible and sensitive. To be feasible a representation must be understood by a learner and the learner must understand how the representation illustrates a given process or concept. Another aspect of feasibility is the teacher's comfort and skill with the representation. The second contextual criterion is sensitivity. Representations, as borrowed likeness, often carry with them other features beyond those which connect them to the context. Certain features may be unacceptable or unhelpful whereas others may mesh particularly well within a given cultural or social context. The cultural or social context may range in receptiveness to certain representations. Teachers need to factor these warrants into their pedagogical reasoning.

Figure 2 summarizes the components of Shulman's (1986) notion of pedagogical content knowledge.

Pedagogical Content Knowledge (PCK)	
Substance of PCK	Syntax of PCK
Knowledge of: <ol style="list-style-type: none"> 1. Ways of representing and formulating mathematical concepts that make it comprehensible to others (representations). 2. Factors that make the learning of topics easy/difficult (cognitive obstacles) 3. Conceptions that students of different ages and backgrounds bring to the learning of the topic. 4. Means of assessing students' understanding of a given concept. 5. Instructional strategies that would enable students to connect what they are learning to the knowledge they already possess. 	MATHEMATICAL WARRANTS <ul style="list-style-type: none"> • conceptual essence • epistemological appropriateness • appreciation of and propensity towards mathematics. WARRANTS BASED ON LEARNING THEORIES <ul style="list-style-type: none"> • Focus • Differentiation • Multifacetedness WARRANTS BASED ON KNOWLEDGE OF LEARNER <ul style="list-style-type: none"> • Accessibility • Interest WARRANTS BASED ON THE CONTEXT <ul style="list-style-type: none"> • Feasibility • Sensitivity



Figure 2: A framework of the substantive and syntactic structure of pedagogical content knowledge (PCK): [Adapted from Shulman (1986) and Ball (2000)]

Figure 3 shows that while Shulman formulated the substantive structure of PCK, Ball added the syntactic component of PCK. The syntactic structure of PCK consists of mathematical warrants, warrants of learning theory, warrants based on knowledge of students and warrants of context.

2.5 Conclusion

The above theoretical framework was used as a tool in focusing my literature review in a bid to understand better what constitutes teacher knowledge of the concept of a function. The theoretical framework outlined above does not specifically relate to teacher knowledge of the concept of a function but teacher knowledge of mathematics teaching in general. The

framework does not explicitly address the following questions: What is the nature of teachers' decompressed knowledge of the concept of a function? What are the various definitions of the concept of a function and when should each specific definition be introduced to learners? What are the cognitive obstacles associated with each representations of the concept of a function? How can we characterize the level of teachers' understanding of the concept of a function? What are some of the warrants that can be used by teachers when judging the worth of the products and processes of pedagogical reasoning that underlie representing the concept of a function to learners?

In the next chapter, a detailed review of literature will be done in order to answer the above questions. Once the above questions have been answered the next step would be to develop a theoretical framework which describes the various components of what constitute teacher knowledge of the concept of a function.



CHAPTER 3

LITERATURE REVIEW

3.1 Introduction

The first section of the literature review focuses on the historical development of the concept of a function and an attempt is made to show how the historical development, as documented Kleiner (1989), Ponte (1984), Malik (1980) and Rutling (1984), can influence the teaching of the concept. Therefore instead of giving a comprehensive historical account the emphasis of the write up would be to discuss its implication for teaching the concept. This is followed by an analysis of the psychological theories on how the concept develops in learners. After highlighting the different conceptions about the concept the chapter ends up by exploring some of the learners' learning difficulties which have been identified in past research studies

3.2 Historical development of the concept of a function

According to Kleiner (1989) the concept of a function went through the following stages of development:

1. Pre-function era
2. A function as a quantity related to a curve
3. A function as an algebraic formula
4. A function as an arbitrary correspondence
5. A function as a set of ordered pairs

3.2.1 Pre-function era

Ponte (1984) pointed out that particular instances of a function could be traced to the ancient civilizations although by then the notion had no name. By then, the idea of function was applied only implicitly or to use Piaget's (1971) term - it was not yet thematized. For example, Ponte (1984) stated that the notion of a function was applied implicitly in:

1. The Babylonian tables of reciprocal, squares, square roots, cubes and cubic roots.

2. The Babylonian numeration system in which counting is viewed as a correspondence between a set of given objects and a sequence of counting numbers
3. The four elementary arithmetic operations, which are functions of two variables.

In view of the remark above it should be the concern of mathematics teachers to consider ways of instilling the notion of a functionality at primary and secondary school levels since learners at those levels encounter mathematical table of values. Lately, Kowski (2005) pointed out that learners could develop profound and broad understanding of the central concepts that characterize functions such as one-to-oneness and bijection if they are asked to focus on the structural properties of mathematical tables.

3.2.2. A function as a variable quantity related to a curve

The word function is said to have been introduced in the 17th century. It is important to understand the context in which the word function was first introduced and used. According to Kleiner (1989) the principal objects of study in 17th century were geometric curves. Malik (1980) pointed out that the 17th century mathematics originated as a collection of methods for solving problems about curves, such as finding tangents to curves, area under curves, lengths of curves and velocities of points moving along curves. For example, the cycloid was introduced geometrically and studied extensively well before it was given as an equation. It is interesting to note that the 17th century mathematics had a geometric flavour and the variables associated with a curve were geometric: abscissas, ordinates, subtangents, subnormal and radii of a curvature of a curve.

According to Malik (1980) Leibniz was the first to introduce the word function in 1673 as a general term for describing quantities related to a curve such as a normal or the slope of the curve. For example, Leibniz is said to have asserted that a tangent is a function of a curve. Leibniz was using the word function to refer to variables related to the curve. Functions related to curves are nowadays called differentiable functions and are still the most frequently encountered types of functions in the calculus courses.

Malik (1980) also pointed out that Newton's "methods of fluxions" applied to "fluents" and not functions. Newton called his variables "fluents" –whose images also were geometric, such as a point flowing along a curve. Newton's major contribution to the development of

the concept of a function was his use of power series. These were important for the subsequent development of the concept of a function.

Kleiner (1989) remarked that since the problem that gave rise to the calculus were geometric and kinematic in nature and that since Newton and Leibniz were pre-occupied with exploiting the marvelous tool they had created, time and reflection was required before the calculus could be recast in algebraic form.

3.2.3 A function as an algebraic formula

Kleiner (1989) described the first half of the 18th century as a period in which there was a gradual separation of the 17th century mathematics from its geometric origin and background. Bos (1980) refer to this gradual separation as the degeometrization of mathematics. This process of degeometrization of mathematics saw the replacement of the concept of a function, applied to geometric objects, with the concept of a function as an algebraic formula. As increased emphasis came to be placed on the formulae and equations relating to the functions associated with a curve, attention was focused on the role of the symbols appearing in the formulae and equations and thus on the relations holding among these symbols, independent of the original curve (Kleiner, 1989).

Malik (1980) pointed out that the first definition of a function as an analytical expression was formulated by Bernoulli in 1718. Bernoulli's definition read:

One calls here Function of a variable a quantity composed in any manner whatever of this variable and of constants

(Cited in Malik, 1980; 4)

However Bernoulli did not explain what "composed in any manner whatever," meant.

According to Rutling (1984) Euler is said to have claimed that mathematics was the science of variables and their functions, thereby endowing the concept of a function a central prominence in mathematics. Euler's entire approach is said to have been algebraic and not geometric and he defined the notion of a function as:

A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities.

(Rutling, 1984:72)

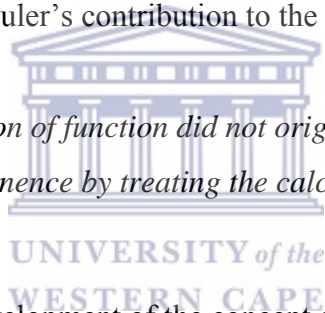
Euler did not define the term “analytic expressions” but tried to give it meaning by explaining that admissible “analytic expressions” involve, the four algebraic operations, roots, exponential, logarithms, trigonometric functions, derivatives and integrals. He classified functions as being algebraic or transcendental; single valued or multi-valued; implicit or explicit (Rutling, 1984).

It is important to observe that Euler

1. Did not consider piecewise defined functions as bona fide functions since they are defined by more than one analytic expressions in different intervals.
2. Had no notion of the univalence property of the concept of a function since he considered implicit and multi-valued relations as function. He was more likely to view the equation of a circle as a function.
3. Did not make a clear distinction between the notion of a function from its representation-the analytical expression.

Hawkins (1970:3) summarized Euler’s contribution to the emergence of function as an important concept as follows:

Although the notion of function did not originate with Euler, it was he who first gave it prominence by treating the calculus as a formal theory of functions.



The main impulse for further development of the concept of a function in the 18th century is said to have come from a controversy over a problem in physics, namely, the motion of a tense string fixed at two ends when it is made to vibrate. In a nutshell, the controversy centred on the meaning of a function and types of functions which could be allowed in mathematics from the standpoint of d’Alembert, Euler and Bernoulli.

The debate did, however, have important consequences for the evolution of the concept of a function. Its major effect was to extend the concept to include:

- (a) Functions defined piecewise by analytic expressions in different intervals. Thus

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad \text{was now, for the first time, considered as a bona fide function.}$$

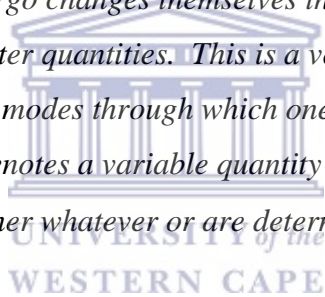
- (b) Functions drawn freehand and possibly not given by any combination of analytic expressions. Thus functions could now be represented graphically.

Lutzen (1978:17) summarized the effect of the debate that surrounded the Vibrating-String Problem as follows:

d’Alembert let the concept of a function limit the possible initial value, while Euler let the variety of initial values extend the concept of a function. We thus see that this extension of the concept of a function was forced upon Euler by the physical problem in question.

To see how Euler’s own view of function evolved over a period of several years, there is need to compare the definition of function he gave in his 1748 *Introductio* with the following definition given in 1755, in which the term “analytic expression” is replaced by “comprises in itself all the modes through which one quantity can be determined by others”.

if, however, some quantities depend on others in such a way that if the latter are changed the former undergo changes themselves then the former quantities are called functions of the latter quantities. This is a very comprehensive notion and comprises in itself all the modes through which one quantity can be determined by others. If therefore, x denotes a variable quantity then all the quantities which depend on x in any manner whatever or are determined by it are called its functions.



(Cited in Kleiner, 1989: 21)

According to Kleiner (1989) Euler went further and classified functions into continuous and discontinuous functions. He classified piece-wise defined functions as discontinuous functions while functions defined by single analytic expressions were referred to as

continuous functions. Hence, in Euler’s sense, the function $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ would be a

discontinuous function while a linear or quadratic function would be considered a continuous function.

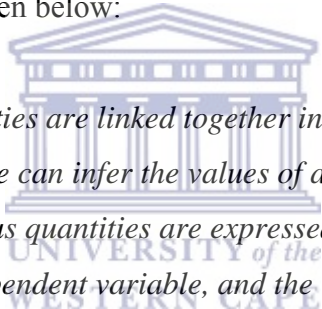
Euler’s new version of the definition of a function shows that he was now using the word function to refer to the value of the input obtained from a given input. Previously he used the word function to refer to the analytic expression.

According to Malik (1980) Cauchy defined the concepts of continuity, differentiability and integrability of a function in terms of limits. In dealing with continuity, Cauchy addressed himself to Euler's conceptions of continuous and discontinuous. He showed that the function

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad \text{(which Euler considered discontinuous) could also be written as}$$

$f(x) = \sqrt{x^2}$ or $f(x) = \frac{2}{\pi} \int_0^\infty \frac{x^2}{x^2 + t^2} dt$ which means that $f(x)$ is also continuous in Euler's sense. Cauchy is said to have claimed that this paradoxical situation could not happen when Euler's definition of continuity is used (Malik, 1980).

Bottazzini (1986) pointed out that Cauchy's conception of function was still Eulerian as seen in his definition of a function given below:



When the variable quantities are linked together in such a way that, when the value of one of them is given, we can infer the values of all the others, we ordinarily conceive that these various quantities are expressed by means of one of them which then takes the name independent variable, and the remaining quantities, expressed by means of the independent variables, are those which one calls the functions of this variable.

(Bottazzini, 1986:104)

Cauchy is said to have classified functions as 'simple' and 'mixed'. The simple functions included linear functions, polynomials, trigonometric functions and their inverses while 'mixed' functions were the composite of the 'simple' ones such as $\log(\sin x)$. Like his predecessors Cauchy's definition is silent about the univalence property, which is one of the key characteristic of the modern definition of the concept of a function.

3.2.4 A function as an arbitrary correspondence

Malik (1980) remarked that mathematicians from Euler through Fourier to Cauchy had paid lip service to the arbitrary nature of functions, but in practice they thought of functions as analytic expressions or curves. In 1829 Dirichlet defined a function as follows:

y is a function of a variable x, defined on the interval $a < x < b$, if to every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way the correspondence is established.

(Cited in Kleiner, 1989; 291)

The novelty in Dirichlet's conception of function as an arbitrary correspondence lies not so much in the definition but in its application. Dirichlet's definition of function was among the first to restrict explicitly the domain of the function to an interval. In the past the independent variables were allowed to range over all real numbers. Again Dirichlet was the first to take seriously the notion of function as an arbitrary correspondence. He showed that he was not paying lip service to the arbitrary nature of functions by adding the phrase 'it is irrelevant in what way the correspondence is established' to his definition of a function. The phrase implied that the requirement for a definite 'law' of correspondence in the definition of a function was not necessary. The 'law' was not supposed to be reasonably explicit, that is, it was not supposed to be understood by and communicable to anyone who might want to study the function. This was made abundantly clear in his 1829 paper on Fourier series, at the end of which he gave an example of a function (the Dirichlet function)

$$D(x) = \begin{cases} c & \text{if } x \text{ is rational} \\ d & \text{if } x \text{ is irrational} \end{cases}.$$

The Dirichlet function:

1. was the first explicit example of a function that was neither given by an analytic expression (or by several such) nor was it a curve drawn freehand.
2. was the first example of a function that is discontinuous (in the modern, not Euler's sense) everywhere.
3. illustrated the concept of a function as an arbitrary pairing.

3.2.5 A function as a set of ordered pairs

Mailk (1980) noted that as the study of higher-level mathematics became more and more abstract, so did the definition of function. The development of abstract algebra and topology gave way to a more set-theoretic definition of function. From 1900-1920 concepts such as metric spaces, topological spaces, Hilbert spaces and Banach spaces were introduced. These developments led to new definitions of a function based on arbitrary sets, not just on real numbers. In 1917 Carathéodory defined a function as a rule of correspondence from a set A to real numbers (Malik, 1980).

In response to the more modern definitions and applications of the concept of a function, Schaaf (1930) stated:

The keynote of Western culture is the concept of a function, a notion even remotely hinted at by any earlier culture. And the concept of a function is anything but an extension or elaboration of previous number concept...it is rather a complete emancipation from such notions



(Cited in Tall 1992:492).

The so-called emancipation from old ideas was evident as the field of mathematics became more abstract. Mathematicians started trying to formalize mathematics using set theory and they sought definitions of every mathematical object as a set. Bourbaki, a well-known proponent of topology introduced a set-theoretic definition of a function. In 1939, Bourbaki offered the following definition:

Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a functional relation in y if, for all x in E there exists a unique y in F which is the given relation with y .

We give the name function to the operation which in this way associates with every element x in E the element y in F which is the given relation x ; y is said to be the value of the function at the element x and the function is said to be determined by the given functional relation. Two equivalent functional relations determine the same function.

(Cited in Kleiner, 1989:12)

Bourbaki also gave the definition of a function as a certain subset of the Cartesian product $E \times F$. This is, of course, the definition of function as a set of ordered pairs. Whereas originally in the previous definitions, the sets E and F above were taken to be sets of numbers, this restriction was removed in Bourbaki's definition. For example, the set of input values could be functions. In general, there is no need at all for either E or F to be a set of numbers. All that is really necessary for a function is two non-empty sets E and F and a rule which is meaningful and unambiguous in assigning to each element x in E a specific element y in F . What is clear in Bourbaki's understanding of a function is that although the sets can be arbitrary the correspondence between elements in the two sets need to be defined by a rule/operation which is meaningful and unambiguous, i.e., the assignment is not arbitrary. While in Dirichlet's definition it is the assignment of elements which is arbitrary, in Bourbaki's definition it's the sets of input and output values which are arbitrary.

The historical development of the concept of a function shows that the meaning and use of the concept of a function has changed as interests of mathematicians changed. The concept unfolded from its modest beginnings as quantities related to a curve, a formula, an arbitrary correspondence and finally as an ordered set. Initially the definition was changing in response to the need by mathematicians to understand and solve important practical problems such as the Vibrating String Problem and the Heat-Conduction Problem. Later the definition of a function was changing in response to changes in the subject mathematics itself. Since the definition of function has been determined by its use in topics of study it implies that in choosing the definition and context in which high school students should approach the concept of a function, teachers should constantly be examining the purpose of studying functions in their own classrooms. Malik (1980: 492) sums this point well by saying:

We note that the definition of function as an expression or formula representing a relation between variables is for calculus or pre-calculus, is a rule of correspondence between reals for analysis, and a set theoretic definition with domain and range is required to study topology.

The above quote have serious implications for the mathematics curriculum at various levels. Since only a small percentage of 'A' Level students eventually study analysis and topology,

the set theoretic definition could be postponed to the beginning of these courses and an appropriate definition of a function, as dependence relationship between two sets of elements should be taught at the school level.

3.3 The psychological development of the concept of a function

In order to teach the concept of a function, teachers need to understand the processes by which learners construct this important mathematical concept and the nature of the cognitive entities constructed in this process. Most theories of how mathematical concepts develop in students focus on the differences between action-oriented conceptions and object-oriented conceptions. An understanding of these theories can be used as a basis for designing a learning program when teaching the concept of a function. This section begins with a quick description of theories on concept formation and then discusses some of the various approaches used to describe a function

3.3.1 Piaget's three forms of abstraction

Piaget (1975) introduced the terms empirical abstraction, pseudo-empirical abstraction and reflective abstraction to describe three different ways in which mathematical concepts develop in learners.



3.3.1.1 Empirical abstraction

When acting on objects in the external world, Piaget speaks first of empirical abstraction where the learner focuses on the objects themselves and the learner deriving knowledge from the properties of the objects. Piaget's example of empirical abstraction is the act of learning how to count which involves pointing at successive objects in a collection and in turn saying out the number names. This may be compressed, for instance, by counting out silently, saying just the last number name. However, counting should be regarded as a particular case of a functional situation. Counting should be viewed as correspondence between a set of given objects and a sequence of counting numbers.

3.3.1.2 Pseudo-empirical abstraction

On the other hand, a focus on actions while acting on objects leads to what Piaget (1975) called pseudo-empirical abstraction in which knowledge is derived from actions on physical

objects. A good example of empirical abstraction is when pupils learn the four elementary arithmetic operations, which are good examples of functions of two variables. To master the number operations the learner focuses on his/her actions and not on the objects being put together.

3.3.1.3 Reflective abstraction

Reflective abstraction involves construction of mathematical concepts by using existing structures to construct new ideas by observing one's thoughts and abstracting from them (Piaget, 1975). Piaget claimed that mathematical entities built productively on one another. In this way:

.the whole of mathematics may therefore be thought in terms of construction of structures ... mathematical entities move from one level to another; an operation on such entities becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternatively structuring or structured by stronger structures

(Piaget, 1975: 703)

Not only physical objects, but also previously developed cognitive structures (whether they are processes or objects of thought) can be subjected to mental operations. Piaget referred to a coherent collection of these structures as a schema.

3.4 Dubinsky's APOS theory of concept development

The Action, Process, Object and Schema (APOS) theory of Dubinsky (1991) sees the development of mathematical concepts in learners in the light of Piaget's theory of reflective abstraction. The APOS theory suggest that a mathematical concept/object is formed when an individual performs actions on already existing objects that are then interiorized into processes latter to become reified into objects to be built into a wider cognitive schema. The notion of reflective abstraction tries to respond to the fundamental questions of "how do mathematical concepts (either as objects or processes) come into being?" and "how do we know an individual deals with a concept as an object or as a process?" According to Dubinsky (1991) the construction of various concepts in mathematics can be described in terms of five forms of construction in reflective abstraction viz: interiorization, reversal, condensation, encapsulation/reification and generalization.

3.4.1 Interiorization

Dubinsky (1991) refers to interiorization as a mental construction of an internal process relative to a series of actions on cognitive objects that can be performed or imagined to be performed, in the mind of an individual, without necessarily running through all the specific steps. An action is any physical or mental transformation of objects resulting in processes. It occurs as a reaction to stimulus that the individual perceives as external. It may be a multi-step response which has the characteristic that at each step, the next step is triggered by what has come before, rather than by the individual's conscious control of the transformation. When the individual reflects upon an action to a point where he/she begin to have control over it, the action is said to have been interiorized and that action becomes a process.

Dubinsky (1991) noted that an important part of understanding a function was to construct a process. For individuals, this means that the learner has to respond to a situation in which a function may appear (via a formula or as an algorithm) and for which the value of the function for a particular value in the domain is obtained, e.g., determining the value of y in the equation $y = 2x + 2$ for a given value of x . Given such a situation, the learner may respond by constructing in his/her mind a mental process relating to the function process. This is a prime example of interiorization. A function as a process is determined as a whole by input – output relationship regardless of the internal procedures of computation. Thus the functions $f(x) = 2x + 2$ and $g(x) = 2(x + 1)$ are one and the same process even though the arithmetic procedures to compute them have different sequences of operations.

3.4.2 Reversal

Interiorizing actions is one way of constructing processes. According to Dubinsky (1991) another way is to work with existing processes to form new ones. This can be done by reversal. For example, a calculus student may have interiorized the action of taking the derivative of a function and may be able to do this successfully with a large number of examples, using various techniques that are taught and occasionally learned in calculus courses. If the process is interiorized the student might be able to reverse it to solve problems in which a function is given and it is desired to find a function whose derivative is the original function. This is anti-differentiation or integration, and it too, is often an action which must be interiorized to become a process. Other examples in mathematics that appear

to involve the reversal of a process include subtraction as a reversal addition, division as the inverse of multiplication and determination of inverse functions.

3.4.3 Condensation

Sfard (1991) refers to the phase of condensation as a period of squeezing sequences of operations into more manageable units. At the stage of condensation a person becomes more and more capable of thinking about a given process as a whole without feeling an urge to go into details. The condensation phase is said to last as long as a new entity remains tightly connected to a certain process. When a function is considered, the more capable a person becomes in playing with a mapping as a whole without actually looking into its specific values, the more advanced the process of condensation has become. Eventually, the learner can combine a couple of composite functions and can perceive the set of composite functions as a single input/output process.

3.4.4. Reification

According to Sfard (1992) a concept in question would have been reified/encapsulated only when a person becomes capable of conceiving a concept as a fully – fledged object. The cognitive mechanism for forming objects is also referred to as entification (Thompson, 1994) or encapsulation (Ayers, et.al, 1988). Whereas interiorization and condensation are gradual, quantitative rather than qualitative changes reification should be understood as a sudden qualitative jump in the way of looking at the concept: a process solidifies into an object, into a stable structure. Anytime a set of functions is considered, it seems necessary to think of the functions in question as objects. Initially, functions are processes so the cognizing agent must have performed an encapsulation in order to consider them as objects.

3.4.5 Generalization

The simplest and most familiar form of reflective abstraction is generalization. According to Dubinsky (1991) a learner's schema, in which functions transform numbers is generalized to include functions which transform other kinds of objects (once they have been encapsulated) such as vectors, sets, or other functions. A learner is said to have generalized the notion of a function if he/she has mastered the arbitrariness property of the concept of a function. The arbitrary nature of the concept of a function refers both to the nature of the sets and the

relationship between the sets. The functional relationships do not have to be defined on any particular sets of objects; in particular the sets do not have to be sets of numbers. Also the relationship between the two sets does not have to be specified by any rule. The correspondence could be arbitrary. Thus, there are at least two ways of constructing objects – from processes and from schemas. Figure 3.1 below is a schematic representation of how the construction of various concepts in mathematics could be described in terms of the five forms of abstraction.

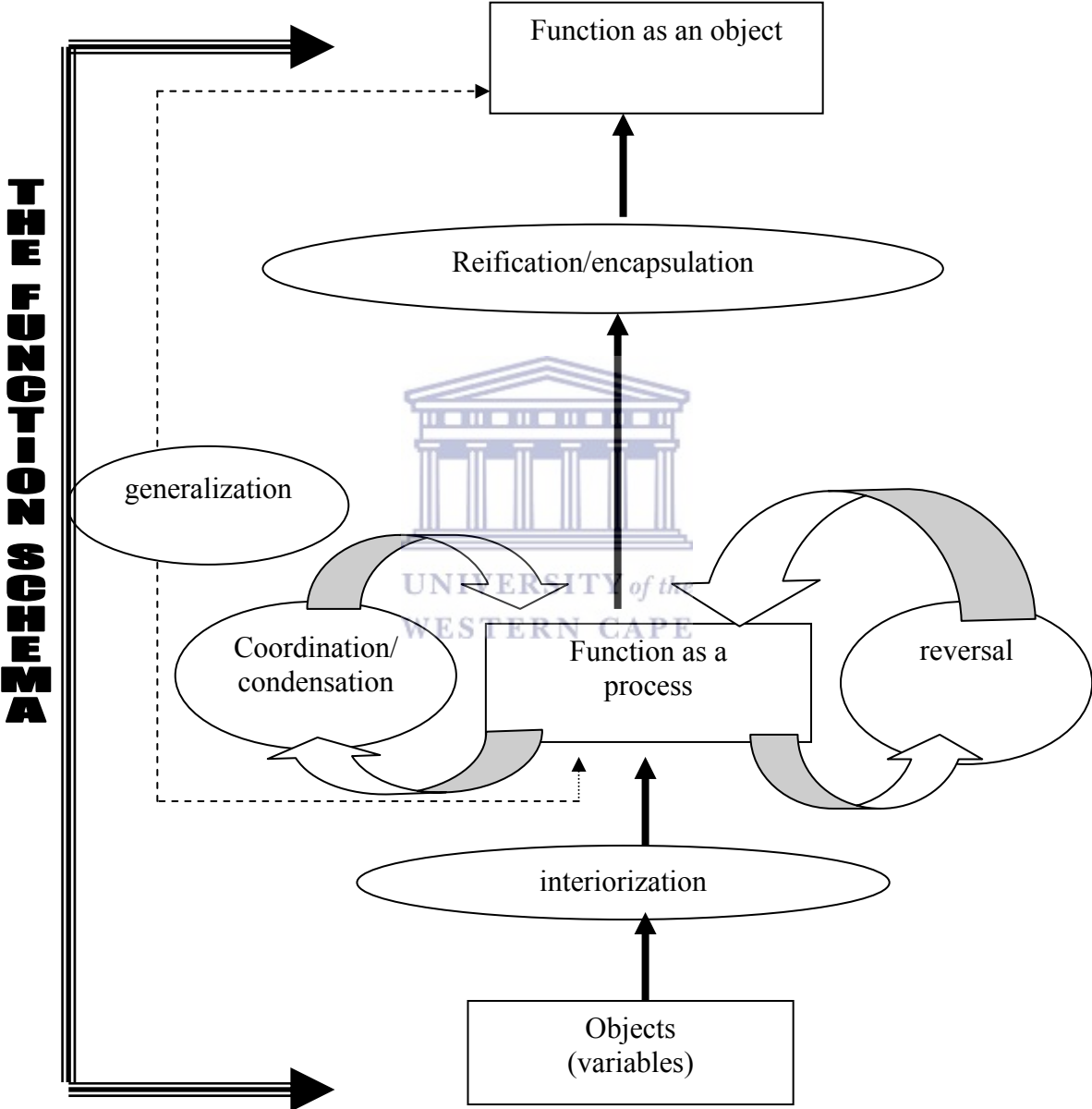
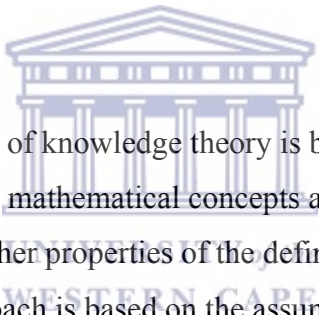


Figure 3.1: The Function Schema: A Theoretical Framework for the construction of the concept of a function [Adapted from Dubinsky, 1991].

3.5 Definition-proof construction of mathematical concepts

Tall (1992) introduced the notion of advanced mathematical thinking to describe the thinking of creative mathematicians imagining, conjecturing and proving theorems resulting in the creation of mathematical concepts, which may be entirely hypothetical, in the mind of the thinker. Thus in advanced mathematical thinking new mathematical knowledge is created by formulating definitions for mathematical concepts as a list of axioms for a given structure, then developing other properties of the structure by deduction from the definition. Advanced mathematical thinking also applies to the thinking expected from students who are presented with the axioms and definitions created by others. The student is often presented with a context where a formal concept is encountered both by examples and by a definition. Each of the examples satisfies the definition, but each has additional qualities, which may, or may not, be shared between the examples. The properties of the formal concept are deduced as theorems, thus constructing meaning for an overall umbrella concept from the definition of the concept. Tall referred to this scheme of concept acquisition as the definition-proof construction of knowledge.



The definition-proof construction of knowledge theory is based on the view that mathematics is a deductive theory and as such, mathematical concepts are given properties as axiomatic definitions and building up the other properties of the defined concepts by logical deduction (Tall, 2001). The deductive approach is based on the assumptions that definitions help in forming concept images and that concepts are acquired by means of their definitions. The idea of giving a verbal definition as a list of criteria and then constructing the concept from the definition is a reversal of the development in elementary mathematics where mathematical concepts are thought to have properties, which can be discovered by studying other objects and related processes.

3.6 Alternative conceptions of a function

In this section different conceptions of a function are going to be described. Knowledge of the different conceptions of a function is important to anyone interested in characterizing teachers' understanding of functions.

3.6.1 Pre-action conception of a function

According to Cotrill et al, (1996) an individual is said to have a pre-action conception of a function if he/she gives a response that appears to indicate little or no conception of a function. A typical response in this category is, for example, a function is an equation (in x) with no y values. Whatever the term means to such an individual, the meaning is not very useful in performing the tasks that are called for in mathematical activities related to functions.

3.6.2 Action conception

In the context of Dubinsky's (1991) action/ process/object theory of concept development an action view involves an understanding of function as a non-permanent construct. An action view pertains to the computational aspects associated with functions such as an arithmetic process. For example one can consider the function $f(x) = 3x^2 - 7$ to be an algorithm used to compute numeric values for a given input. This conception does not require an awareness of patterns and regularities that may exist between numeric values of successive inputs and outputs, nor attention to casual and dependency relationships between inputs and outputs. An action conception is concerned with the computation of a single numeric value via a given algorithm or rule of association. When asked to define a function an individual with an action conception would give a response which indicates a replacement of a number for a variable and then computing a number where there is no indication of an overall process of transforming a number to obtain another number (Dubinsky and Harel, 1992). It is a static conception in that the subject will tend to think about a function as a one step algorithm.

3.6.3 Pre-process conception

Sfard (1991) use the term pre-process conception to refer to a conception of a function which is in transition from an action conception to a process conception, that is, a conception of a function that has not fully developed into a process conception.

3.6.4 Process or operational conception

Operational or the process conception occurs when a person refers to a function as a process rather than an object. A process conception of a function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will

always produce the same transformed quantity. An individual with a process conception of a function is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done (Dubinsky and Harel, 1992). Such an individual is also able to combine a process with other processes, or even to reverse it. For instance, a function as a process is determined as a whole by coordinating the input-output pairs disregarding the internal procedures or computations. Thus the functions $f(x) = 2x+2$ and $g(x) = 2(x+1)$ are one and the same as processes even though the arithmetic procedures to compute them have a different sequence of operations. When a function is considered as a process the intermediate stages or procedures are considered as a single process without needing to carry out the intermediate steps.

3.6.5 Pseudo-structural conception

Sfard (1991) defined the pseudo-structural conception as a tendency to identify a given mathematical concept with the symbol which represents it. It manifests itself mainly in students' inability to make transitions from one representation of the concept to another. Someone with a pseudo-structural conception of a function would regard an algebraic formula as a function in itself and not a representation of a function. Euler, a renowned 18th century mathematician, seemed to put the equal sign between the idea of a function and the algebraic formulae which he referred to as 'analytic expressions'.

3.6.6 Object or Structural conception

Breidenbach et al (1992) claim that the process conception provides an entryway into an object-oriented understanding of a function while Sfard (1991) describes an object-oriented conception as the reification of the process view. Seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing. It also means being able to recognize the idea at a glance and to manipulate it as a whole without going into details. For example, one can consider the expression $3(x + 5) + 1$ in $y = 3(x + 5) + 1$ as a process which produces various outputs. However, one can also 'see' the expression as a certain number in its own right. Student difficulties in simultaneously comprehending these meanings of an expression have been referred to as the 'process-product dilemma' (Davis, 1984). However, Lin and Cooney, (2001) remarked that being capable of somewhat 'seeing' the invisible mathematical objects appears to be the essential component of mathematical ability and that

lack of this capacity is one of the major reason why mathematics appears practically impermeable to so many. A function is conceived as an object if it is possible to perform actions on it, in general perform actions that transform it. It is possible to ascertain whether an individual has constructed an object conception of a function by the way that individual talks about or defines a function. A function when conceived as a set of ordered pairs rather than as a computational procedure is such an abstract object.

3.6.7 Procept conception

Sfard (1991) reiterated that the structural and operational conception of a mathematical concept should be considered as different sides of the same coin and that the ability of seeing a concept, such as the concept of a function, both as a process and as an object was indispensable for deep understanding of the concept. In other words, the operational and the structural conceptions of a function although seemingly incompatible, are in fact complementary. Tall (2001) introduced the term procept to refer to an amalgam of three components: a process that produces a mathematical object, an object produced by that process and a symbol that represents either the process or the object. After the object conception Tall added the procept conception to indicate a conception of a function in which an individual is flexible enough to move back and forth between the process and the object conceptions as required by the task at hand. It is important to note that mathematical concepts can be conceived in two fundamentally different ways: structurally as objects and operationally as processes.

3.6.8 Property-oriented view of a function

While the theory of reification set forth that students come to develop more permanent abstracts through the enrichment of understandings of associated mathematical processes and actions, Slavit (1997) suggested an alternative route to the reification of a function. He formulated the theory of a property-oriented view of a function. The theory suggests that students conceive of functions as entities possessing various properties of a local and global nature. Global properties involve an analysis of the entire function while local properties are involved with individual or selected input-output pairs. Global properties include symmetry, periodicity, and monotonicity while local properties include intercepts, points of inflection and asymptotes etc. When discussing properties of a general function, one must consider all the characteristics associated with an object satisfying the definition of a function.

The property-oriented view of function therefore deals with the gradual awareness of specific functional properties, followed by the ability to recognize and analyze functions by identifying the presence or absence of these properties. Once a student has become familiar with these functional properties through various experiences, he/she can ‘see’ a function as an object either with or without these properties (Slavit, 1997). The property-oriented view is said to develop through two types of experiences.

First, the property-oriented view involves an ability to realize the equivalence of procedures that are performed in different notational systems. Noticing that the processes of symbolically solving $f(x) = 0$ and graphically finding x -intercepts are equivalent (in the sense of finding zeros) demonstrates this awareness. Second, students develop the ability to generalize procedures across different classes and types of functions. Here, students can relate procedures across notational systems, but they are also beginning to realize that some of these procedures have analogues in other types of functions. For example, one can find zeros of both linear and quadratic polynomials as well as other types of functions, and this invariance is what makes the property apparent.

Development of a property-oriented view of functions takes time since it is dependent on the knowledge of several functional properties, notational systems and classes of functions. For example, if linear and quadratic functions are studied almost exclusively as is the case of many algebra courses, then a student’s mental schema of functional properties will be quite small. To this student, functions are certainly well behaved and continuous, are either monotonic or change the direction and are zero at most once. At this stage a property-oriented view of function would have developed but it is certainly limited. When other polynomials, logarithmic, exponential, trigonometric functions are added, as is the case in many advanced algebra courses, the student’s library of functional properties will increase and a property-oriented conception of function strengthens.

3.6.9 Point-wise conception of a function

Calson and Oehrtman (2001) alluded that an individual with a point-wise conception of a function conceives a function as being made up of more or less isolated values or of input-output pairs. Such an individual is not able to recognize that information about input-output

pairs as belonging to a continuum of data points neither is he able to coordinate the amount and direction of change of one variable with changes in the other variable

3.6.10 Covariance view of a function

Calson and Oehrtman (2001) describe the covariance conception as the ability to coordinate the amount and direction of change of one variable with changes in the other variable. Ability to analyze, manipulate and comprehend the relationship between changing quantities illustrates the covariance view

3.7 The proceptual divide

De Marois and Tall (1996) described, the pre-procedure, procedure, process, concept and procept conceptions of a function as layers of increasing understanding of the concept of a function. Note that Tall and DeMarois prefer to use the term ‘concept’ rather than the term ‘object’ because they feel that the terms such as number concept and fraction concept are more common in ordinary language than ‘number object’ or ‘fraction object’. Pre-procedure assumes that the student is on the ground floor, so to speak, with respect to the concept of a function. A procedure is a coherent sequence of actions – a schema of actions such as the binary operations on numbers. Learners at the procedure layer can do routine mathematics accurately. A procedure is a specific algorithm such as determining the value of a function for a given input. On the other hand, a process is a cognitive entity, not dependent on individual steps, but rather on the result produced from the original input. For example, the expressions $2x + 6$ and $2(x + 3)$ represent two different procedures although they are identical processes. The results of applying each procedure to a given input are the same. Students who view these as different functions might be classified at the procedure layer while those who classify these as the same function might be placed at the process layer. Students at the process layer can cognitively accept the existence of a process between input and output without needing to know the specific algorithm. These students would have interiorised the specific procedures into a process and these learners at the process layer can perform mathematics flexibly and efficiently.

The concept layer aligns closely with the ability to treat the mathematical idea as an object to which a procedure (e.g. determining the derivative of a function) can be applied. After the concept layer a procept layer is placed to indicate the flexibility to move easily between the

process and the object layers as required. Students reach the most depth (the procept layer) when they can demonstrate flexibility in viewing any representation of a mathematical concept either as a process or as an object, as required by the problem situation.

Gray and Tall (1994) claim that a major source of the generative power of mathematics is in the use of symbols which are used ambiguously to evoke both a process of calculation and the product of that calculation. They related the divergence of success between the high achievers and the low achievers to the development of the procept conception by suggesting that the interpretations of mathematical concepts and their related symbolic representations as processes or objects leads to a proceptual divide between the less successful and the more successful. In their work with students they found out that the more able students were able to treat mathematical symbolism flexibly as processes or as objects, whichever was more appropriate in a given context while the less able tended to conceive mathematics as separate procedures to be done.



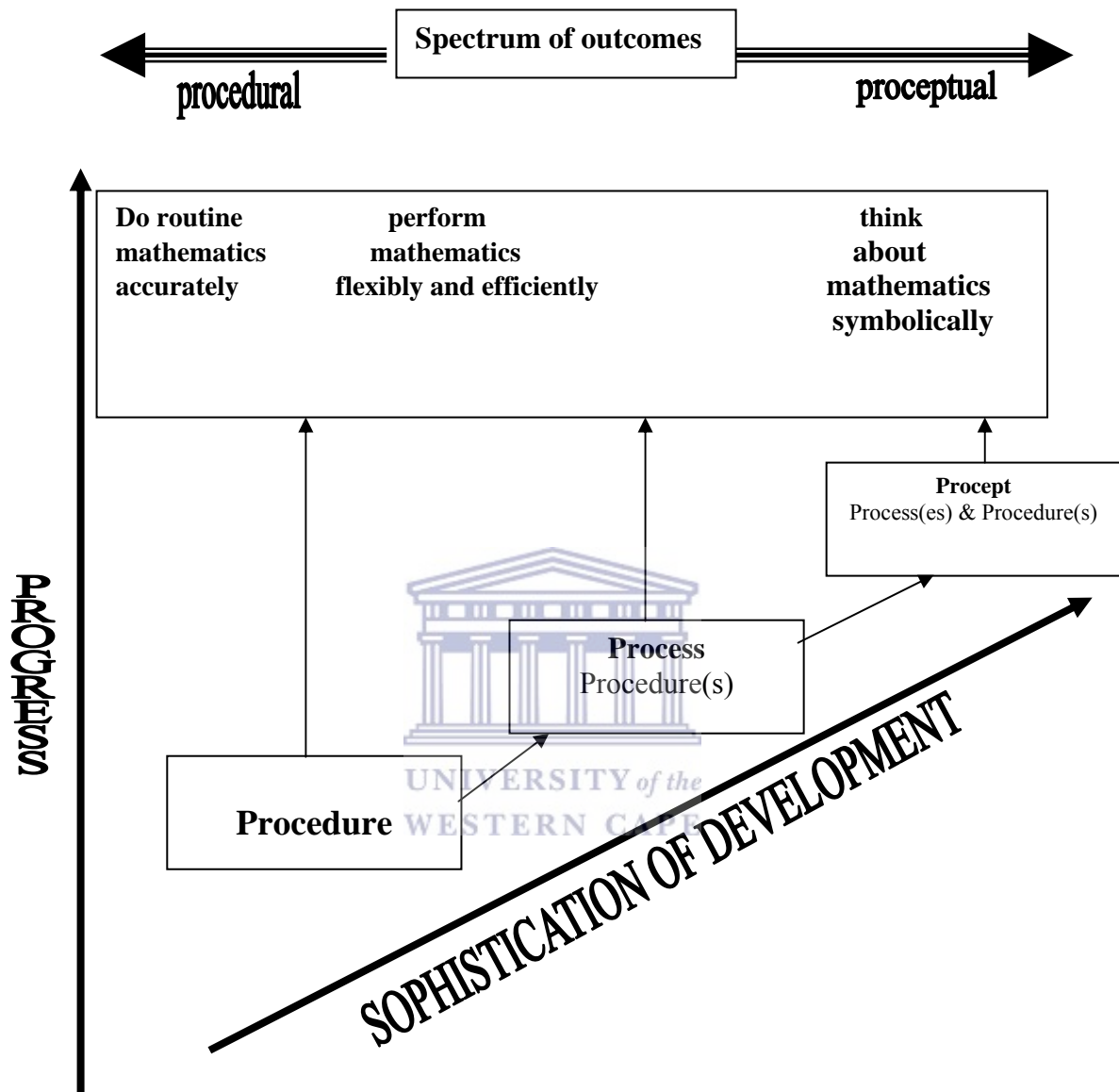


Figure 3.2: A spectrum of performance using mathematical procedures, processes and precepts (Adopted from Tall, 2001)

Tall (1999) illustrated diagrammatically (Figure 3.2) the broad spectrum of performance between individuals at different levels of compression through the procedure, process, and precept layers. Figure 3.2 shows that:

- Procedural or action conception allows an individual to do routine problems

- Process conception allows an individual to perform mathematics flexibly
- Procept conception allows an individual to think about mathematics symbolically

Tall attribute the low achievement to the fact that procedures occur in time and they take up mental space, whereas procepts can be conceived and manipulated as mental objects and are easy to manipulate for those who can be flexible thinkers.

3.8 Representations of the concept of a function

Teaching involves representing mathematical concepts to learners in a manner that makes it comprehensible to students (Shulman, 1986). An emerging theoretical view on mathematical learning that has been growing in significance is that multiple representations of concepts can be utilized to help students develop deeper, more flexible understanding of mathematical concepts (Skemp, 1987). An official document of the NCTM (2000), the Standards, allude to the fact that one of the major goals of algebra is that students should understand the relationships among tables, graphs, and symbol and to judge the advantages and disadvantages of each way of representing relationships for particular purposes. Because of the complex nature and manifold uses of functions, functional situations lead to a variety of representations including:

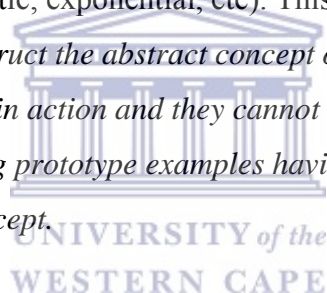
- verbal representations of a function in formal or function machine representation (everyday) language
- a set diagram (representing a function by two sets and arrows between them)
- a function box (representing an input output relationship)
- a set of ordered pairs (considered set theoretically)
- a table of values (often computed using a formula or a computer procedure)
- a graph drawn by a computer or by hand
- a formula or an equation

Each of these representations has its own peculiarities that contribute to the complication of the student's concept image of the concept of a function (Tall, 1999). An understanding of a function in one representation will not necessarily correspond to an understanding in another representation but ability to translate among varied formats is necessary to effectively interpret problem situations. When combined, the information gleaned from diverse representations contributes to a deeper, more comprehensive understanding of the

underlining functional situations (Even, 1990). According to Kaput's (1989) theory of linked representation systems the common representations of functions form the basis of a concept image. Because an individual can develop multiple concept images, which can exist in both complementary and contradictory ways, the more tightly connected the representations, the more robust and compatible the system of concept images will be. An integrated concept image that highlights the associations among representations and facilitates flexible moves among them is particularly beneficial in that one has control over the representations one wants to use (Dreyfus, 1992). Because each representational format has varying limitations or strengths in different contexts, it is beneficial for the teacher to have the choice of which representations to employ and the knowledge needed to make such a choice.

Tall's (1999) analysis also suggests that different representations of a function are presented and interpreted in subtly different ways. For instance, set diagrams are often introduced as prototypes to represent general ideas, whilst graphs and formulae are met successively in clusters (linear functions, quadratic, exponential, etc). This creates a dilemma for the teacher:

The learner cannot construct the abstract concept of a function without experiencing examples of the function in action and they cannot study examples of functions in action without developing prototype examples having built –in limitations that do not apply to the abstract concept.



(Baker & Tall, 1992: 13)

Thompson (1994; 39) questioned the meanings given to representations seemingly shared by the mathematics education community by saying:

...the idea of multiple representations, as currently construed, has not been carefully thought out, and the primary construct needing explication is the very idea of a representation.... The core concept of a function is not represented by any of what are commonly called multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance...it maybe wrongheaded to focus on graphs, expressions, or tables as representations of a function. We should instead focus on them as representations of something that, from students' perspective, is representable, such as aspects of a specific situation

Thompson (1994) claimed that if students do not realize that something remains the same as they move among different representations, then they begin to see each representation as a topic to be learned in isolation.

3.9 The concept of a function and its associated cognitive obstacles

The focus of this section is to discuss some of the cognitive obstacles and learning difficulties which have been found to be experienced by learners when learning the concept of a function. Thompson (1994) distinguishes cognitive obstacles from instructional obstacles as follows. A cognitive obstacle is knowledge which functions well in a certain domain of activity and therefore becomes well established, but then fails to work satisfactorily in another context where it malfunctions and leads to contradictions. An instructional obstacle is instruction that promotes new cognitive obstacles or support or is neutral in regard to students' existing cognitive obstacles. Students' learning difficulties with the notion of a function are widely reported and well known (Vinner, 1989; Even, 1990 and Dreyfus, 1990). Students have been found to have problems in making links between different representations of a function, in manipulating symbols related to functions, and reifying the concept. While some of the learning difficulties are attributed to inappropriate teaching approaches and can be avoided, some of the learning difficulties are due to the inherent complexities of the concept itself and therefore cannot be avoided.

3.9.1 Inherent complexities of reifying the concept of a function

It has been stressed that the operational and the structural conceptions of a function are complementary, namely that both of them are necessary and both must be used in the process of learning and in problem solving. However learners have been found to experience problems in reifying the concept of a function. Sfard (1992) highlighted two major reasons why reification of mathematical processes is difficult, namely; the problem of semantic concessions that have to be made to enable reification and the reification-encapsulation vicious circle.

3.9.1.1 Problem of semantic concessions

First, there is the problem of semantic concessions that usually must be made to enable reification. More often than not, a new abstract object comes as a generalization of an

already well known process which, before reification was interpreted in terms properties of a well-known process. During reification some properties of the process conception of a mathematical concept are lost and, as Sfard noted, the new entity is suddenly detached from the process that produced it and it begins to draw its meaning from the fact of its being a member of a certain category of objects. As Piaget and Garcia (1989:204) put it, “when we move up in the hierarchy of mathematical notions certain initial properties of objects can no longer be accepted, or else they lead to contradictions in interpretive schema”. The problem is that these inevitable concessions are sometimes difficulty to make. What must be given up in passing from a process conception of a mathematical concept to its object conception is the very process which until then was the main source of its meaning and the properties which until then seemed to be the most essential properties of the concept in question. In the case of the concept of a function it is the algorithmic feature (plugging in values in a given formula to produce unique outputs) which, although it is firmly established when a function is perceived as a process, has to be given up.

3.9.1.2 Reification-interiorization vicious circle

The second source of the problem of reification highlighted by Sfard is an inherent vicious circle whereby the lower level reification of a concept and the higher-level interiorization, in which the concept in question is being operated upon, are prerequisite for each other. Dubinsky's (1991) theory of a concept formation shed light on these difficulties. According to Dubinsky's model reification or encapsulation of a given process occurs simultaneously with the interiorization of higher-level processes. In order to perceive a function as an object, one must try to manipulate it as a whole: there is no reason to turn a process into an object unless we have some higher-level process. But there is a vicious circle: on one hand, without an attempt at the higher level interiorization, the reification will not occur; on the other hand existence of objects on which the higher level - processes are performed seems indispensable for the interiorization – without such objects such processes must appear quite meaningless. In other words: lower level reification and higher-level interiorization are prerequisite for each other (Sfard, 1991). In the case of the concept of a function reification of the concept occurs when some processes are performed on these functions (e.g., determining the limit of a function), which is the higher-level interiorization of the concept of a limit. It follows therefore, that at the crucial junction in the development of the concept of a function, learners may become entangled in the vicious circle. To get out of the

entanglement may not be easy for the learners. No doubt Cornu (1991) attributed learners' problems in understanding the notion of a limit to learners' inability to reify the concept of a function.

However, Sfard (1991) presented the structural thinking as a very powerful weapon against the limitations of the working memory. According to Kaput (1992) one of the psychological justifications for forming conceptual entities lies in their role in consolidating or chunking knowledge to compensate for the mind's limited processing capacity; especially with respect to working memory. To avoid loss of information in the working memory large units of information can be chunked into single units or conceptual entities (objects). Thus thinking of functions as processes would require more working memory space than if they were encoded as objects. As a result functions would be more difficult to retrieve, process or store if functions are viewed as processes.

3.9.2 Compartmentalization of the definitions and concept images of a function.

Vinner (1992) describes the compartmentalization phenomena as a situation in which two items of knowledge, which are incompatible with each other, exist in one's mind without the individual being aware of it. Vinner's (1992) experimental work with students showed that the majority of the students did not use definitions when working on cognitive tasks in technical contexts. When asked about the definition of a function, the students came up with the Bourbaki formulation, but when working on identification construction tasks their behaviours were based on the formula conception. Since students in Vinner's study had compartmentalized the definitions and the concept images of a function they exhibited inconsistent or non-coherent behaviours. The set of mathematical objects considered by the students to be examples of the concept of a function were not necessarily the same as the set of mathematical objects determined by the definition. For example, one of the students defined a function as a correspondence between the elements of two sets while at the same time claiming that a discontinuous graph did not represent a function. Vinner attributed compartmentalization of the definitions and concept images of a function to the use of the formalist approach to the teaching of mathematics in which learners are expected to construct the concept of a function from the definition. Commenting on the formalist approach to the teaching of functions Cuoco (1994:255) stated:

experience seems to show that a function as a class of ordered pairs approach is one which imposes severe limitations upon the student and provides a poor preparation for any further work with functions, either in school or later.

Vinner (1989) and Dreyfus (1990) concurred that definitions created problems in mathematics learning and that they are a clear testimony of the conflict between the structures of mathematics as conceived by mathematicians and the cognitive processes of concept acquisition. The idea of giving a verbal definition as a list of criteria and then expecting learners to construct the concept from the definition is a reversal of most of the developments in elementary mathematics where mathematical concepts are thought to have properties which can be discovered by studying other related concepts.

3.9.3 Lack of flexibility in switching representations

A major source of learners' difficulties with functions is their lack of flexibility in switching representations or working on the relationships between them. Schwartz and Dreyfus (1992) attributed the lack of flexibility to the treatment of graphs, tables and formulae representations as separate static entities, as mathematical objects in their own right, instead of as distinct representations of a single object namely a function. Dreyfus and Eisenberg (1983) uncovered students' difficulties with several transformations such as stretches and shifts as well as with change of variables. Students tended to view algebraic data and graphical data as being independent.

In Graham's (1991) study, first semester calculus students were not able to provide any type of general definition of a function but readily gave examples of functions by writing formulae. There was little evidence that students saw functions as objects of study in mathematics, rather functions were given in equation form, usually where one was expected to do something to it such as substituting a value. This part of studying functions (plugging in values) seemed to be firmly established.

3.9.4 Notational complexities

Although there are many facets to mathematical anxiety, notational complexities are often obstacles in preventing understanding of the concept of a function (Eisenberg 1991). This problem is not in the mathematics but in the representation of mathematics. Sierpiska

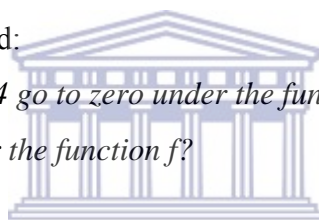
(1992) remarked that the symbols used in connection with functions are not helpful. Eisenberg (1991) chronicled only a few of the pitfalls associated with initial notions of functions. For example, he noted that the $f(x)$ notation itself is confusing because $f(x)$ stands both for the name of a function and for the value of the function for a given input value.

Eisenberg (1991) speculated that students would have less trouble with understanding composite functions if they were presented in the form $x \xrightarrow{g} g(x)$, $g(x) \xrightarrow{f} f(g(x))$. He argues that this notation would help students understand the meaning of the argument. For example, seventy percent of beginning calculus students at Ben-Gurion University could not solve the following problem:

If 2 and 4 are the values of x for which $f(x) = 0$. What are the values of x for which $f(4x) = 0$.

This same problem was rephrased:

Only the values of 2 and 4 go to zero under the function f . what values multiplied by four will go to zero under the function f ?



Seventy percent of the students could answer the problem in this form and they seemed to have a basic understanding of what they were doing. This is a common phenomenon even in simple word problems: that the notation in the question greatly affects the students' ability to answer it. Students often do not realize that functions transform every point in the domain to a new position. While this is understood, problems such as finding the zeros of $f(kx)$ given the zeros of $f(x)$ are not fully understood.

Eisenberg (1991) also found out that defining a function in terms of an integral such as

$\ln x = \int_{t=1}^x \frac{1}{t} dt$ was beyond most students in elementary courses - a common difficulty in the

first stages of advanced mathematics where the idea of a definition, rather than a description is new. Likewise visualizing functions in parametric form also proved to be intolerably difficult, especially when the representation moves from two dimensions to three.

3.9.5 Cognitive obstacles associated with forms of representation

Since the concept of a function can be represented using different representations in a variety of contexts, depending on the context, various cognitive obstacles surface from the onset. Each representation of the concept of a function, as with any other representations of a mathematical concept gives rise to a range of cognitive obstacles requiring cognitive reconstruction in subsequent learning episodes (Tall, 2001). What is important is that teachers should be aware of the cognitive obstacles associated with the use of each representation and the associated reconstructions required thereafter. In order to overcome a cognitive obstacle it therefore becomes necessary to destroy the original insufficient, malformed knowledge and replace it with new conceptions, which operate satisfactorily in the new domain. Sierpiska (1992) consider the process of rejection and overcoming an obstacle is an essential part of knowledge construction. The transformation cannot be performed without destabilizing the original idea by placing it in a new context where it is clearly seen to fail. This therefore requires a great effort of cognitive reconstruction on the part of the learner.



3.9.5.1 Cognitive obstacles associated with the function machine

The function machine, as with any other initial starting point, gives rise to a range of cognitive obstacles requiring cognitive reconstruction in later developments.

The function box is often used in the early stages of the high school curriculum in Zimbabwe. However this is usually as a ‘guess my rule’ problem, to guess the internal formula expressing the rule. This activity, besides giving rise to the epistemological obstacle that all functions are given by a formula (Tall, 2001), learners get the impression that every relationship should be expressed by an explicit rule and yet the association between elements in the domain and range could be arbitrary (Kaput, 1992).

According to Tall et al (2001) a major weakness of the function box as a representation of the concept of a function is that it does not have an explicit range and domain. The domain can be introduced in a natural way as a set of possible inputs and in context such as the real function there is a natural range namely the real numbers. This may embody a belief that a function will always have a natural domain and range rather than the domain and range being specifiable in the definition.

3.9.5.2 Cognitive obstacles associated with the algebraic representations

Schwartz and Dreyfus (1992) noted that algebraic representations of a function were ambiguous for two possible reasons. First, there is no unique algebraic formula which represents a given formula. For example, the formula $y = 4x - 12$ and $y = 4(x - 3)$ represent the same linear function. Similarly, the expression $y = |x|$ and $y = \max(x, x)$ represent the same function. Second, the ambiguity arises due to failure to specify the domain of a function in the algebraic representation. For example whether or not $y = x + 3$ and

$y = \frac{x^2 + x - 6}{x - 2}$ represent the same function depends on how the domain of the function has been defined.

Eisenberg (1991) remarked that when functions are identified with the algebraic representation only, learners end up perceiving functions as rules with regularities. Also a change in the independent variable is seen as causing a change in the dependent variable with the consequence that constant functions are often not considered as functions. Consequently, students in Barnes' (1988) study claimed that $y = 4$ was not a function since y did not depend on x . However according to the arbitrary property of the concept of a function the correspondence between the elements in the domain and the range need not be determined by a rule. The correspondence is arbitrary.

3.9.5.3 Cognitive obstacles associated with the graphical representations

Schwartz and Dreyfus (1992) noted that the problem with the graphical representation of a function is that the information about a function given in graphical form is always partial because of the choice of a viewing window. In the graphical setting a representation of a function is obtained by choosing a viewing window characterized by the bounds of the x -values and the y -values Figure 3.3 below shows two representations of the function

$$f(x) = x^2(10 - 2x).$$

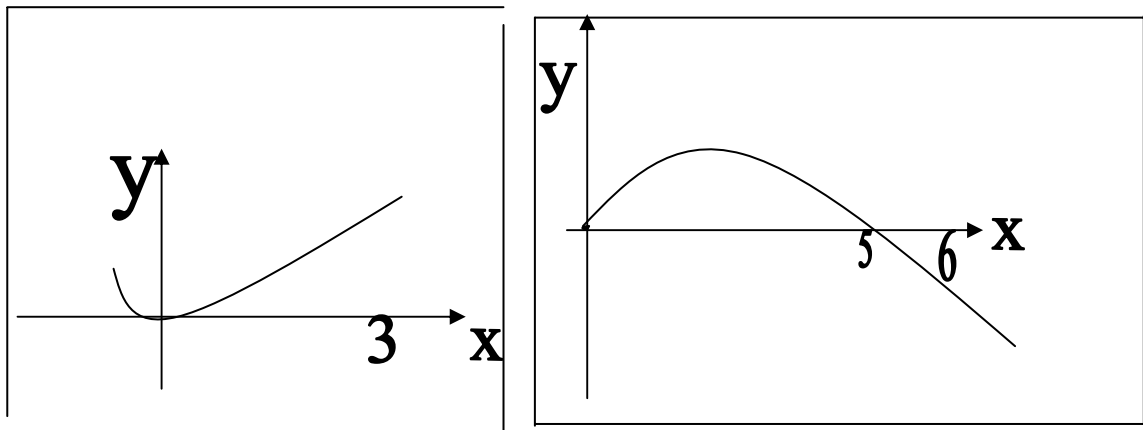


Figure 3.3 Graphs of $f(x) = x^2(10 - 2x)$ in the ranges $-1 \leq x \leq 3$ and $0 \leq x \leq 6$ respectively

Each of the graphical representations above is a partial representation of the function $f(x) = x^2(10 - 2x)$. It is important to note that new representations of the function can be obtained simply by changing the units of the axis. Eisenberg (1991) pointed out that when a mapping is represented graphically the vertical line test is used almost exclusively in determining whether a given example is a function or not thereby giving the learners the impression that all functions can be represented graphically.

The other learners' difficulty relates to the representation of information by a graph. Monk (1992) demonstrated that learners have an inclination to be over-literal in interpreting the visual information on a graph. Learners tend to expect a too-close resemblance between the shape of the graph or its other prominent visual aspects and the real situation that the graph refers to. Kaput (1989) reports a study in which students are shown the speed-time graph for a car going round a racetrack. They were asked to describe the probable shape of the racetrack or to choose from alternatives provided. The overwhelming response was to draw a diagram of a racetrack that strongly resembled the speed-time graph. In another study Monk (1992) reported that when students were shown a side view of an individual cycling up and over a hill and were asked to draw a graph of the speed vs position along the path. Fifty percent of the students drew graphs which incorporated the visual features of the contour of the hill. Monk refers to the students' impulse to act as if the graph were much more literally a picture as the problem of iconic translation.

3.9.5.4 Cognitive obstacles associated with the tabular representations

The tabular representations of a function are simply all the possible tables obtained by choosing a set of x-values with corresponding y-values. As is the case with the graphical representation of a function, information about a function in the tabular representation is always partial since only parts of the domain and range are given (Schwartz and Dreyfus, 1992). The Zimbabwean ‘O’ and ‘A’ Level Mathematics syllabuses refer to graphs and table of values as representations of a function in spite of them being partial representations of functions. However the assumptions that graphical and tabular representations give a rounded picture of a function is not valid, hence, students and teachers have to learn how to deal with the problem of partiality. Students and teachers who have not learned how to deal with the problem of partiality would not be able to recognize that discrete numerical and/or graphical information about the domain and range belong to a continuum of data points.

3.10 Conclusions

The literature review shows that there are similarities between the historical development of the concept and the psychological explanation of how the concept develops in learners. Historically the concept was conceived as a process before it was conceived as an object. This historical development was initially necessitated by the need to solve practical problems but latter it was a result of the developments in the mathematics. A theoretic analysis of Dubinsky’s model of concept development, as well as a closer look at the history of the concept of a function show that a three –step pattern can be identified in the successive transition from operational to structural conceptions. First there must be a process performed on the already familiar objects, then the idea of turning this process into a more compact, self contained whole should emerge, and finally an ability to view this new entity as a permanent object in its own right must be acquired. These three components of concept development are referred to as interiorization, condensation and reification.

The literature review also shows that functions can be represented symbolically, graphically and numerically. To develop a deep understanding of the concept of a function learners must be able to switch from one representational form of a function to another. Each representation of the concept of a function, as with any other representations of a mathematical concept gives rise to a range of cognitive obstacles requiring cognitive reconstruction in subsequent learning episodes

CHAPTER 4

THEORETICAL FRAMEWORK OF THE STUDY

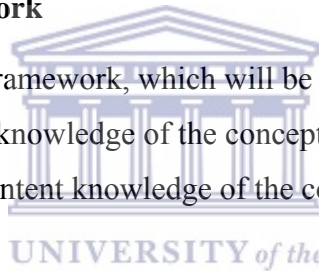
4.1 Introduction

One of the major purpose of doing a literature review and defining a provisional theoretical framework of teacher knowledge of mathematics was to prepare the ground for developing a theoretical framework for understanding what constitutes teachers' knowledge of the concept of a function. The developed framework was also used as an analytical tool for analyzing prospective teachers knowledge of the concept.

4.2 Components of the framework

The central components of the framework, which will be elaborated in this chapter, are:

1. Teachers' subject matter knowledge of the concept of a function.
2. Teachers' pedagogical content knowledge of the concept of a function.



Pedagogical content knowledge (PCK) is further divided into substantive and syntactic components. The schematic summary of the framework shown in figure 4.1 below show that knowledge of conceptions, definitions, examples, images, and representations of a function as components of teacher knowledge of the concept of a function are classified either as sub-components of subject matter knowledge of or as sub-components of the substantive component of PCK. When a component of teacher knowledge is classified under subject matter knowledge, the focus of interest in knowing how that knowledge is organized in the teachers' mind. On the other hand, when the component is classified under PCK, the focus will be on how that knowledge is used in the context of teaching. In short, there are two ways of looking at teachers' knowledge: how it is organized in the teachers mind and how it is used in the practice of teaching.

4.2.1 Subject matter knowledge

Ball and Bass (2000) suggested that mathematical knowledge for teaching is different from the mathematical knowledge held by other specialists, just as mathematical knowledge used for engineering is different from mathematical knowledge needed for teaching. Similarly, teachers' knowledge of the concept of a function should be different from the physicist's or the chemist's knowledge of the concept. The differences between teacher knowledge of the concept and other specialists' knowledge of the concept should be evident in their conceptions of a function, knowledge of definitions and examples of a functions, concept images evoked at the mention of the word function and familiarity of the different representations of a function.

4.2.1.1 Conceptions of a function

Whereas a physicist or a chemist might be limited in their conceptions of a function the mathematics teacher should be familiar with the different conceptions of a function. Tall (1999) considers the alternative ways of approaching the concept – the procedure, process, object and procept conceptions as increasing levels of understanding of the concept. De Marois and Tall (1996) also describe the procedure, pre-procedure, process, object and procept conceptions of a function as reflecting increasing levels of understanding the concept. Ideally a mathematician or a mathematics teacher should have a fully developed procept conception of a function in order to deal with a function both as a process and as an object and be flexible enough to move between the two conceptions depending on the problem at hand.

However, according to Ball and Bass (2000), what distinguishes a mathematician's mathematical knowledge from that of the mathematics teacher is that while the mathematician's knowledge can remain highly compressed, the mathematics teacher is expected to decompress one's mathematical knowledge into less polished form such that the elements of the any mathematical domain in question remain accessible and visible. In the case of the concept of a function the mathematics teacher should not just rely on the procept conception of a function whereby the concept would be in its most compressed form. The mathematics teacher should be able to decompress the concept so that not only the range and the domain of the function become evident, but be in a position to describe, where possible, how the assignment from the domain to the range is done.

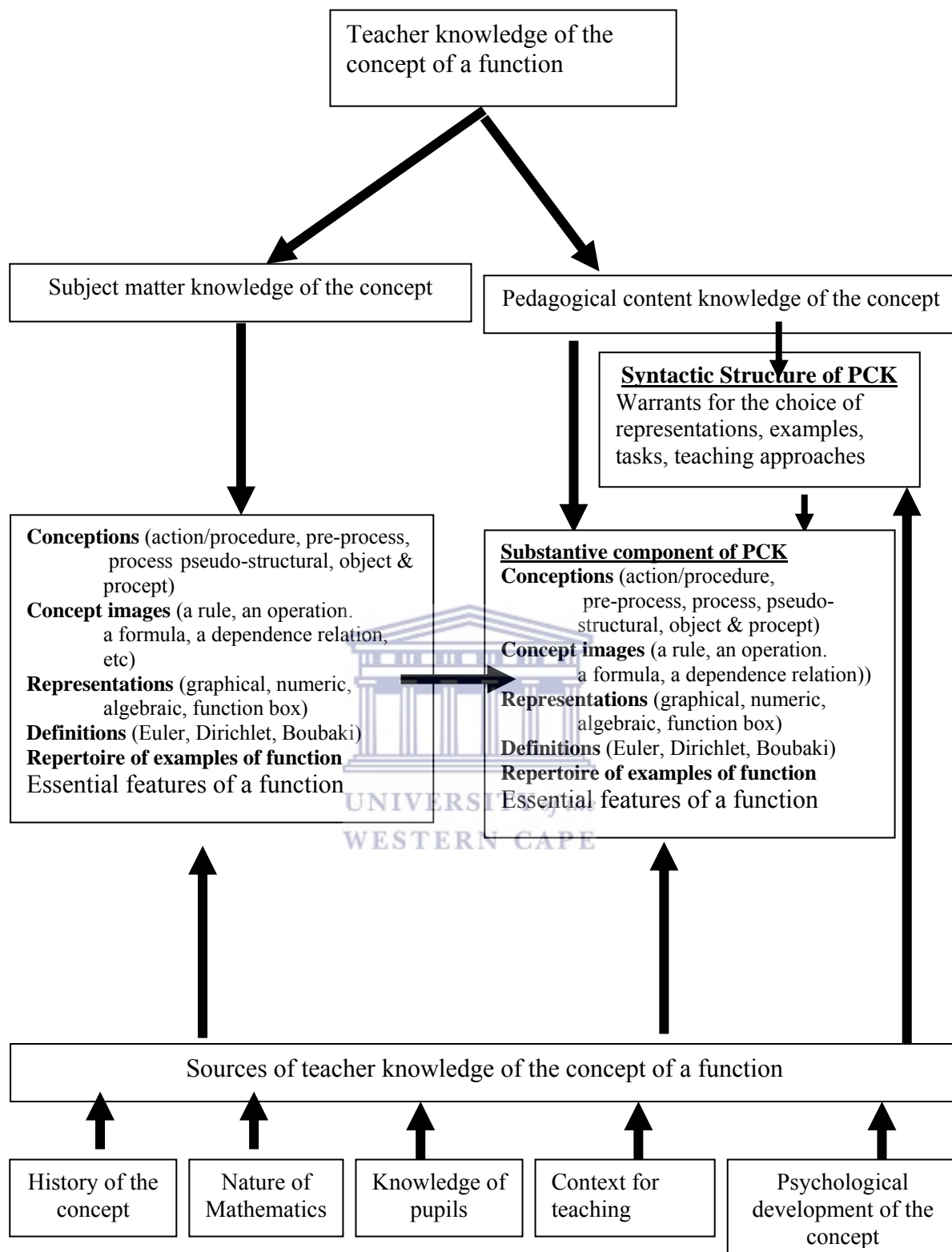
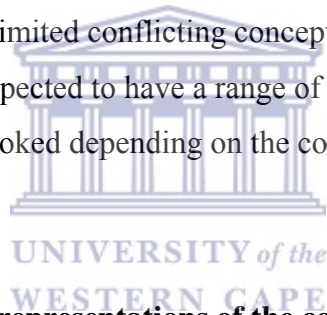


Figure 4.1 Schematic summary of the analytical framework teacher knowledge of the concept of a function

4.2.1.2 Concept images of a function

An individual's conception of a concept can be inferred from the concept image evoked in the individual's mind by the concept name. Vinner (1989) introduced the term concept image to describe the total cognitive structure that is associated with the concept, which include all the mental pictures and associated properties and processes. Vinner (1989) suggested that concept images associated with the concept of a function are not formed by definitions but by experiences. This explains the diversity of concept images associated with the concept of a function: a correspondence between variable, a rule of correspondence, a manipulation or an operation, a formula, an equation and a graph. Additional diversity and refinement of these images is suggested when given graphs of functions are all continuous resulting in people perceiving continuity as an inherent characteristic of a function.

Tall and Vinner (1981) observed that at different times, seemingly conflicting concept images may be evoked in an individual. Whereas other specialists such as physicists and chemists might have seemingly limited conflicting concept images of a function, the high school mathematics teacher is expected to have a range of concept images of a function each of which can be appropriately evoked depending on the context in which a function is presented or discussed.



4.2.1.3 Knowledge of different representations of the concept

Work with functions, as with any other mathematical topic, is conducted via different representations such as tables, algebraic expressions and equations. Familiarity with different representations and the ability to translate and form linkages among them create insights that allow a better, deeper, more powerful and more complete teacher understanding of the concept (Kyvatinsky and Even, 2004). Thompson (1994) remarked that the core concept of a function is not represented by anyone of what are commonly called multiple representations of functions, but instead our making connections among representational activities produces a subjective sense of invariance.

Mathematics teachers must be able to choose appropriate representations depending on the context and need. In doing so, teachers need to know that information about a function given in graphical form is always partial because of the choice of the viewing window. Likewise

the tabular representation is always partial since only parts of the domain and range are given.

4.2.1.4 Knowledge of definitions of a function.

According to Ball (2000) teacher knowledge of mathematics includes knowledge of mathematical definitions. In the case of the concept of a function, familiarity with the different definitions of a function constitutes teachers' knowledge of the concept. Besides, general knowledge of the origin and historical development of the concept of a function is essential if teachers are to appreciate that the definition of a function has been changing in response to changes in mathematics as a subject. Malik (1980) pointed out that while Bernouli's definition of a function as an expression or formula representing a relation between variables is for the study a Calculus and Dirichlet's definition of a function as a correspondence between real numbers is for the study of Analysis, Bourbaki's set-theoretic definition with domain and range is required for the study of Topology.

4.2.1.5 Repertoire of examples of functions.

Part of teacher knowing and understanding the concept of a function is to know and have easy access to specific examples, which constitute the basic repertoire. A basic repertoire includes powerful examples that illustrate important ideas and properties such as the univalence and the arbitrariness properties. Some of the examples are simple and illustrate a simple aspect while others are complicated and present several terms. For example while the linear functions involves the idea of constants which might be confused with the variable quantities, the piece-wise functions shows that the rule of correspondence can vary for different ranges of the domain.

Mathematics teachers' basic repertoire of examples of a function should include important examples of functions at all levels of the mathematics curriculum. Teachers should think of the basic operations as functions in which, for example, the multiplication and the addition tables are representations of two-real valued functions which can be represented algebraically as $f(x, y) = xy$ and $f(x, y) = x + y$ respectively. Beyond calculus, teachers should not have trouble in recognizing functions when they appear as vector fields in vector calculus and as differential operators in which both the input and the output of the function are functions.

4.2.1.6 Essential features of a function.

Part of teachers' knowledge of the concept is knowing the essential features of a function. Freudenthal (1983) considered arbitrariness and univalence properties of a function as key characteristics of the concept. The arbitrary nature of a function refers to both the relationship between the two sets on which the function is defined and the sets themselves. The arbitrary nature of the relationship means that functions do not have to exhibit some regularity that can be described by any specific expression or particular shaped graph (Even, 1992). The arbitrary nature of the two sets means that functions do not have to be defined on any specific sets of objects, in particular, the sets do not have to be sets of numbers.

Whereas the arbitrary nature of functions is implicit in the definition of a function, the univalence requirement, that for each element in the domain there be only one element in the range, is explicitly stated in the modern definition of a function.

4.2.2 Pedagogical content knowledge

The schematic summary of the framework shows that teachers' pedagogical content knowledge for teaching the concept of a function have both the substance (i.e. knowledge of how the different representations, conceptions, concept images, definitions, examples, essential features of a function, cognitive obstacles are used in the teaching of the concept) and the syntax, i.e., the standards or the warrants that guide the pedagogical reasoning entailed in choosing representations, examples, definitions of a function in an endeavour to foster desired conceptions and concept images of a function and at the same time overcoming associated cognitive obstacles.

4.2.2.1 Knowing definitions of a function for teaching

According to Ball (2000) the most important criterion for a good definition is whether or not a definition is usable or operable by pupils at a particular level. Definitions of a function must be based on elements that are themselves already defined and understood, hence, knowing mathematical definitions for teaching requires more than learning mathematically acceptable definitions. What is needed is being able to understand and work with definitions in the classroom, with pupils, treating them in a way that respects the role definitions play in doing and knowing mathematics. Knowing how definitions function and what they are

supposed to do, together with also knowing a well acceptable definition in a discipline, would equip a teacher for the task of developing a definition that has mathematical integrity while being acceptable to pupils. A definition of a mathematical object is useless, no matter how mathematically refined or elegant, if it includes terms that are beyond the prospective user's knowledge (Ball, 2000). Mathematics teachers should appreciate that the definition of a function as an expression or formula representing a relation between variables is for calculus or pre-calculus, is a rule of correspondence between real numbers was developed for the purpose of studying analysis, and a set theoretic definition with domain and range is required to study topology (Malik,1980).

4.2.2.2 Knowing representations of functions for teaching

Each representation of the concept of a function, as with any other representations of a mathematical concept gives rise to a range of cognitive obstacles requiring cognitive reconstruction in subsequent learning episodes (Tall, 2001). What is important is that teachers should be aware of the cognitive obstacles associated with the use of each representation and the associated reconstructions required thereafter.

4.2.2.3 Using learners' concept images of a function in teaching

Because an individual can develop multiple concept images of a function, which can exist in both complementary and contradictory ways, teachers need to know the different concept images of a function which can be evoked in the learners' mind at different situations. Also a learner's concept image may be at variance with one's definition of a function thereby creating a cognitive conflict (Vinner and Dreyfus, 1989). Tall (1999) remarked that it is not sensible to expect students to be able to argue logically from concept definition without expecting interference from their individual concept images.

The knowledge of these particular instances where conflicting concept images are evoked in the learner's mind or where the concept image of a function is at variance with the learner's definition of a function may make the teacher more sensitive to learners' reactions. The sheer variety of individual's concept images of a function suggest that the choice of examples for use by learners should be made in such a manner that cognitive conflicts on the part of the learner are reduced and at the same time the learner is helped to reconstruct his or her concept image of a function.

4.2.2.4 Using learners' conception of a function to recast the teaching of functions

Studies (Schifter and Fosnot, 1993) suggest that effective teachers need to attend to students' ways of thinking about mathematical tasks or concepts. If teachers are to move learners from the action conceptualization, a view of a function as a repeatable mental or physical manipulation of objects; to that of a process conceptualization, the interiorization of actions so that the total action can take place entirely in the mind of the learner; and finally to the object conceptualization, an encapsulation of the process in its totality, then teachers need to identify the learners current conceptions of a function and then use the learners current conception of a function to recast the teaching of functions.

4.2.2.5 Knowing examples of functions for teaching

Baker and Tall (1992) pointed out that learners cannot construct the abstract concept of a function without experiencing examples of the function in action and at the same time they cannot study examples of a concept of a function in action without developing prototype examples having built-in limitations that do not apply to the abstract concept. Because each example of a function has varying limitations or strengths in different contexts, it is beneficial for the teacher to choose appropriate examples for use with learners and the knowledge needed to make such a choice. Teachers need to know thoroughly different examples of a function that are in the high school mathematics curriculum and have at one's disposal a set of examples much wider than those in the high school mathematics curriculum.

4.2.2.6 Knowledge of cognitive obstacles associated with the learning of functions

Mathematics teachers need be knowledgeable with cognitive obstacles faced by learners when learning the concept of a function. While some of the cognitive obstacles are attributed to inappropriate teaching approaches and can be avoided, some of the cognitive obstacles are due to the inherent complexities of the concept itself and therefore cannot be avoided (Sierpinska, 1992). Depending on the examples or representational forms used certain cognitive obstacles surface from the onset. However unavoidable cognitive obstacles should be seen in the positive sense since they are opportunities for teachers to create what Ausubel (1968) referred to as cognitive dissonance in the mind of the learners in subsequent learning.

Acts of overcoming an obstacles or resolving a cognitive dissonances are considered as critical moments since they result in deeper understanding of a function.

What is important for the teacher is know the cognitive obstacles associated with the use of each representation or example of a function and the associated reconstructions required on the part of the learner in subsequent learning. For example, introducing the concept of a function as a function machine, not only gives rise to the cognitive obstacle that all functions are given by a formula (Tall, 2001), but learners get the impression that every relationship should be expressed by an explicit rule and yet the association between elements in the domain and range could be arbitrary (Kaput, 1992).

4.2.3 Syntactic component of PCK

Ball (2000) argued that an extensive repertoire of pedagogical representations, as conceived by Shulman (1986), was not sufficient for teaching since there was no repertoire of representations that could possibly suit all the possible teaching contexts. Substantive pedagogical content knowledge does not primarily serve to organize the teaching of the concept of a function; hence, a new type of professional knowledge for mathematic teachers, the syntactic component of PCK shown in figure 4.1 is needed. The syntactic component of PCK is arising from the realization that mathematics teaching is complex and multidimensional and that teachers need to be active decision makers who determine their own priorities rather than implementing standard directions, plans and routines (Sullivan and Mousley, 2000).

Ball's (2000) notion of syntactic component of PCK imply that teachers do not have to rely on a given repertoire of pedagogical representations, examples and definitions of a function as they would not be sufficient to enable a teacher to teach the concept of a function effectively since the teaching contexts vary. Teachers need to rely on some considerations or warrants that would be the basis for appraising, evaluating and modifying representations, examples and definitions of a function as well as guiding their generation. The warrants for judging the worth of products and processes of pedagogical reasoning include mathematical warrants, warrants based on learning theories, knowledge of learners and the context (Ball, 2000).

4.3 Sources of teacher knowledge for teaching the concept of a function

The schematic summary of the framework in figure 4.1 shows that the syntactic component of PCK, just like the teacher's subject matter knowledge of the concept of a function and substantive component of PCK is informed by the teacher's knowledge of the nature of mathematics, the historical development of the concept, the contexts for teaching the concept, learners background characteristics and the psychological development of the concept. Mathematics teachers are expected to reconstruct justifications for instructional sequences for teaching the concept of a function by considering the implications of the above-mentioned domains of knowledge.

4.3.1 Knowledge of the psychological development of the concept

Knowledge of psychological theories on the development of the concept of a function is essential in planning learning activities for the pupils as these theories try to explain how the concept develops in the learners' mind. The theories seem to emphasize that new learning is best built on prior learning. For example, Dubinsky's (1991) APOS theory suggest that a mathematical concept/object is formed when an individual performs actions on already existing objects that are then interiorized into processes latter to become reified into objects to be built into a wider cognitive schema. Similarly in Sfard's model, concepts, such as the concept of a function, are assimilated into the schema in the last stage of three-step abstraction process:

A constant three step pattern can be identified in the successive transition from operational to structural conceptions: first there must a process performed on already familiar objects, then the idea of turning this process into a more compact, self contained whole should emerge, and finally an ability to view this new entity as a permanent object in its own right must be acquired. These three steps will be called interiorization, condensation and reification.

(Sfard, 1992:64-65)

These theories are suggesting the concept of a function can be introduced to the learners by asking them to respond to situations in which a function may appear (via a formula or as an algorithm) and for which the value of the function for a particular value in the domain is obtained, e.g., determining the value of y in the equation $y = 2x + 2$ for a given value of x . Given such a situation, the learner may respond by constructing in his/her mind a mental

process relating to the function process. This is a prime example of interiorization. Hence the action becomes a process when the individual can describe or reflect upon all the steps in the transformation without necessarily performing them. To ensure that learners are then capable of conceiving the notion of a function as a fully – fledged object perhaps teachers need to design activities in which learners perform some operations on functions such as combining a set of functions.

4.3.2 Knowledge of the historical development of the concept

Although high school learners might not be required to learn the historical development of the concept of a function teachers need to know the historical development of the concept since that knowledge is likely to help teachers when formulating definitions of a function for use by learners at different levels of the mathematics curriculum. The historical development of the concept of a function shows that the definition of function as an expression or formula representing a relation between variables is adequate for calculus or a pre-calculus course; is a rule of correspondence between elements is for use in Analysis and a set theoretic definition with domain and range is required in studying of topology (Malik, 1980). Since the modern abstract set-theoretic definition of a function is the most widely quoted definition in school mathematics text books, teachers who do not know the historical development of the function are likely to motivate it even in cases where the classical definition would have been the appropriate definition for use by learners.

4.3.3 Knowledge of the nature of mathematics

Knowledge about the nature of mathematics is interrelated with knowledge of the concept of a function. Knowledge about the nature of mathematics is a more general knowledge about the discipline, which guides the construction and use of different types of knowledge. It includes ways, means and processes for the establishment and creation of truths as well as the relative importance of different ideas (Kvatinsky and Even, 2004). Knowing the relative importance of the concept of a function would go a long way in helping teachers in designing mathematical learning sequences in which the function is a foundational concept and at the same time an organizing principle.

The nature of mathematics also includes it being a creation of the human mind, which is influenced by forces inside and outside mathematics, and the characteristics of the constant

change of mathematics. Knowledge of how the conceptions of a function have been changing due to the developments within and outside mathematics can be used as cues when designing learning activities for the learners. Teachers also need to know how the general knowledge supports the knowledge of the concept of a function. For example, work with mathematical concepts, which are abstract, requires the use of models, each of which is limited and presents only facets. Similarly in learning the concept of a function, different models or representations have to be used, depending on the situation.

4.3.4 Knowledge of learners' conceptions of a function.

One aspect of teachers' knowledge that has been found to influence significantly their teaching of mathematical concept such as the concept of a function is teachers' knowledge of the landscape of learners' conceptual development (Doerr and Lesh, 2002). This includes knowledge of students' social and cultural contexts, the mathematics they know and use, their preferred ways of learning, and how confident they feel about learning the mathematical concept in question. Schifter and Fosnot (1993) remarked that effective teachers needed to attend to students' ways of thinking about mathematical ideas. Teachers' understanding of students' conceptions of a function and how these conceptions might be developed further seem to be the basis for the teacher to support the students in ways (such as using appropriate representations, models and language) that will promote student learning.

4.3.5 Knowledge of the context for teaching the concept

Since the concept of a function is considered a unifying idea in algebra and other school mathematics courses, teachers need to know the different contexts in the mathematics curriculum in which the concept manifest itself. Currently school curricular changes and research studies focus almost exclusively on real-valued functions of a single continuous variable (Kawski, 2005).

Teachers need to know the different contexts in which the concept of a function can be taught and revisited thereby developing a comprehensive understanding of the concept in learners. Teachers need to understand that learners first encounter functions in arithmetic, where functions appear as the usual operations on numbers in which a pair of numbers corresponds to a single number, such as the product of two numbers. For example, they need

to consider the multiplication table as a representation of a function, which can also be represented algebraically as $f(x, y) = xy$. In geometry, teachers should appreciate the existence of functions that relate sets of points to their images under motions such as flips, slides and turns and in probability, they should perceive events and their likelihood as inputs and outputs of the probability function. Beyond calculus teachers should not have problems in recognizing functions when they appear as vector fields in vector calculus and as differential operators in the Differential Equations course. In a vector field the input is not a number, but a point on a plane or a three-dimensional space on a curve or surface but also its output is a vector. In the second case both the input and the output of the function of the differential operator are functions themselves. Teachers need to know the different contexts for teaching the concept of a function so that they can capitalize on what learners already know about the concept in a given setting and then devise learning activities for the learners in which learners would be re-visiting the concept in a different context.

4.4 Conclusions

The concept of a function is considered as a unifying idea in the mathematics curriculum. Consequently a growing number of countries have included the concept in their school curricula. Research on learners' understanding of the concept conducted in the last two decades indicates that learners experience difficulties when learning the concept. Since teachers have a crucial role in supporting student learning there is need to assess teachers' knowledge for teaching the concept. Teachers' ability to fulfill this role is connected to their subject matter knowledge and no one would argue with the claim that teachers need adequate subject matter knowledge. Although Shulman (1986) argued that teachers needed a special kind of knowledge, which he referred to as pedagogical content knowledge, what adequate knowledge for teaching the concept of a function might mean is not clear.

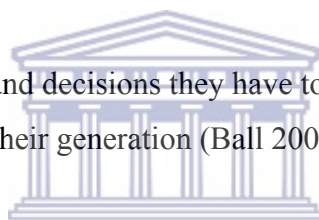
In this chapter a theoretical framework for describing teacher knowledge for teaching the concept of a function was proposed. The framework includes teachers' subject matter knowledge and pedagogical content knowledge for teaching the concept. Subject matter knowledge includes knowledge of the different conceptions and concept images associated with functions, different ways of defining and representing the concept, examples of function and the essential features of a function and how these knowledge components are organized in the teacher's mind. When the above mentioned knowledge domains are considered from

the point of view of how the teacher uses them in fostering a comprehensive understanding of the concept in learners, they constitute the teachers' substantive component of pedagogical content knowledge. The schematic summary of the framework shows that teachers' subject matter knowledge of the concept influences directly the teachers' substantive component of pedagogical content knowledge for teaching the concept.

The syntactic component of PCK is a major component of the framework. The syntactic component of PCK is arising from the realization that the teaching of any mathematical idea such as the concept of a function entails:

- Making choices as to which representation, definition or example to use with the learners
- Figuring out learners misconceptions and the possible sources of such misconceptions
- Deciding on how to help learners to overcome cognitive obstacles associated with the learning of functions.

As teachers make these choices and decisions they have to appraise, evaluate and modify representations as well as guide their generation (Ball 2000).



The schematic summary of the framework in figure 4,1 also show that the component of teacher knowledge for teaching the concept of a function are informed by the teachers' knowledge of the historical development of the concept, knowledge of the nature of mathematics, knowledge of pupils as learners of mathematics, their knowledge of the psychological development of the concept and the contexts in which the concept manifests itself. In a high school mathematics class, teachers are not necessarily teaching the historical development of the concept of a function to their learners. Neither are they expected to be teaching the nature of mathematics nor the psychological development of the concept to high school learners. However, these knowledge domains are considered as sources of teacher knowledge for teaching the concept since they are the basis for appraising, evaluating and modifying representations, definitions and teaching approaches as well as guide their generation.

The above framework of teacher knowledge of the concept of a function was used to explore the prospective teachers' knowledge of the concept of a function. To guide the investigation of teacher knowledge of the concept the following questions were posed:

- What are the prospective teachers' understandings of the definition of a function?
- What is the breadth and depth of prospective 'A' level mathematics teachers' understanding of the concept of a function?
- How do the prospective teachers' ways of knowing and their concept images of the concept of a function influence how they intend to teach the concept?
- Which warrants are being used by the prospective teachers in their hypothetical lesson plans for teaching the concept to an 'A' level class?

The next chapter will focus on how the task-based interviews, teaching scenarios and reflective interviews were used to collect data to answer the above questions.



CHAPTER 5

RESEARCH METHODOLOGY

5.1 Introduction

The purpose of the study was to investigate:

1. The Zimbabwean prospective teachers' knowledge of the definition a function and their concept images evoked by the concept name
2. The extent to which the prospective teachers can translate from one representation of a function to another and the extent to which they would have compressed the concept
3. The influence of the prospective teachers' conceptions and concept images of a function on the process of transforming the subject matter knowledge for the purpose of teaching
4. The warrants being used by the participants in appraising, modifying or generating representations, examples and definitions of a function in the hypothetical lesson plan for teaching the concept.

The empirical component of this study drew on interviews with prospective 'A' Level mathematics teachers. I interviewed six prospective teachers at the point they were about to graduate as 'A' level mathematics teachers. My goal was to learn about the knowledge for teaching the concept of a function they had at the end of their pre-service mathematics education degree programme. In this chapter I discuss the structure of the interview used, the characteristics of the participants, how data was analyzed and the strategies taken to ensure that the collected data is valid and reliable.

5.2 The Research Design

Basing on the assumptions that an individual's conceptions of mathematical concepts are dynamic, contextual and can be revealed through in-depth investigations this study followed a case study design.

5.3 The participants

The participants in this study were all the six undergraduate students majoring in mathematics, at Masvingo State University, with the intention of completing a four-year degree programme leading to certification as secondary mathematics teachers. Of the six prospective teachers four were females and two of them were males. At the time of study the prospective teachers had completed university mathematical modules on Calculus, Algebra, Analysis, History of Mathematics, Differential Equations, Statistics and a module titled Introduction to Higher Mathematics which focused on formal proof in advanced mathematics. All these modules were taught and examined in the Mathematics Department. Besides, the prospective teachers had done a high school methods course and a Professional Studies course which were offered by the Curriculum Studies Department. The methods course sensitized prospective teachers on teaching strategies for secondary school mathematics while the Professional Studies course was a blend of selected topics from Educational Psychology, Sociology of Education and Philosophy of Education courses.

5.4 The interviews

The primary purpose of the interviews was to learn more about prospective teachers' cognitions about the concept of a function as well as how they envisioned teaching the concept of a function to 'A' level learners. The interviews were designed on the basis of the theoretical frameworks discussed in Chapters 3 and 4.

All the six prospective teachers participated in the interviews. Because it was long and demanding for participants, the interviews were conducted in three sessions, each lasting about two hours. Interviews in the first session were task-based interviews. According to Wilkerson and Lang (2004) task based interviews involve a problem solver and the interviewer interacting in relation to one or more tasks introduced to the problem solver by the interviewer. Task-based interviews were used to explore the prospective teachers' knowledge of the concept of a function. The interviews included questions involving interpreting functions represented by graphs, situation descriptions, formulae and tables and translating among multiple representations of the concept. The tasks and questions in the second session were grounded in scenarios of classroom teaching. I presented the prospective 'A' level teachers with scenarios constructed out of common tasks of teaching the concept. For example, prospective teachers were asked to decide what to do in response

to a student's question or when helping a group of students with a common misconception. Below I explain each of these parts of the interview. The complete interview protocols can be found in appendix A, B and C.

5.4.1 Task-based interviews

The task-based interviews focused on prospective 'A' level teachers' content knowledge of the concept of a function. The tasks were divided into two categories. The first category of tasks was meant to probe prospective teachers' knowledge of the definitions of a function, namely the arbitrariness and the univalence properties of the function and the relationship of the different definitions of a function with the other topics in the mathematics curriculum. The tasks were also meant to probe prospective teachers' conception of a function. For instance, I asked the prospective teachers to explain in a sentence or so what they thought a function is and to give their own definition of a function.

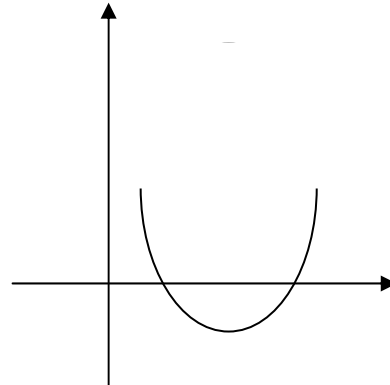
The second category of tasks was designed to measure the breadth and depth of prospective teachers' understanding of the concept of a function. The breadth of understanding of the concept was assessed by the prospective teachers' ability to translate among the various representational forms (tables, graphs, etc) while the depth of understanding of the concept was characterized by the prospective teachers' levels of comprehension through the procedure, process, object and concept layers. I also used non-routine tasks that were designed to elicit thoughtful responses. Generally, the non-routine tasks could not be solved by the routine application of taught procedures. Implicitly they involved the use of multiple representations. They were represented using one or more representations, but working in a different representation would facilitate progress toward the goal. For example task 2 (c) below was meant to assess the prospective teachers' ability to translate among the symbolic, tabular and graphical representation of a function.

Task 2.c

An equation, a table and a graph are displayed on the card (card provided) for the same function.

$$F(x) = x^2 - 3x - 10$$

x	y
-5	30
-4	18
-3	8
-2	0
-1	-6
0	-10
1	-12



- What is the output if the input is -1 ? Did you use the equation, the table or the graph to answer the question? Is there any other approach?
- What is the output if the input is 12 ? Did you use the equation, the table or the graph to answer the question? Is there any other approach?
-
- What is the output if the input is 4 ? Did you use the equation, the table or the graph to answer the question? Is there any other approach?
- What are the inputs if the output is 0 ? Did you use the equation, the table or the graph to answer the question? Is there any other approach?

Figure 5.1. A translation task.

5.4.2 Interviews based on teaching scenarios

A major assumption of the study's theoretical framework is that, in carrying out the activities of teaching, teachers' understanding of the subject matter interacts with their knowledge and beliefs about teaching, learning and the contexts of learning. Therefore in order to learn about prospective teachers' knowledge of teaching the concept of a function and the role that knowledge plays in the thinking about teaching and learning the concept, I designed interview tasks which were typical of those tasks that arise in teaching the concept. These tasks were ones that all teachers, whatever their view of teaching, perform although the way in which they would deal with these tasks varies in relation to their view of teaching. These tasks include responding to unanticipated student questions or novel ideas, examining

student's written work and planning approaches of teaching the concept of a function. Each question was cast in the form of a scenario or sketch of a teaching situation. After describing the scenario to the prospective teachers, I asked them what they would do or say if that situation arose in their own teaching and why. Figure 5.2 below shows an example of a teaching scenario

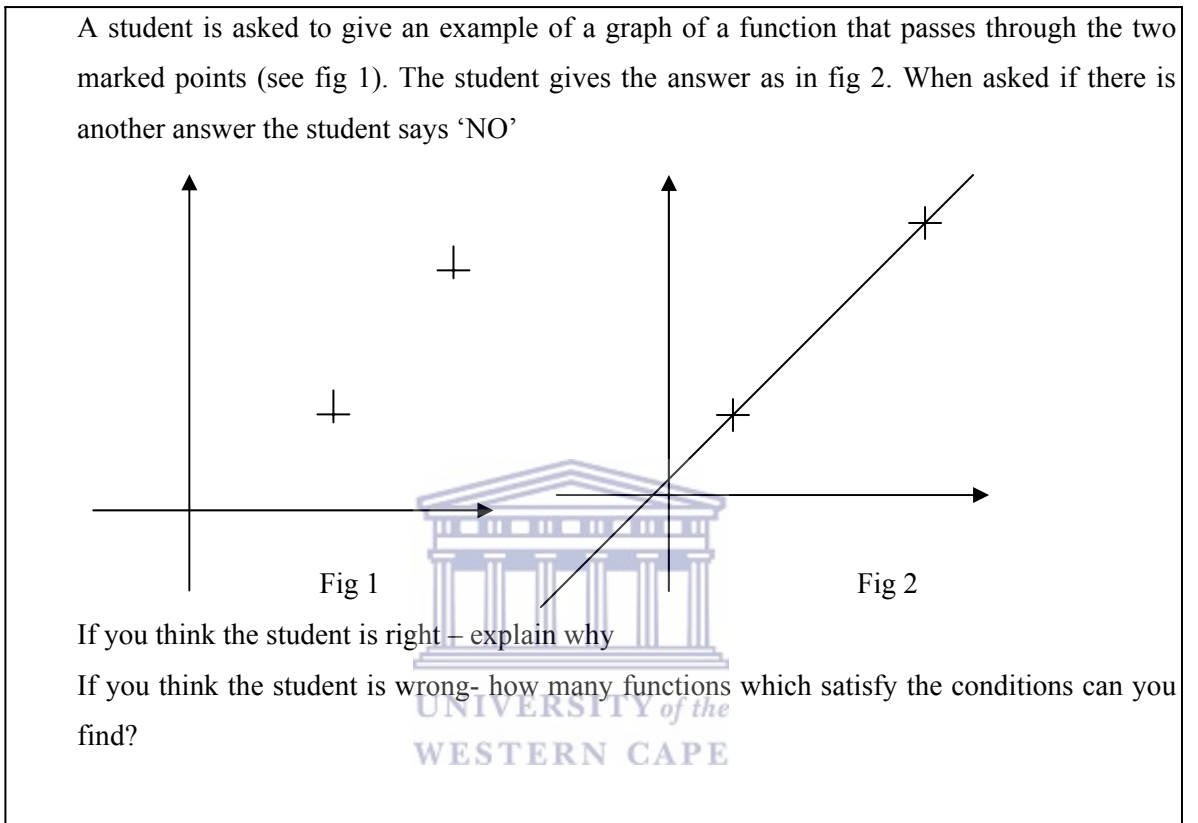


Figure 5.2. An example of a teaching scenario

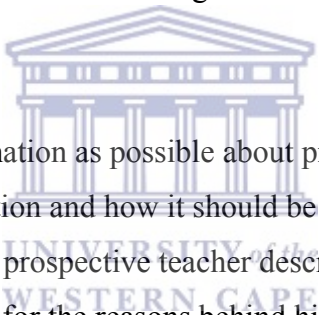
5.4.3 Reflective interviews

In the last interview the prospective teachers were first asked to plan a hypothetical teaching sequence which could be used in teaching the concept of a function to an 'A' level class. The hypothetical lesson plans were analysed with the intention of identifying aspects of the prospective teachers' intended actions that seemed to be significant in the sense that they could have been influenced by the prospective teachers' knowledge of the concept of a function or any one of the sources of teacher knowledge for teaching the concept of a function.

The reflective interviews were based on their lesson plans. In the reflective interviews the prospective teachers were asked to reflect on their lesson plans and then give valid reasons for their choice of teaching approaches, examples, representations or definitions of a function they intended to use as indicated in their hypothetical lesson plans.

5.5 Interview procedures

I interviewed each prospective ‘A’ level teacher three times within a period of three months. Each session lasted approximately two hours. The three months data collection cycle allowed me to begin analysing what I was learning from earlier interviews. Basing on the findings of the preliminary analysis I was able to sharpen the focus of my probes during the later interviews and modify some of the interview tasks in relatively minor ways. The changes included adding probes and altering the set up of the classroom scenarios. Probing was critical in order to learn why prospective ‘A’ level teachers said what they said about the concept of a function, how they envision teaching the concept and the criteria they used to justify their teaching decisions.

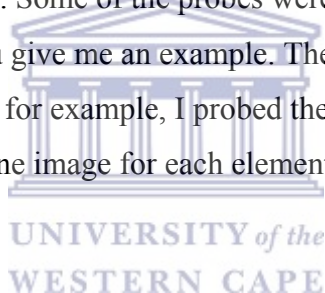


In an effort to get as much information as possible about prospective ‘A’ level teachers’ ideas about the concept of a function and how it should be taught, I used some standard probes. For example whenever a prospective teacher described something he or she would do with a student, I always asked for the reasons behind his/her actions. I probed to find out if the prospective teacher had other options in his/her repertoire i.e., alternative definitions and interpretations of a function, courses of action in cases of teaching scenarios and if so what the choice depended on. For example, after a prospective teacher had given me his definition of a function, I asked him to suggest an alternative definition he would have for a pupil who would not have understood the initial definition of a function.

The aim of the interviews was to understand the underlying thinking of an individual, to enter the individual’s mind rather than take the written responses from a test, by asking questions like, “What is your way of deciding whether a graph is a function or not?” A second feature of the interview was to determine the strength of conviction behind what the prospective teachers were saying. As Ginsburg (1997) discusses, Piaget noted that children tend to say what they believe the adults want to hear, so he used methods of “repetition” and “counter-suggestion” to gain insight into the strength of conviction (Ginsburg, 1997).

Therefore, prompts like, ‘It is not important to answer correctly or wrongly. Try to tell me what is going on in your mind’ were repeated throughout the interviews. If a prospective teacher explained successfully why a given item is a function or not, he or she was asked a non-function item as a counter-suggestion. When a prospective teacher gave a successful explanation, he or she was asked the same question from a different angle with a counter-suggestion to seek persistency in the responses. If a prospective teacher seemed to be reluctant, he or she was encouraged to say what came into his mind irrespective of whether the response was right or wrong.

Sometimes the prospective ‘A’ level teachers asked me questions. For example some wanted to know if their knowledge of the concept of a function was ‘OK’. When this happened, I tried to deflect their query and use it instead as an opportunity to learn more about their thinking by asking. “What is it about this that feels confusing to you?” That probe was often both effective and helped me learn more about the individual prospective teachers’ ideas about the concept. Some of the probes were standard, e.g., “Why?” ‘What do you mean by that?’ and ‘Can you give me an example. There were times when the probes were specific to given situations, for example, I probed the students to suggest reasons for why the requirement of having one image for each element in the domain is inherent in the definition of a function.



The interview sessions were audio taped. The audiotape was set up to be as unobtrusive as possible. The interviews were tape-recorded so that I could obtain verbatim transcription. In analysing the data I wanted to examine the metaphors and terms prospective teachers used as well as their capacity to articulate their understanding of the concept of a function. I also wanted a record of my role: How I probed and how I influenced the pace or direction of the interview. The interviews were transcribed and edited to be as faithful as possible to prospective teachers’ exact words, inflections and tones. The participants were asked to read the transcripts in order to check whether the transcripts reflected a true picture of what transpired during the interviews.

5.6 Data Analysis

In this section I discuss the methods used to analyze the data. Since the purpose of my study was to describe prospective ‘A’ level teachers’ knowledge of the concept of a function I

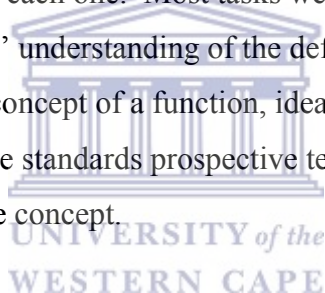
focused on analytic comparisons and contrasts across individuals, tasks and themes. To do this I analysed the interview transcripts from three particular perspectives: by person, by task and by analytic theme.

5.6.1 Reduction and analysis by person

Data analysis began with editing the transcripts. I listened to each interview and edited the transcripts to ensure that they corresponded with the audiotapes, adding emphasis and correcting transcriptions where necessary. Prospective teachers' written responses to the interview tasks were also inserted accordingly in the transcripts.

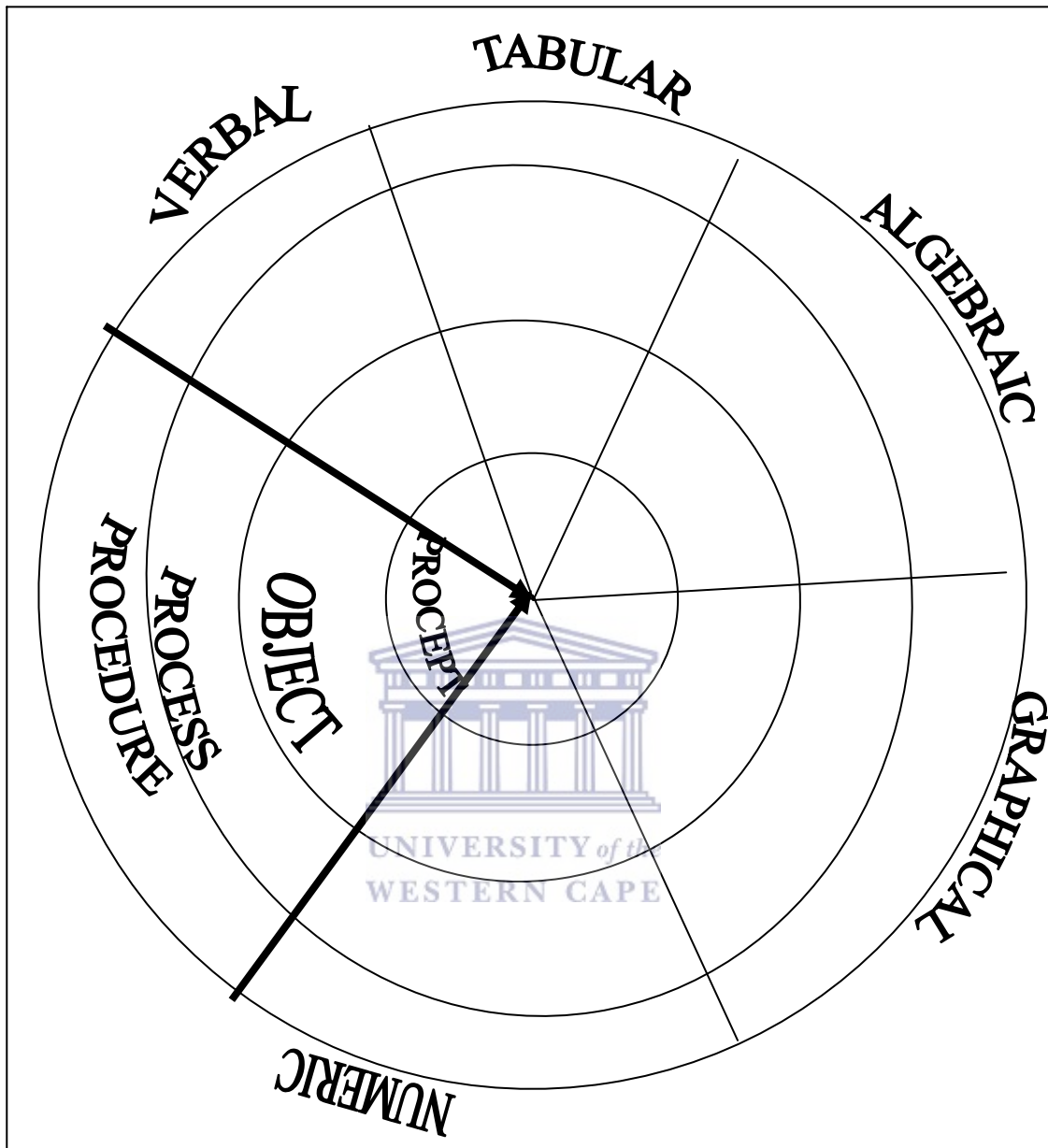
5.6.2 Reduction and analysis by task

Another step in the process of reducing and analyzing data was to read the transcripts by interview task. Drawing from careful substantive analysis of each question, I created a set of projected response categories for each one. Most tasks were cross – analyzed on several dimensions: prospective teachers' understanding of the definition of a function, prospective teachers' concept images of the concept of a function, ideas about teaching or learning the concept, the teacher's role and the standards prospective teachers use to appraise, evaluate and modify representations of the concept.



5.6.3 Analysis by themes

One more level of analysis was thematic, using themes that I had brought to the study such as subject matter knowledge of the concept of a function and prospective teachers' ideas about the teaching of the concept, but modified and elaborated by the person and question level analyses.



**Figure 4.3 Representations and conceptions of the concept of a function
(Adopted from DeMarois and Tall 1999)**

To understand prospective teachers' breadth and depth of understanding of the concept of a function, I adopted an analytical model shown in figure 4.3. The model was developed by DeMarois and Tall (1999) in which conceptions and representations of a function are combined diagrammatically, with conceptions as concentric circles representing increasing depth of understanding the concept. The sectors in figure 4.3 represent the various

representations of a function. To allow each representational form to be linked to any other, the above picture should be seen as having sectors that can be moved and connected in any way.

It is important to note that this visualization oversimplifies the complexity of the cognitive structure. Some representational forms such as tabular might be essentially more primitive than other representational forms such as symbolic. In fact it has already been pointed out that some representational forms (e.g., graphical) provide partial information about the function while the algebraic representation could be capturing more information related to the function.

The above model allowed me not only to characterize prospective teachers' cognitions of the concept of a function but also to characterize prospective teachers' versatility and adaptability by noting the variety of cells in which prospective teachers function and their ability to connect their work in different cells. Arcavi (1993) believes that having a flexible understanding of a function means that an individual is able to work within each cell of the model as well as move throughout the cells of the model by changing representations and/or perspectives as needed to complete a mathematical task involving the concept of a function.

The model was used to record the perspectives and representations being used at particular points in time in the individual prospective teachers' oral responses to the various tasks. Then by using the model as a whole, flexibility was documented in the following manner. Instances of versatility were identified when the individual was able to view a function from either perspective (procedure, process and object) as well as in any representation (tabular, algebraic, etc). Instances of adaptability were identified when the individual showed the ability to access specific cells (particular perspectives combined with particular representations) depending on the task at hand.

In order to uncover prospective 'A' level mathematics teachers' knowledge for teaching the concept of a function, basic thematic questions were used, namely:

- What ideas and commitments do prospective teachers draw on in responding to each classroom scenario?
- How do prospective teachers weave together the substance of PCK in the course of teaching the concept of a function?

- What ideas did the prospective ‘A’ level teachers have about teaching, learning and the teacher’s role with respect to the teaching of the concept of a function?
- What did the prospective teachers know and assume about students as learners of the concept of a function?
- What did the prospective teachers seem to consider as standards that guide their pedagogical reasoning entailed in representing the concept of a function to their students?

5.7 Reliability

Fundamental concerns I had when collecting data revolved around the degree of confidence I could place in what I saw on prospective teachers’ written protocols or heard when they gave their verbal responses to the task based interviews and whether other people could recognize categories that I had generated.

5.7.1 Consistence of responses

To strengthen on the reliability of the responses I got from the six prospective teachers I interviewed each respondent twice using the same tasks. Initially each respondent was interviewed individually in the mathematics seminar room. The second round of interviews were focused group interviews. The six prospective teachers were divided into two groups of three so I had two focused group interviews. The tasks used for the individual and focused group interviews were the same. This arrangement was done to check on the consistencies of the prospective teachers’ responses, especially their reasons behind their responses to each classroom scenario.

5.7.2 Inter-rater reliability

Inter-rater reliability was more concerned with whether other people could systematically and with intersubjective agreement apply the categories I had generated similarly to the task-based interview data. Improved inter-rater reliability on the interview categories was determined by asking fellow doctoral students to code parts of the interview data using the categories I had generated either from the literature review or from the theoretical framework.

5.8 Validity

5.8.1 Descriptive validity

This refers to the factual accuracy of the researcher's account of what happened in the research scene. This kind of validity seeks to ensure that the researcher is not engaged in fiction writing. The implication of this is that there should be evidence of the account. Hence audio visual and documentary evidence is critical so that anyone can access the evidence to authenticate the accuracy of the account. In this study descriptive validity was addressed through the availability of audiotapes and a number of documentary sources like the prospective 'A' level mathematics teachers written work and interview reports.

Validity is concerned with the question "Am I measuring what I intend to measure?" Two steps were taken to ensure the content validity of the interview tasks. Firstly, two doctoral students who were in the same programme with me were asked to review the interview tasks and to indicate the form of teacher knowledge the respective interview tasks were assessing. This helped me to establish face validity of the interview tasks and to identify ambiguous or unclear tasks. Secondly, I pilot tested the interview tasks with the outgoing third year students, a process that also led to item modification. The students who participated in the pilot study and the two doctoral students were asked to comment on the interview tasks and I also discussed with them on whether the use of the tasks was a fair approach for assessing teacher knowledge of the concept of a function.

5.8.2 Interpretive validity

This refers to the meaning which research subjects attach to all the objects, events and behaviours in the research setting. This type of validity is not physical but mental. This includes all thoughts, feelings, beliefs and perceptions of the respondents. According to Denis (1996) the language of interpretive validity is not that of the researcher but that of the subjects of the research. It is their account, in their own words. Interpretive validity therefore is a matter of making deductions from what the prospective teachers say.

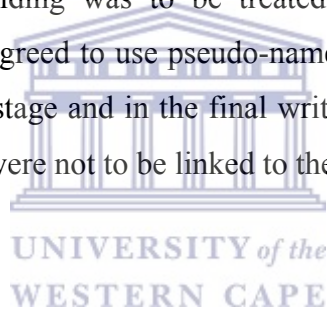
Interpretive validity in this study has been addressed through the availability of audio-tapes and a number of documentary sources like the prospective teachers written responses to task-based interviews.

5.8.3 Respondent validation/ Member checking

This refers to the taking of the research findings back to the subjects that have been studied for their own verification and perceptions about the findings. By the time I was halfway with data analysis the respondents had graduated and had been deployed in schools so it was not possible to take the research findings back to the respondents.

5.8.4 Ethical considerations

When pursuing any qualitative study on human participants, it is important to consider respecting their rights under international law. In this study, the prospective teachers were told in advance that the goal of the study was to understand how they think about the concept and not to assign a grade. As a result they were quite ready to share their thinking. They were informed that their participation was voluntary and that anyone was free to withdraw from the study at any point during the course of the study. To guarantee the participants that the information that they providing was to be treated as confidential information, the participants and the researcher agreed to use pseudo-names instead of the real names of the participants at the transcription stage and in the final write up. The researcher promised the participants that the real names were not to be linked to the data.



5.9 Conclusion

This chapter has described the purpose of the study, characteristics of the participants, the structure of the interviews and how the data was analysed. The next chapter will present and discuss the prospective teachers' cognitions of a function and how they envisioned the teaching of the concept.

CHAPTER 6

DATA PRESENTATION AND DISCUSSION

6.1 Introduction

In this chapter I present the data that characterizes prospective “A” level teachers’ knowledge of the concept of a function. More specifically I report on characterizations of the prospective teachers,’ knowledge of the definition of a function, the extent to which they can use the definitions, their understanding of the properties of a function and their ability to translate from one representational form of a function to the other.

6.2 Prospective “A” level teachers’ knowledge of the definition of a function

In Item 1 the prospective ‘A’ Level teachers were asked to first give a definition of a function, which they would teach their “A” level class, and then an alternative definition that might help a student with difficulties. Table 6.1 shows that the initial definitions of a function, which the “A” level teachers would teach their learners, are versions of the new formal definition.

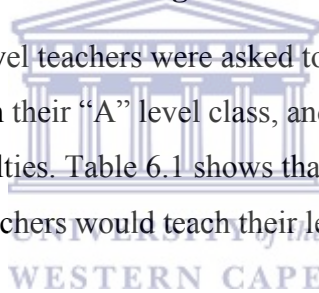


Table 6.1 Prospective “A” level teachers’ initial and alternative definitions of a function

Student	Initial definition	Alternative definition
Alice	A function f from a set X to a set Y is a formula that assigns to each element x in X a unique element y in Y . The set X is called the domain of f . The set of corresponding elements y in Y is called the range of f , e.g., $y = \sin x$	An <u>operation</u> done on certain numerical values of x that assigns to every value of x a value of $y = f(x)$ - it's like - given group of numbers you perform some operation on the numbers. The operation you do is called a function.
Ben	A function f from set X to a set Y is a rule that assigns to each element x in X a unique element y in Y , e.g., $y = 2x+4$	A <u>process</u> that can be performed on any number and is represented in algebraic form using x as a variable.
Chipso	A function is a dependence relation between two variables which can be described by a formula or an equation	A mapping where one x value is mapped to only one y - value. The mapping is done by substituting the x value in a given equation/ formula to get the y value.
Daniel	A function consists of three objects: two non-empty sets X and Y and a rule f which assigns to each element x in X a single fully determined element y in Y , e.g., $y = x^2+2x$	A function is an equation which has variable inputs, process the inputted number and gives an output, e.g., $Y = x^2$ or $y = \sin x$
Edith	A function is any correspondence between two sets that assigns to every element in the first set exactly one element in the second set	A mathematical expression or equation that gives a connection between two factors. One can substitute the first factor to get the second factor $y = x$
Fari	A function is a relationship based on a certain algebraic formula in which one set of variables depends on another set of variables. The dependent variable y is said to be a function of the independent variable x if for every value of x there is a corresponding value of y	An expression that gives a range of answers with different values of x , e.g., $y = x^2 - 1$

Ironically these versions of the modern definition of a function are not appropriate for the “A” level class since the modern definition was developed to make Analysis and Topology manageable. Although the new formal Bourbaki definition of a function is too abstract for high school students the same definition is cited in the Zimbabwean “A” level mathematics syllabus and in most “A” level mathematics textbooks. Markovits, Eyton & Bruckheimer (1986) made similar observations in their study of U.S. high school mathematics curriculum.

Also the initial definitions given by these prospective teachers seem to indicate that these teachers had a structural view of a function. To them a function was either a formula (Alice), a set of ordered pairs (Ben), a correspondence or a dependence relation (Edith and Fari).

In their alternative definitions of a function for the student with difficulties in understanding the first definition the prospective teachers shifted from a structural view of a function to an operational view of a function suggesting that the prospective teachers think that the process view of a function is more comprehensible to the “A” level students than the structural view of the concept of a function. While the initial definitions given by all the prospective teachers portray the impression that they have a structural understanding of a function their alternative definitions portray an image of a function as a process that takes a number, does certain things to it and gives back a result or an output (Alice, Ben and Chipu) or identify a function with its representation such as an equation or an algebraic expression (Daniel, Edith and Fari). Euler, a renowned mathematician, seemed to have had a similar conception of a function since he too referred to functions as “analytical expressions” (Malik, 1980). The tendency of identifying a function with its representation such as an equation or an algebraic expression is referred to by Sfard (1991) as a pseudo-structural conception of a function. One possible reason why prospective teachers would develop such a conception is that the examples used to illustrate and work with functions in their calculus courses are exclusively functions whose rule of correspondences is given by a formula. The order in which the prospective teachers gave their definition of a function is a reversal of the order the concept of a function developed historically and the psychological explanation of the development of the concept of a function. Although the historical and the psychological development of the concept of a function shows that the concept is conceived operationally before it is conceived structurally the prospective teachers prefer to start by teaching the structural definition first before the operational definition.

6.2.1 Operability of the definition of a function

The second set of task-based interview tasks focused on the operability of the definition of a function by the prospective “A” Level mathematics teachers. A mathematical definition is said to be formally operable for a given teacher if that teacher is able to use it in creating or reproducing a formal argument (Tall, 1999). The definition of a function is operable for a given teacher if he/she can use the properties outlined in the definition in assessing the correctness of pupils’ responses to tasks involving the concept of a function. The definitions given by the prospective teachers fell in one of the four categories. Below are the four categories of the definition of a function and some of the prospective “A” level teachers’ responses to pupils’ answers to some hypothetical identification tasks involving the concept of a function.

6.2.1.1: A function as a rule of correspondence

Two of the prospective “A” level teachers’ definition of a function fell under this category. Prospective teacher Ben defined a function as follows:

A function f from set X to a set Y is a rule that assigns to each element x in X a unique element y in Y

Prospective teacher Daniel’s definition was as follows:

A function consists of three objects: two non empty sets X and Y and a rule f which assigns to each element x in X a single fully defined element y in Y .

The above definitions eliminate the possibility of an arbitrary correspondence since a rule and an arbitrary correspondence are contradictory. A rule is expected to have some regularity whereas a correspondence may be arbitrary. The aspect of a rule was dominant in both definitions. The dominant idea of a rule was also expressed by the two teachers in their responses to whether $\{(1,10);(2,20);(3,31)\}$ was a function or not. They felt that the above was not a function since a “rule which connects the x-coordinate and the y- coordinate cannot be found” (Teacher Ben). The following extract illustrates how dominant the idea of a rule was in prospective teacher Ben’s definition of a function.

Researcher: What changes would you make to the set of ordered pairs in order to come up with a function?

Teacher Ben: The last ordered pair should be (3,30).

Researcher: What would be your reason to consider the new set of ordered pairs a function?

Teacher Ben: The ordered set would be a function since there would be a rule which connects the x-coordinate with the y-coordinate

Researcher: What would be the rule? Can you express it in your own words?

Teacher Ben: In this case the rule is –multiply the given input x by 3 in order to get the corresponding output y

The emphasis on the rule was also evident in teacher Ben's response to what he thought would be the conditions which had to be fulfilled if the two sets of ordered pairs are to represent the same function.

Researcher: Do the two sets of ordered pairs $\{(1,4);(5,20);(3,12)\}$ and $\{(2,8); (4,16); (6,24)\}$ represent one and the same function or not?

Teacher Ben: Yes they represent the same function

Researcher: Can you justify your answer?

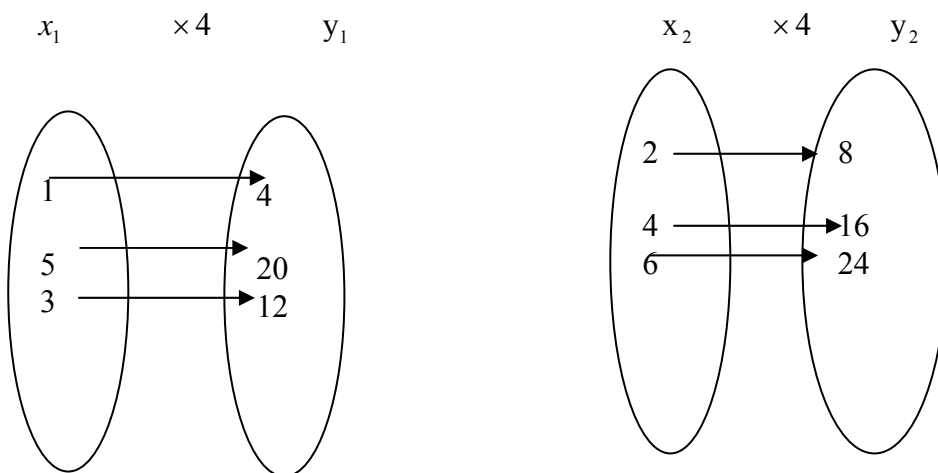
Teacher Ben: In each case the x and the y –coordinates are connected by the same rule which can be written algebraically as $y = 4x \dots$ so the two sets of ordered pairs represent the same function.

Since he places undue emphasis on the rule Teacher Ben thinks the two sets of ordered pairs represent the same function although the two sets of ordered pairs do not represent the same function. The equality of functions depends upon the domains being equal, the ranges being equal and the actions being equal (Acavi, 1987). Equality of functions do not require that the 'rules' should be equal. For instance, $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ are equal, where

$$f(x) = \max(-x, x) \quad \text{and}$$

$$g(x) = |x|$$

Prospective teacher Daniel did not consider the two sets of ordered pairs as representing the same function. When asked to give reasons for his answer teacher Daniel re-represented the functional relationship using set diagrams to obtain:



Pointing at the sets teacher Daniel remarked that:

..the corresponding sets are not identical... you see X_1 and X_2 variables are different ... the same applies to the Y_1 and Y_2 variables.... So the two sets of ordered pairs do not represent the same function although the rule happen to be the same

Researcher: How best can you describe the rule?

Teacher Daniel: The rule can be represented by the algebraic formula $y = 4x$

Prospective teachers Ben and Daniel used the words rule and function synonymously. These two teachers could not use their definitions to distinguish functions from non-functions. For example teacher Ben classified the piecewise function

$$f(x) = \begin{cases} x + 2 & \text{for } x \leq 1 \\ x^2 & \text{for } > 1 \end{cases}$$

as a non function since it had ‘two rules’. However Ben classified the function

$$f(x) = |x| = \begin{cases} x & \text{for } \geq 0 \\ -x & \text{for } x < 0 \end{cases} \text{ as a bona fide function. Although Ben considered functions as}$$

defined by rules he changed his behaviour when confronted with the modulus function. The modulus function was regarded as a function on the basis of familiarity with the function.

6.2.1.2 A function as a correspondence

Prospective teacher Edith defined a function as “ any correspondence between two sets that assigns to every element in the first set exactly one element in the second set”. This definition is referred to as the Dirichlet –Bourbaki definition. Again prospective teacher Edith gave a definition of a function, which is found in most Analysis textbooks. To avoid

the term correspondence, one may talk about a set of ordered pairs such that no two pairs have the same first member. The correspondence in the Dirichlet –Bourbaki definition is arbitrary and need not be defined by a rule. However, although prospective teacher Edith gave the Dirichlet –Bourbaki definition of the concept of a function she did not use the arbitrariness and univalence properties of the definition of a function in evaluating the correctness of pupils’ responses to the hypothetical identification tasks. Prospective teacher Edith felt that the correspondence was defined by a definite rule as evidenced in the following extract:

Researcher: You are saying a student who says ‘a correspondence that associates -1 with each negative number, $+1$ with each positive number, a 3 with zero’ is not a function is correct. Why do you think the student is correct?

Teacher Edith: It’s not just a single function. There are three functions .one of them gives -1 for all negative numbers; the second gives $+1$ for all positive numbers and the third one gives a zero when the input is 3

Researcher: Do the following functional representations $f(x) = \max(-x, x)$ and $g(x) = |x|$ define the same function or not? Give reasons for your answer.

Teacher Edith: No they do not represent the same function since the rules of correspondence are not the same.

Teacher Edith seems to think that the rule of correspondence is unique, thus, to her $f(x) = \max(-x, x)$ and $g(x) = |x|$ represent two different functions. Also if the correspondence between the numbers looks arbitrary teacher Edith speaks of infinitely many functions since for her each ‘element would have its own rule of correspondence’.

6.2.1.3 A function as formula or an equation

Teacher Alice defined a function as

A function f from a set X to a set Y is a formula that assigns to each element x in X a unique element y in Y .

The set X is called the domain of f . The set of corresponding elements y in Y is called the range of f .

Teacher Alice's alternative definition of a function was

An operation done on certain numerical values of x that assigns to every value of x a value of $y = f(x)$ - its like –given a group of numbers you perform some operation on the numbers. The operation you do is called a function.

What is evident from teacher Alice's definitions is that for her there is no difference between the idea of a function and the representation of the idea. Teacher Alice tends to identify a function with the mathematical representation (i.e., the formula) of a function. Such a conception of a function is referred to by Sfard (1991) as a pseudo-structural conception of a function. Teacher Alice seems to have a process view of functions as verified by her consistent reference to functions as entities that accept inputs to produce outputs where the only possible inputs and outputs are numbers. As a result she doesn't consider the determinant function, in which every square matrix is mapped to a unique number, as a bona fide function. Teacher Alice's examples of functions were the special functions such as linear, quadratic and trigonometrical functions.

6.2.1.4 A function as a dependence relation

The word function was used by prospective teachers Chipo and Fari to suggest a relationship or a dependence of one quantity to another as illustrated below:

Teacher Chipo: A function is a dependence relation between two variables which can be described by a formula or an equation

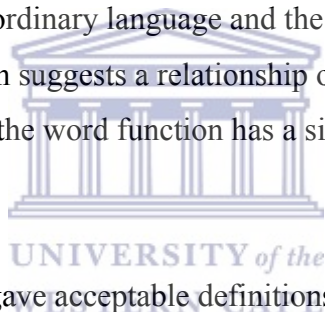
Teacher Fari : A function is a relationship based on a certain algebraic formula in which one set of variables depends on another set of variables. The dependent variable y is said to be a function of the independent variable

x if for every value of x there is a corresponding value of y .

Unlike prospective teachers Ben and Daniel who regarded a function as a rule, for prospective teacher Chipo and Fari a function is not just a rule but a relationship between two sets of variables where the relationship was described or represented by means of an algebraic formula or an equation.

It is true that a function is a relationship in which each independent variable is matched to a unique dependent variable. The two definitions given by the two prospective teachers are silent about this important property of mathematical functions. As a result the two prospective teachers regarded an equation of a circle as representing a function. A probable reason why these teachers regarded an equation of a circle as a function might arise from the use of language in the mathematics classroom. Many authors, e.g., Backhouse et al, 1987 and Zill, 1985, still use the term 'implicit function' to describe equations which can be differentiated by a process known as implicit differentiation.

Vinner (1991) drew attention to two modes of the use of definitions - the everyday use and technical mode required in formal reasoning. Definitions given by the prospective teachers Chipu and Fari suggest that these two teachers are using the word function in the literal sense and not in the more restrictive mathematical sense. Although mathematical language builds on the existing structure and logic of common language there is sometimes a mismatch between the use of words in the ordinary language and the mathematical language. In everyday usage the word function suggests a relationship or a dependence of one quantity on another whereas in mathematics the word function has a similar meaning but slightly more specialized interpretation.




All the six prospective teachers gave acceptable definitions of the concept of a function. The participants' definition or explanations of a function were categorized as acceptable if they made reference to

- the arbitrary nature of functions
- the univalence property of functions, i.e., the uniqueness of the image of each element in the domain
- that all elements in the domain have an image in the range

The arbitrary property of functions implies that functions do not have to exhibit some regularity or be described by any specific expression or particular shaped graph. The arbitrary nature of the two sets means that functions do not have to be defined on any specific sets of objects in particular, the sets do not have to be sets of numbers.

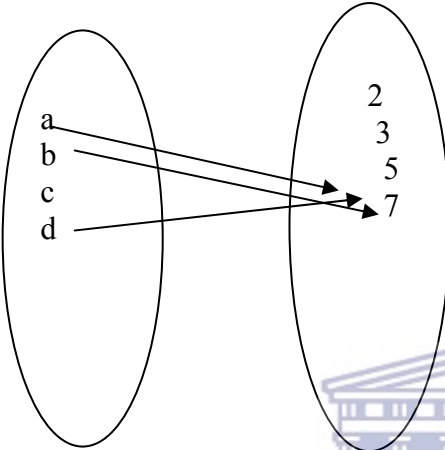
The univalence requirement that for each element in the domain there be only one element in the range is stated explicitly in the modern definition of a function. All the six prospective teachers seem to have little or no understanding of the use of quantifiers i.e. the phrases 'for

every element' and 'there exists only one element' which make the univalence of functions explicit. Although these quantifiers were explicitly stated in the prospective teachers' definitions of a function, they did not use these quantifiers as criteria for checking whether a given mathematical object was a function or not. For example the prospective teachers were given different representations of functions in isolation. They were given the following and were asked whether they were functions or not

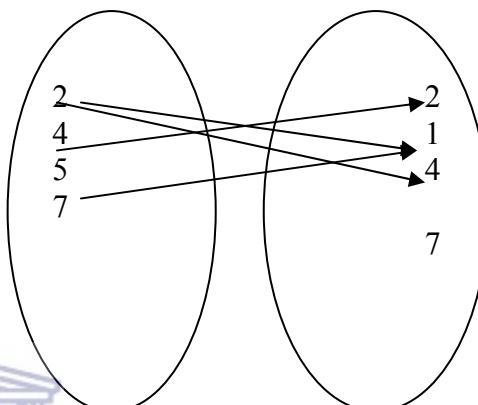


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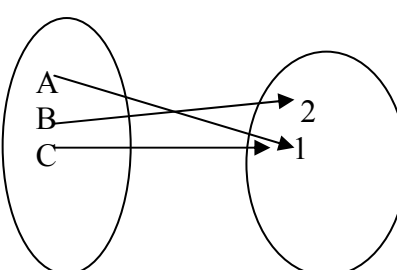
(a)



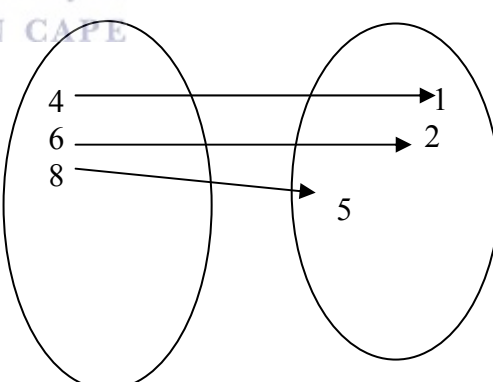
(b)



(c)



(d)



Formulae

(a) $f : \mathfrak{R} \longrightarrow \mathfrak{R}$ where $f(x) = \sqrt{x}$

(b) $f : \mathfrak{R} \longrightarrow \mathfrak{R}$ where $f(x) = x^{-1}$

In the interviews the prospective teachers were asked to explain why they considered or did not consider the above as functions. The following responses show that the prospective teachers do not always use the universal quantifier 'for every element in set X' to distinguish functions from non-functions, especially when the functions are represented algebraically.

- (a) is not a function because there is an element d left in the domain
- (b) is a function because for every element in the domain there is an element in the codomain
- (c) not a function because there are two values for a
- (d) a function because for every element in the domain there is an element in the range

However all the prospective teachers regarded

$$f : \mathbb{R} \longrightarrow \mathbb{R} \text{ where } f(x) = \sqrt{x} \text{ and}$$

$$f : \mathbb{R} \longrightarrow \mathbb{R} \text{ where } f(x) = x^{-1}$$

as functions even though not all elements in the respective domains have images in the corresponding codomains. In the case of $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = \sqrt{x}$ the negative numbers in the domain have no images in the codomain while the second mapping would only be a function if zero is excluded from the domain.

Difficulty with quantifiers is very common among students. Schechter (1999) found out that Italian graduate students had an easier time avoiding errors with quantifiers if they used symbols \forall and \exists instead of words ‘for every’ and ‘there exists’ respectively.

The findings of the study show that, although the six prospective teachers knew the set-theoretic definition of a function, they could not use that definition to distinguish functions from non-functions. They seemed to have developed at most a process conceptualization of a function and they tend to use their concept images of a function when deciding whether a given mathematical object was a function or not. They viewed a function as a repeatable mental manipulation of objects where the only manipulable objects are numbers. It is generally expected that high school treatment of the function moves student understanding of it from the action conceptualization to that of a process conceptualisation, the interiorization of action so that the total action can take place entirely in the mind of the learner, and finally to the object conceptualization which is the encapsulation of the process in its totality. However, five of the six prospective teachers came up with the modern definition of a function which might suggest that the prospective teachers view a function as an object. However the results seem to indicate that reproducing the modern definition of a function from memory does not guarantee clear understanding of a function. The prospective teachers do not use the definition of a function which they would have memorized to assess the

correctness of the learners responses to identification tasks involving the concept of a function.

Knowing mathematical definitions for teaching requires more than learning mathematically acceptable definitions in various mathematical courses. Being able to reproduce the modern definition of a function is not enough. What is needed is being able to understand and work with definitions in classrooms, with pupils, treating them in a way that respects the role definitions play in doing and knowing mathematics. Knowing how definitions function and what they are supposed to do, together with also knowing a well-accepted definition in the discipline, would equip teachers to develop a definition of a function that has integrity and is also comprehensible to students

6.3 Breadth and depth of prospective “A” Level mathematics teachers’ understanding of the concept of a function

The second research question focused on prospective “A” Level mathematics teachers’ understanding of the concept of a function in terms of its breadth and depth. The breadth dimension of the concept of a function is conceived by Tall (1999) as consisting of the various representations of the concept of a function namely: verbal, graphical, table of values, algebraic formulae, and set diagrams. Hence one’s breadth of understanding the concept of a function is determined by an individual’s ability to link different representations of a function. Kaput (1992) says that the use of more than one representation or notation system illustrate a better picture of a mathematical concept or idea since complex mathematical ideas are seldom adequately represented using a single notation system. Zachariades et.al. (2002) argue that representational systems are the keys for conceptual learning and determine, to a significant extent, what is learned.

The terms pre-procedure, procedure, process, object and procept conceptions were introduced by Tall (1999) to refer to various layers of increasing depth of understanding the concept of a function. Pre-procedure denotes that the student has not attained the procedural layer. Students who have a procedural conception of a function are dependent on carrying out a sequence of step-by-step actions. Students with a process conception of a function can accept the existence of a process between input and output without needing to know the specific steps and can view two procedures with the same input and outputs as the same

process. The object layer denotes the student's capacity to treat the concept of a function as a manipulable mental object to which a process can be applied while the procept layer indicates the ability to move between the process and the object conceptions in a flexible way.

Task-based interviews were used to measure the breadth and depth of prospective "A" Level mathematics teachers' understanding of the concept of a function. Prospective 'A' level mathematics teachers' breadth of understanding by the concept of a function was assessed through their ability to translate among the various representational forms while their depth of understanding the concept of a function was assessed by determining their level of comprehension through the pre-procedure, procedure, process object and procept layers.

Non-routine tasks were designed to elicit thoughtful responses. Generally non-routine tasks could not be solved by the routine application of taught procedures. They were worded using one or more representations but working in a specific representation would facilitate progress towards the solution.

6.3.1 The function machine as a representation of a procedure, process or a mental object

In order to determine the prospective "A" level mathematics teachers' depth of understanding the concept of a function they were asked to determine whether pairs of function machines, algebraic equations or formulae represented the same function or not. A pair of function machines (Figure 6.1) provided data on prospective "A" Level mathematics teachers' depth of understanding the concept of a function as well as their ability to translate from the function machine representation to the algebraic representation. The function machine is often introduced as a visual representation of the concept of a function seen as an input/output process, in which there is a single output for any given input.

(a) Write down equations of the functions represented by the function machines A and B and give reasons why you think the two function machines represent/do not represent the same function

(b) A continuous function f satisfies
 $f(1) = 3$; $f(1.1) = 3.1$; $f(1.2) = 3.3$; $f(1.3) = 3.6$; $f(1.4) = 4$; $f(1.5) = 4.5$;
 $f(1.6) = 5$; $f(1.7) = 5.4$; $f(1.8) = 5.7$; $f(1.9) = 5.9$; $f(2) = 6$
 Represent the information graphically

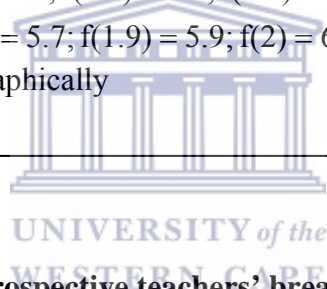


Figure 6.1: Items measuring prospective teachers' breadth and depth of understanding the concept of a function

The six prospective “A” Level mathematics teachers were asked to write down equations for each function machine and to give reasons why they thought the two functions machines represented/ did not represent the same function. The responses to the task are shown in Table 6.2 below.

Table 6.2: Function Machines as representations of procedures, processes or mental objects

Prospective teacher	Machine A	Machine B	Are functions A and B equal?
Alice	$y = 3x + 6$	$y = 3(x + 2)$	Yes. If I distribute 3 in machine B I get the same function as A
Ben	$y = 3x + 6$	$y = 3(x + 2)$	Yes – but different procedures are carried out
Chipo	$y = 3x + 6$	$y = (x + 2)3$	No- they are two different procedures used to get the output for a given input
Daniel	$x \times 3 + 6$	$[x + 2] \times 3$	No- they are different functions since they have different ways of getting the output for every given input
Edith	$(y)=3x + 6$	$(y)=3(x + 2)$	Yes-the corresponding inputs and outputs are the same
Fari	$3x + 6$	$x + 2(3x)$	No. you come up with the same y value for the given x values, but they are different processes.

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The responses show that Alice and Edith were referring to a function as a mental object while prospective teachers Chipo and Fari viewed functions as procedures. Alice and Edith could easily link the function machine representation and the algebraic representations. Daniel gave a literal translation of both function descriptions showing less flexibility when moving from the function machine representation to the algebraic representation. Fari also gave a literal translation of the second function as “ $x + 2(3x)$ ” revealing that she is less comfortable relating the function machine representation to the algebraic representation. While Alice, Ben and Chipo wrote down equations of functions which were being represented by the function machine, Daniel, Edith and Fari gave algebraic expressions instead of functions. However Edith was quick to change her algebraic expression $3x + 6$ to an equation $y = 3x + 6$.

The different conceptions of a function held by the prospective teachers were also evident when they were asked to identify the input and the output in their equations or expressions they had given. All the prospective teachers referred to x and y as the input and the output respectively. However although the algebraic symbols $3x + 6$ and $3(x + 2)$ had been equated to y by three of the six prospective teachers some did not consider the algebraic expressions as representing any output. The following extract show the different responses to the question ‘What does $3x + 6$ and $3(x + 2)$ represent?’

- *the output*-Chipo
- *procedures for calculating the output*-Ben
- *processes for getting the output y* -Alice
- *calculations performed on the given input in order to get your output*-Daniel
- *both represent the output*- Edith
- *processes for getting the output*-Fari.

Again it is evident that Edith and Ben view the algebraic expressions in their equations as objects while for the other prospective teachers the algebraic expressions evoke a process or a method of obtaining the value of the function for a given input. The fact that four of the prospective teachers viewed the algebraic expressions in the equations as processes or procedures might have contributed to the development of the process or procedural conception of a function in these teachers. Tall (1999) remarked that students who do not understand the proceptual nature of notations as representing both a process and an object are unable to encapsulate high order processes into objects. The above results show that McGowen et.al’s(1999) generalization that the use of function machines as an input/output box enables students to have a mental image of a box that can be used to describe and name various processes without the necessity of having an explicit process defined is not valid since only two of the six prospective teachers viewed a function machine as a representation of a mental object. Tall et.al (2000) argued that the function machine has the iconic, visual aspects, embodying both an object-like status and also the process aspect from input to output.

The second item in figure 6.1 was meant to assess whether the prospective teachers had gone beyond the point-wise conception of a function to the covariance conception. In solving the task one needed to translate the numerical information which was given in algebraic notation

into graphical information which is given in terms of points or slopes. Moreover, one needed to deal with the fact that the given numerical information is partial. It contained only eleven points out of a continuum. In order to come up with the correct curve, one needed to coordinate the instantaneous rate of change, direction of concavity and the inflection points of the function with the continuous changes in the independent variable for the entire domain of the function.

All the prospective teachers could plot the points on the graph although in some cases the scales used were not uniform. These findings seem to suggest that all the prospective teachers in this study had a clear conceptualisation of the point-wise view of a function. They all appreciated that the function was monotonically increasing for the given range of the input values, thereby giving the impression that the prospective teachers have also developed a covariance conception of a function. However the covariance conception of a function had not fully developed in four of the six prospective teachers who could not draw an appropriate curve connecting the six points. Figure 6.3 below shows the different curves drawn by the six prospective teachers.

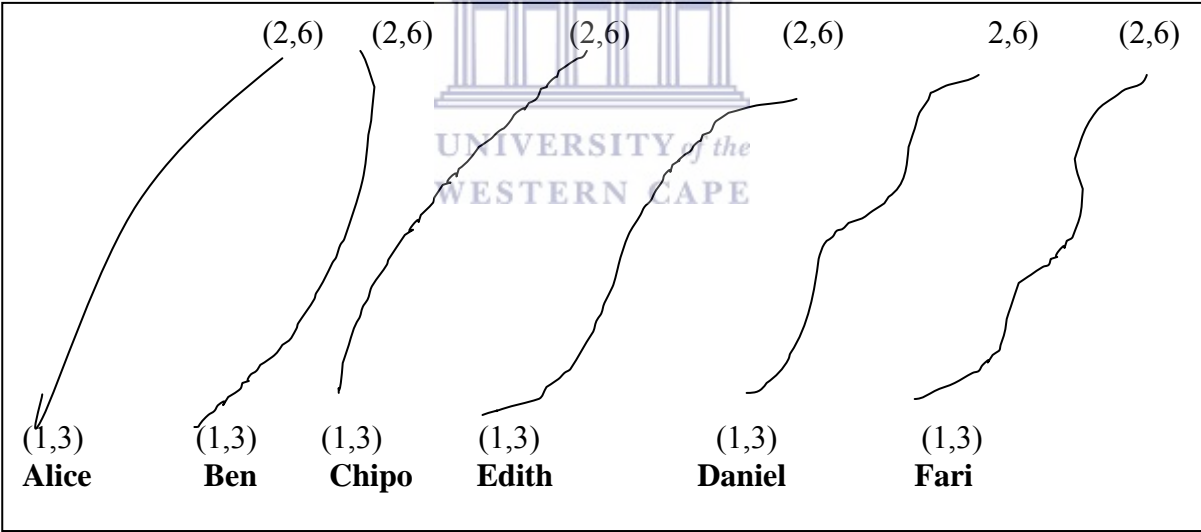


Figure 6.3 Prospective teachers’ graphical representation of the function f

The only prospective teachers who could draw graphs showing clear understanding of the concavity changes were Alice and Edith. The rest of the prospective teachers could not draw a graph whose gradient increases initially from one to five then decreasing gradually to one again.

6.3.2 The algebraic equation as a representation of a procedure, process or an object

In order to assess the prospective “A” level mathematics teachers’ depth and breadth of the concept of a function when it is given in algebraic form they were asked to identify algebraic expressions (Figure 6.2 question 1) which defined the same functional relationships. In order to respond to this item prospective teachers were expected to sketch the six functions on the same Cartesian plain. Their ability to draw the graphical representation of functions was a measure of their flexibility in moving from the algebraic to the geometrical/ graphical representations of a function.

1 Which of the following algebraic expressions represents the same functional relationships

- (a) $f(n) = \sum_{k=1}^n (2k - 1)$ $k, n \in \mathbb{N} \dots$ (b) $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$
- (c) $f(n) = n^2$ $n \in \mathbb{N}$ (d) $f(x) = \max \{ -x, x \}$
- (e) Recursively defined relation
 $g(x+1) = g(x) + 2x + 1$ where $g(0) = 0$ and $x \in \mathbb{N}$

2 Which of the following propositions describe functions? (x and y are real numbers)

(a) $y = \begin{cases} 2x + 5 & \text{if } x \text{ is even} \\ 1 - 3x & \text{if } x \text{ is odd} \end{cases}$

(b) $y = 4$

(c) Are all equations representations of functions? Give reasons for your answer

Figure 6.2: Algebraic expressions as representations of procedures, processes or objects

All the six prospective teachers, with the exception of prospective teacher Chipu who thought that it was not necessary to sketch the function in order to determine their equality, could sketch the functions on the same Cartesian plain. In coming up with the graphical representations of the functions the prospective teachers had to draw up a table of values before drawing the corresponding graphical representations. Only prospective teachers Alice and Edith seemed to conceive a function as an object as shown by the following extract.

Researcher: Which sets of algebraic equations represent the same function?

Alice: Of the five algebraic equations I noticed that two of them are continuous while

the other three are discrete. I figure $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$ and

$f(x) = \max \{ -x, x \}$ represent the same function.

Researcher: How can you convince anyone they represent the same function?

Alice: Well- I know that the modulus function and its graphical representation and I can easily sketch it. [She draws a rough sketch of the modulus function on a paper] I then checked whether the points of $f(x) = \max \{ -x, x \}$ lie on the graph of the modulus function or not. [She evaluates the value of the function at $x = -2, -1, 3$ and 4] - the points happen to coincide hence the two formulae represent the same function.

Researcher: You have not said anything about the other three equations.

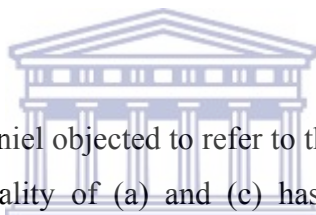
Alice: The algebraic formula $f(n) = \sum_{k=1}^n (2k-1)$ $k, n \in \mathbb{N}$ is referring to what one

gets when the first n odd terms are added and I know that the sum is equal to n^2 where n is the number of odd terms. So the algebraic formulae

$f(n) = \sum_{k=1}^n (2k-1)$ $k, n \in \mathbb{N}$ and $f(n) = n^2$ $n \in \mathbb{N}$ represent the same functional relationship.

Prospective teacher Chipó's response to item 1 shows that she had an action view of a function. For her the notion of a function was tied to a specific rule, formula or computational procedure and this involved the completion of specific computations. She did not even bother to re-represent the functions graphically or in tabular form. Her reason for saying that each of the formulae represented different functions was that each of the formulae defined specific computations when determining the output for a given input. However she considered the algebraic expression $y = 4$ as a representation of a function although she could not state a specific computation for getting the output. The algebraic expression $y = 4$ was considered a function on the basis of familiarity. Sfard (1992) attributed the students' difficulty with the constant function to the students' implicit belief that in order to speak about a function a change in the independent variable must be followed by a change in the dependent variable. Carlson (1998) remarked that understanding the basic idea of equality of two functions requires a generalization of the input-output process, the

ability to imagine the pairing of inputs to unique outputs without having to perform or even consider the means by which it is done. Prospective teachers need to understand that any means of defining the same relationship gives the same function. That is, a function need not be tied to specific computations or rules that define how to determine the output for a given input. For example the formulae $f(n) = n^2$ $n \in \mathbb{N}$, $f(n) = \sum_{k=1}^n (2k - 1)$ $k, n \in \mathbb{N}$ and the recursively-defined relation $g(x + 1) = g(x) + 2x + 1$ where $g(0) = 0$ and $x \in \mathbb{N}$ provides the same results on natural numbers and thus define the same function. The same applies to, $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$ and $f(x) = \max \{ -x, x \}$. Sfard (1992) considered the ability to recognize the same concept under many different disguises as one of the most important characteristics of thinking of functions as abstract objects. Surprisingly the same prospective teachers who have not reified the concept of a function expected their pupils to learn the set-theoretic definition of a function in which the concept of a function is viewed as an abstract object.



Prospective teachers Ben and Daniel objected to refer to the above sets of algebraic formulae as the same. Although the equality of (a) and (c) has been proved by induction these prospective teachers did not consider them as equal. Daniel remarked, “*You can say they give the same values, but not that they are that same*”. These two prospective teachers had a process conception of a function. Prospective teacher Ben could come up with sets of input variables and their corresponding output variables with respect to each algebraic formula. After comparing the sets of possible inputs and outputs obtained from the formulae he concluded that ‘*although the sets of formulae (a) and (c) also (b) and (d) respectively produced the same outputs for the given inputs but, since the processes involved in coming up with the outputs were different the formulae were representing different functions.*’

On the other hand prospective teacher Fari had the tendency of referring to an algebraic representation of a function as the function itself. To him the function was the representation of the function hence since all the algebraic expressions were worded differently he thought that they were six different functions. Kieran (1989) noted similar tendencies among students and teachers who considered a formula or an algebraic expression as an object in

itself and not standing for anything else. Sfard (1991) refers to such a conception of a function as a pseudo-structural conception of a function.

The relationship between functions and sequences proved to be problematic to the five of the six prospective teachers, who considered functions and sequences as two disjoint mathematical ideas. Daniel was the only one who correctly viewed sequences as a subset of functions although he had a process view of a sequence:

-sequences are a special set of functions in which the positive integers – counting numbers – are inputted to give some unique output. For example, if we have a sequence $a_n = \frac{1}{n}$, the output of the sequence is obtained by evaluating $\frac{1}{n}$ when $n = 1, 2$ and so on. So a sequence, just like any other function, is a process where you input something to go through some sort of process and you get an output.

Researcher: What is the process in your case?

Daniel: The process is evaluating the reciprocal of the given natural number.

Researcher: Is the sequence 2, 4, 8, 16.... still a sequence?

Daniel: It is still a sequence where 1 correspond with 2, 2 with 4, 3 with 8 and so forth

The above extract shows that Daniel can construct in his mind a process of accepting a positive number, seeing it as an ordinal specification of one of the quantities in the sequence, and taking that quantity as the result of that process. The other prospective teachers dissociated sequences from functions for the following reasons:

- *there are no identifiable sets of input and output variables* (Alice, Ben and Edith).
- *it's just an array of numbers* (Fari, Chipu).

By itself a sequence of quantities would not represent a function until an individual adds something to the structure of the situation. It is necessary to think in terms of the first term, second term and so forth. When there is an indication that this is what the individual is doing, then one can say a process conception of a function is at play.

All the prospective teachers could not accept that the recursively-defined relation $g(x+1) = g(x) + 2x + 1$ where $g(0) = 0$ and $x \in \mathbb{N}$ was a bona fide function which is equal

to $f(n) = n^2$ $n \in \mathbb{N}$ and $f(n) = \sum_{k=1}^n (2k-1)$ $k, n \in \mathbb{N}$ respectively. The following responses

show that, to these prospective teachers, when one deals with a function, he or she has to perform an explicit manipulation to an input in order to obtain the corresponding output value.

- *the new output i.e., $g(x+1)$ is a combination of the previous output $g(x)$ and the function $2x+1$ whereas in a given function it's only the input which should determine the value of the output. (Chipo and Fari)*
- *the output is not strictly obtained from a given input (Daniel and Edith)*

These findings seem to suggest that the question of whether or not an explicit manipulation on a given input can actually be performed is a crucial factor in determining whether a given situation can be described by a function

Prospective “A” level mathematics teachers’ responses to item two in Figure 6 shows that like many mathematicians before them (e.g., d’Alembert), today’s prospective teachers cannot accept the idea of a function defined on split domains. This finding is in line with data collected by Markovits et al (1986) and Vinner and Dreyfus (1989). The rejection of a function represented by more than one formula seems only natural in a student who does not distinguish between a symbol and the abstract entity behind it. Some of the prospective “A’ level mathematics teachers said that in item 2 (a) “*two functions have been defined, not just one*” (Teacher Alice, Edith, Chipo, and Fari). The same phenomenon was observed by Vinner (1992) with respect to graphs where some students insisted that a discontinuous curve represented several functions rather than one.

6.3.3 The verbal description of a function as a procedure, process or a mental object.

The questionnaire items given in Figure 6.3 also provided information about the prospective ‘A’ level mathematics teachers’ ways of thinking about the concept of a function when represented verbally. Through such direct question as the first one I hoped to find whether their conceptions of a function are closer to the operational or the structural conception

1. Which of the following sentences is, in your opinion, a better description of the concept of a function?
 - (a) A function is a computational process which produces some value of one variable (y) from any given value of another variable (x).
 - (b) Function is a kind of (possibly infinite) table in which to each value of one variable corresponds a certain value of another variable.
2. True or False?
 - (a) Every function expresses a certain regularity (the values of x and y cannot be matched in an completely arbitrary manner).
 - (b) Every function can be expressed by a certain computational formula (e.g., $y = 2x + 1$ or $y = 3\sin(\pi + x)$)
3. For every value of x we choose the corresponding value of y in an arbitrary way (e.g., by throwing a dice)

Figure 6.3: Verbal description of a function as either a procedure, process or an object

In response to the first questionnaire item two of the prospective teachers (Ben and Daniel) agreed that a function is a stable construct, composed of infinitely many parts. The other four chose the description which associated functions with a computational process. Since in the formal definition of a function which is found in Calculus textbooks and often taught to “A” level students, no computational procedure is mentioned, the strong preference for the operational version shows that the propensity for the operational thinking develops even when it is not deliberately promoted.

Not only do prospective “A” level mathematics teachers seem to think about functions in terms of processes rather than permanent mental objects, but they also believe that the process must be algorithmic and reasonably simple. Indeed responses to question 2 in figure 7 indicate quite a narrow range of processes fitting in with the prospective “A” level mathematics teachers’ operational conception of a function. First and foremost, such a process must display ‘certain regularity’. This opinion was expressed by five of the six prospective teachers. All the prospective teachers, with the exception of Alice, rejected the idea of an arbitrary defined function. The prospective teachers’ need of a well defined rule which can be defined by a formula is a motif which repeats itself many times in other studies (Vinner, 1992; Dubinsky and Harel, 1992).

6.3.4 Determination of functionality

In the interview tasks prospective teachers were presented with functions they had seen in previous mathematics courses. These items were meant to assess the extent to which the prospective teachers would have extended the range of application of the notion of a function. Below are some of the interview items involving functions from previous mathematical courses.

1. Determine whether the following are functions or not. If they are functions state the range and domain of the function.
 - (a) $f(x, y) = x + y \quad x, y \in \mathfrak{R}$
 - (b) $f(x, y) = 2x + 1 \quad x, y \in \mathfrak{R}$
2. One of the differential operators is what you know as the derivative. Instead of writing y' or $\frac{dy}{dx}$ we just use a D . You may recall this sort of notation from your differential equations course. For example, $D(x^2) = 2x$ or $D(\cos x) = -\sin x$. Is D a function in any way?
3. Would you consider the following function machine as a representation of a functional relationship?

Square matrix

Figure 6.4 Special functions

Prospective teachers' responses to the items in figure 8 show that these teachers had a narrow view of the concept of a function since they tended to link the notion of a function to sets of real numbers. The prospective teachers, who, since their initial entry into schools, had been used to addition as an essential part of their elementary intellectual inventory, could not view a correspondence between pairs of numbers and single numbers as a function. Although the function $f(x, y) = x + y \quad x, y \in \mathfrak{R}$ is a bona fide function, which takes two numbers x and y and assigns to them their sum $x + y$, it was not considered as a function by the prospective teachers for the following reasons:

- *we have no functions with two inputs (Alice and Fari)*
- *there should be a unique input (Ben)*
- *there is no formula linking y to x (Edith)*
- *is not a function but a means of stating the addition operation on real numbers (Daniel).*

Although the prospective teachers had studied functions of several variables, where a pair of numbers corresponds to a single number such as the basic number operations, a function of two real variables like the one given above did not evoke the notion of a function in their mind. To them, the notion of a function is tied to the belief that in order to speak about a function, a change in one variable must be followed by a change in the dependent variable. The prospective teachers found it difficult to conceptualise the function in 1 (b) above. The prospective teachers felt that the y in $f(x, y)$ was redundant. They were more comfortable if the function was a linear function.

The question about the differential operator did not go over very well. Only prospective teacher Daniel really appeared to understand the basic idea of function spaces, that is, an operation acting as a function on a domain space whose elements themselves are functions. The following extract shows that Daniel had an action conception of the differential operator:

Daniel:

The differential operator D is a function since it produces a unique function which is called the derivative from the original function by means of fixed mathematical rules. The original function could be viewed as the input with the corresponding derivative as its output.

Daniel's response also show that he conceives the functions which are being differentiated and their corresponding derivatives as objects since he refers to them as inputs and outputs respectively. The responses from other prospective teachers were as follows:

- *there is no way we can show graphically that the differential operator is a function (Fari, Ben).*
- *How can we have functions as inputs and outputs – there are no numbers involved (Edith, Alice)*

The above responses show that five of the six prospective teachers had not yet generalized their function schema. According to Dubinsky (1991) an individual is said to have generalized the notion of a function if he or she has mastered the arbitrariness property of the concept of a function. The arbitrary nature of the concept of a function refers to the nature of the inputs and outputs. The functional relationships do not necessarily have to be defined on numbers but can also be defined on other mathematical objects such as vectors or other functions as long as these mathematical notions have been reified by the individual into objects. Just like the 18th and 19th century mathematicians the prospective teachers still think that functions are defined on sets of numbers. As a result they all think that a mapping, which maps all square matrices to their corresponding determinants, is not a function. Their reasons for considering the determinant function as a non-function were:

... it would not be possible to plot matrices against their determinants since the domain of a function should be a set of real numbers (Alice)

... there is no equation in which you can substitute a matrix to get a number (Chipo)

...the strategies (rules) for obtaining the determinant of a matrix depend on the order of the matrix, e.g., for a 2 by 2 matrix the determinants equal to the difference between the product in the leading diagonal and the product of the elements in the other diagonal... you need a different strategy to evaluate the determinant of a 3 by 3 matrix (Fari and Edith)

The last response above suggests that prospective teachers (Fari and Edith) also think that the correspondence between elements in the domain and the range should be defined by a specific rule whereas the correspondence could be arbitrary. One interesting observation is that although the prospective teachers had studied differentiation and determinants of matrices at secondary and tertiary levels the notion of a determinant function and differential operator as a function defined on matrices and other functions respectively is never evoked in their minds.

6.4 Influence of prospective teachers' conceptions and images of a function on their pedagogical content knowledge.

The third research question focused on whether prospective "A" Level mathematics teachers' conceptions and images of a function was going to influence how they intended to teach the concept to an 'A' level class. Task-based interviews were used to capture prospective teachers conceptions and images of a function. The second source of data to answer the same research question was the reflective interviews data based on the prospective teachers' reactions to hypothetical lesson plans.

The findings collaborate the previous researchers' (Sanchez and Llinares, 2003) findings that teachers' conceptions of the nature of the mathematical content influences the way they would approach the teaching of the mathematics. The influence of the prospective teachers' ways of knowing the concept and their concept images on their pedagogical content knowledge was noted in the prospective teachers' use of examples of functions, their reasons for the choice of teaching contexts and the extent to which the prospective teachers' compartmentalized knowledge of functions manifested itself as they plan to teach the concept to an 'A' level class.

6.4.1 Influence of teachers' concept images a function on choices of examples for use with learners

Table 6.1 shows that when the prospective teachers were asked to give examples of a function all their examples were algebraic equations in which the output variable (y) was expressed as a function of a variable (x) Such responses show that the function as an algebraic equation was a dominant concept image among the prospective teachers. The influence of the prospective teachers' concept images of a function on how they intended to teach the concept was evident in their hypothetical lesson plans. Table 6.4 below also show that in their hypothetical lesson plans the prospective teachers gave a priory role to algebraic representations of a function. Chipu and Edith incorporated the use of the algebraic equation in modeling real situations. Daniel and Ben emphasized the algebraic representation as defining a rule of obtaining the output for a given input. For these prospective teachers the algebraic formulae were considered as a means of describing either the rule of correspondence or the dependence relation.

6.4.2 Influence of teachers' conceptions a function on choices of learners' tasks

What emerged from the task-based interviews with prospective teachers was that the prospective teachers had a process or an operational conception of a function. This influenced the prospective teachers' organization of the content and the choice of learners' tasks.

Table 6.4. A comparison of prospective teachers' conceptions and concept images of a function and the examples and tasks intended for use with pupils.

Name	Prospective teacher's dominant conception and concept image of a function		Examples of a function and tasks for the 'A' Level class	
	Conception of a function	Concept image of a function	Examples	Pupils' Tasks
Alice	operational	A formula/ equation	algebraic formulae	-Translation activities involving switching from one representational form to another -Determine range and domain of given functions -Use definition of a function to distinguish functions from non-functions
Ben	operational	A rule of correspondence	algebraic formulae	-Define a function -Evaluate the output of a function when given the input -Determine the domain and range of given functions - Identify the dependent and the independent variables in the functional relationship.
Chipo	operational	A dependence relation	algebraic models	- Identify the domain and range of the functions they would have formulated
Daniel	operational	A rule of correspondence	algebraic formulae	-Identify the independent and the dependent variables in the given algebraic formulae -Use the definition to distinguish functions from non functions
Edith	operational	A correspondence	algebraic models	-Derive functional relationships from everyday practices
Fari	operational	A dependence relation	algebraic formulae	-Identify the independent and the dependent variables in the given functional relationships -Evaluate the input when given the output of a given formula Evaluate the output when given the input of a given formula

The prospective teachers' operational conception of a function manifested itself as they were planning their hypothetical lesson plans for use with 'A' level learners. Table 6.4 shows that the prospective teachers placed undue emphasis on computational activities over the

problems of interpreting graphs. For all the six prospective teachers their conceptions and images of a function as a teaching-learning object influenced what they considered important for the learner and affected their use of the modes of representations in teaching the concept.

6.4.3 Influence of teachers' ways of knowing the concept of a function on their pedagogical content knowledge.

The results seem to show that the prospective teachers' cognitions of the concept have a bearing on how they intended to teach it. For example, the prospective teachers in this study were not very flexible in translating from the numerical to the graphical representation of a function. As a result this inflexibility in using different representations was evident in the way they intended to teach the concept. This result collaborates with Even (1998) assertion that flexibility in moving from one representation to another is intertwined with flexibility in using different approaches in teaching functions.

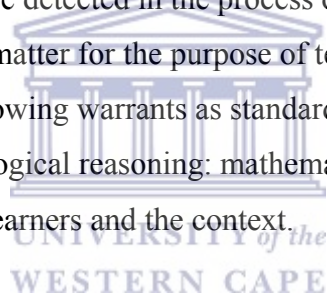
Evidence coming from the task-based interview indicated that the prospective teachers' knowledge of the definitions of a function and their concept images of a function were compartmentalized. This compartmentalization phenomenon was evident when prospective teachers classified some mathematical object as non-functions on the basis of their concept images of a function although the same mathematical object could have classified as a function on the basis of the definition of a function they would have formulated. The prospective teachers compartmentalized knowledge of a function seemed to have had an influence on how they intended to teach the concept to the 'A' level class. For example, although they intended to teach the modern definitions of a function, the examples used to illustrate and work with the concept were exclusively functions given by a formula.

6.5 Prospective teachers' pedagogical reasoning on functions

In the last interview the prospective teachers were asked to plan a hypothetical teaching sequence for the concept of a function. The hypothetical lesson plans were analysed with the intention of identifying aspects of the prospective teachers' intended actions that seemed to be significant in the sense that they could be construed to be informed by the prospective teachers' knowledge of the concept of a function or mathematics pedagogy. An analysis of the lesson plans revealed the following nine aspects of the teachers' intended actions:

1. Adherence to the textbook (ATD)
2. Use of context (UC)
3. Choice of examples (CE)
4. Making connections (MC)
5. Choice of representations (CR)
6. Teacher demonstration (TD)
7. Pupil Demonstration (PD)
8. Conception of concept of a function emphasised (CCFE)
9. Concentration on procedures (COP)

In the interviews the prospective teachers were expected to provide reasons for their decisions. In other words the aim of the interview was to assess prospective teachers' pedagogical reasoning. Llinares (2003) used the term pedagogical reasoning to portray the transformation of content knowledge for the purposes of teaching. He argued that the influence of one's conception of a mathematical concept and the mental images of how the concept should be taught could be detected in the process of pedagogical reasoning, i.e., when one transform the subject matter for the purpose of teaching and give arguments about it. Ball (2000) identified the following warrants as standards for judging the worth of products and processes of pedagogical reasoning: mathematical warrants, warrants based on leaning theories, knowledge of learners and the context.



6.5.1 Pedagogical reasons for the choice of functional representations

Different prospective teachers used different representations of a function. Fari was the only one who used the function box as a representation of a function. She justified the use of the function box by saying:

- *the concept of a function is too abstract for learners to grasp unless they are given an object which is responsible for the change to focus on.. The function box also help pupils to understand the algebraic representations such as $x \rightarrow 3x + 1$ or $f(x) = 3x + 1$. .. so the use of the function box enables learners to have a mental image of a factory that can be used to describe and name various processes without the necessity of having an explicit process defined (Fari)*

The citation above show that Fari was being sensitive to pupils' learning difficulties hence the choice of a function box as a representation of a function. The function boxes were

supposed to help the learners to master the algebraic representation of functions. Earlier on Fari had defined a function as a succession of operations. This conception of a function might have influenced Fari to use the algebraic and function box representations of a function. Her image of a function might also have influenced her choice of the tasks for the learners. The learners were expected to evaluate the output given the algebraic formula and the input.

The second prospective teacher Chipo had earlier on defined a function as a dependence relation between two variables. This conception of a function had an influence on how she intended to teach the concept of a function. In her hypothetical lesson plan Chipo emphasized the mathematical use of the word function and the associated mathematical notations. The influence of the conception of a function as a dependence relation was also evident in the use of real world contexts in which one set of variables depended on another set of variables. Chipo placed undue emphasis on functions as models of real life situations and as indicated in her lesson plan pupils were expected to study functional relationships in real life situations. When asked why she preferred to start with real life situations she responded:

- *in practice... when there is a real problem I will have to covert it into a mathematical model and look at the behavior of the function mathematically. For most functions, the real problem does not appear, but it is a model of a possible real problem (Chipo)*

Edith intended to introduce the function by asking the pupils to model a real world problem, i.e., determining the number of handshakes (H) when there are n people at a party. The pupils were as if they were supposed to follow Bruner's stages in representing the functional relationship. The inactive representation of the functional relationship was going to be characterized by pupils acting out the arrival scene, determining the number of handshakes as numbers of the invited people vary. This was to be followed by the symbolic representation, skipping the iconic representation of the functional relationship. This approach was to result in representing the relationship between the number of handshakes (H) and the number of people (n) in a tabular form, followed by deduction of the algebraic equation $H = \frac{n(n-1)}{2}$ which was to be represented graphically. Edith seemed to have

thought seriously about how the lesson was supposed to proceed and had already prepared

the graphical representation of the function $H = \frac{n(n-1)}{2}$. When asked to provide the motivation behind his approach Edith responded:

- I believe that connecting the mathematical topic with real life motivates pupils. The approach also helps pupils to appreciate that mathematics is a way of understanding the world - for example, the table of value, the graph and the algebraic formula are different ways of modeling the party scenario. Presenting the three modes of representation helps pupils to understand the links between the modes of representation (Edith)

Edith's response show that she deliberately intended to use the three modes of representing the function in order to motivate the pupils since she described the modes as different ways of modeling the scenario. The use of the three modes was probably motivated by the need to ensure that there is coherence between the three modes of representing a function.

One conception of a function which is emerging from the prospective teachers' responses is that functions are perceived as models which are being used for organizing the physical world. In Chipo and Edith's approaches functions are introduced as models of relationships. In fact, this is how the concept of a function came into being historically. The notion of a function can be regarded as a result of the human endeavour to come to terms with changes observed and experienced in the surrounding world (Sierpiska, 1992).

Ben emphasized the algebraic representation of a function for the single reason that

- almost all functions studied at 'A' level are given in equation form.... The syllabus requires pupils to determine the domain and the range, differentiate and integrate function given in algebraic form (Ben)

Teachers in Zimbabwe cannot ignore the demands of the syllabus and the examination system. Previous studies (Nyagura, 1992) have shown that teaching in the schools was examination oriented. Alice emphasized the algebraic and the graphical representation of a function and the pupils' ability to translate from the algebraic to the graphical representation of a function. The choice of these modes of representation of a function was influenced by her strict adherence to the textbook. Although teachers are not supposed to follow the

textbooks, Alice was slavishly following the textbook examples in which the algebraic and the graphical modes of representing the function were common.

6.5.2 Pedagogical reasons for the pursuit of mathematical connections

One interesting observation made when analysing the lesson plans was that the prospective teachers made very little use of concrete learning aids. In almost all cases the teaching of the concept of a function was based on what learners were bringing to the learning of the concept of a function. The prospective teachers referred to the knowledge, which learners were bringing to the learning of the concept of a function, as the assumed knowledge. There was a deliberate effort by the prospective teachers to connect the learning of the concept of a function to the stated assumed knowledge. Huckstep et al (2002) remarked that rich mathematical connections – just like the selection of examples- can be determined pre-actively in the planning phase of teaching but were quick to remind us that a mathematically informed teacher should take advantage of the pupils' responses to make connections interactively.

The prospective teachers put forward the following reasons for basing their teaching of the concept of a function on the stated assumed knowledge:

- *mathematical concepts are hierarchical in which the learning of new concepts is based on previously learned concepts. Since the concept of a function is defined on sets of variables-using sets of variables would be a good starting point when teaching the concept of a function (Alice)*
- *pupils would find it easy to grasp the concept of a function if we relate the concept of a function to functional relationships they have met in their everyday experiences (Fari)*
- *learners have drawn graphs and have calculated the output given an input and a formula connecting the two sets of variables. Teaching the concept of a function will be a question of giving a name to a familiar process i.e., plugging in numbers in a formula to get an output (Chipo and Edith)*
- *contexts like studying the relationship between the yield and the amount of fertilizer used to produce the yield provide an easy gateway to the leaning of the concept of a function – such a context help the learners to appreciate the*

usefulness of the mathematical concept being learnt thereby motivating learners to learn (Ben)

- *pupils understand a new concept such as the concept of a function better if they can link it to what they already know (Daniel)*

Alice's reason for linking the teaching of the concept of a function to other mathematical concepts such as sets of variables is based on her knowledge of the hierarchical nature of mathematical knowledge. On the other hand Fari and Ben believe that the use of familiar contexts enhances pupils' understanding of new concepts hence the need to contextualise the teaching of the concept of a function. However the decision to contextualise the teaching of the concept of a function did not stem from the fact that the concept of a function evolved as early mathematicians were resolving their problems such as the vibrating string problem. Ben's response also seems to imply that the use a context would motivate the learners to learn. Daniel's reason for linking the learning of new concepts to previously learnt concepts is based on his knowledge of learners whereby learners are said to learn by assimilating new knowledge to what they already know. Skemp (1971) referred to the learning approach described by Daniel as schematic learning.

The pursuit of mathematical connections in mathematics teaching has intensified in recent years. Askew et al (1997), for example, have singled out teachers with the so called "connectionist orientation" as those who are more likely to be effective teachers of numeracy than those with other beliefs about teaching and the nature of mathematics. Ball (2000), for example, argued for this vital element of teaching and goes on to say that it is imperative that teachers must appreciate and understand the connections among mathematical ideas.

6.5.3 Pedagogical reasons for the choice of examples

Two distinct different uses of examples in teaching the concept of a function were noted. The first was inductive – providing examples of a function. The examples were particular instances of the general notion of a function. All the prospective teachers intended to start the lesson by giving the pupils the definition of a function followed by examples and non-examples of a function. Fari intended to refer to the definition from the 'A' level mathematics textbook and then explain the qualifiers 'to each' and 'exactly' in the definition of a function. On the other hand Chipu wanted to use examples to clarify the mathematical

use of the word function from its everyday usage. Ben and Daniel intended to define a function and then cite examples from real world contexts in which the dependent variable was a function of the independent variable. When asked to justify the logic behind her approach Alice gave a response which is atypical of all the other prospective teachers' responses:

- *for learners to understand the concept of a function they should understand that it is made up of the domain, the range and a mapping which maps every element in the domain to a unique element in the range. The non-examples are meant to show that the function ceases to exist either when some elements in the domain are not paired with elements in the range or when an input is paired to more than one output. These examples act as building blocks of the concept of a function.*

The above responses show that the choice of these examples was influenced by the prospective teachers' knowledge of the concept of a function and its sub-concepts (range, domain, and the uniqueness property). These examples and non-examples were meant to provide or facilitate abstraction of the concept of a function. Rowland et al (2002) pointed out that a set of examples could be unified resulting in the formation of a concept which can assimilate subsequent examples of that concept. In other words once a concept has been formed and named by an individual as a result of noting the invariant features from the given examples, he or she is able to entertain examples of the concept outside the realm of personal experiences. Skemp (1979) called this psychological phenomenon reflective extrapolation.

The second set of examples chosen by the prospective teachers were what are often called exercises or tasks. These tasks were formulated to assist the student to master a given procedure or the notion of a function and also to develop fluency with the procedure. The demands of the 'A' level mathematics examinations seemed to have a significant influence on the prospective teachers Alice, Chipo and Ben's choices of the practice exercises planned for the pupils. These prospective teachers planned tasks in which learners were expected to determine the domain and range of given functions. In order to get the solution to the tasks one has to evoke previously learnt ideas such as:

- negative numbers do not have square roots
- division by zero is undefined
- solution of inequalities

The following episode illustrates the reasons behind the choice of the tasks

Researcher: Was the choice of these tasks deliberate? If so, can you give reasons why you have these tasks

Ben: Tasks like ‘Determine the domain and range of the functions $y = 3 + \sqrt{4 - x^2}$ and

$$y = \frac{x^2 + x + 5}{x^2 - 3x - 4}$$

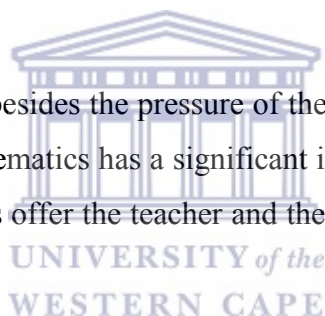
are very popular in the ‘A’ level mathematics examinations – so

these tasks are meant to prepare students for the exam.

Researcher: Is that the only reason?

Ben: The tasks also give me the opportunity to integrate the new work on function with the other topics – for example, in finding the domain of the first function learners have to use the fact that $4 - x^2$ is never negative resulting in the need to remember how the inequality $4 - x^2 \geq 0$ - a topic they would have learnt before. Similarly the learner have to remember that division by zero is undefined when finding the domain of the second function.

The responses above show that besides the pressure of the examination system an awareness of the integrative nature of mathematics has a significant influence on the choice of the tasks planned for the learners the tasks offer the teacher and the pupils to link the new work to the earlier concepts.



6.5.4 Pedagogical reasons for the different levels of teacher and pupil involvement

The hypothetical lesson plans prepared by the prospective teachers showed that at one time or the other either the teacher or the pupils were expected to demonstrate a skill or one's understanding of the concept of a function, pose a question, respond to a question or participate in a discussion. The hypothetical lesson plans could be segmented into three distinctive and readily identifiable phases. In the first phase, the introductory phase, the prospective teachers intended to revisit work which pupils would have learnt prior to the learning of the concept of a function. This work was viewed as some good starting point when introducing the concept of a function.

The second phase which the prospective teachers referred to as the lesson development phase, typically started with the introduction of the definition of a function by the prospective teacher, followed with some examples and non-examples of a function after

which pupils were expected to work on tasks given by the teacher. In the final stage, the lesson closure, the prospective teachers expected pupils to take a leading role in summarizing the main ideas covered during the lesson.

During the interview the prospective teachers indicated that at the introductory phase they were more interested in assessing the mathematical knowledge which their learners were bringing to the learning of the concept of a function. However, there was no evidence that this knowledge was used at the lesson development stage. The lesson development stage was characterized by teacher exposition – giving the definition of a function followed by examples and non-examples of a function. All the prospective teachers viewed teaching as telling. From the position, these prospective teachers held an image that mathematics teachers had the knowledge and the responsibility for transmitting it and that their pupils would assimilate it without any difficulty as shown in the following assertion about the most important thing for teaching the concept of a function

- *the first thing is that I have to know what a function is and its features...and then know how to transmit it to pupils ...after defining the function the examples and non-examples I give should help the pupils to understand the idea of a function (Daniel)*
- *to start off, I would explain what a function is then I would go on to linear functions (Fari)*

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The above prospective teachers' assertions were consistent with the espoused teacher behaviors as documented in their hypothetical lesson plans. The lesson plans show that the verbs which the prospective teachers used to describe their behaviors were-explaining, illustrating, defining and demonstrating. For them learning was a question of knowing the information previously provided by the teacher. From this perspective the prospective teachers were going to attribute the pupils' inability to define and highlight the key features of a function to

- *lack of grasping the notion of a function (Chipo)*
- *lack of paying attention to what I would have taught (Daniel)*

The hypothetical lesson plans also show that the learners were expected to respond to questions which required them to provide factual information or to tasks which required them to follow well laid out steps which would have been demonstrated by the teacher

6.5.5 Decisions informed by pedagogy versus pedagogy-free decisions

In the theoretical framework of the study it was pointed out that the syntactic structure of pedagogical content knowledge consisted of the warrants for judging the worth of products and processes of pedagogical reasoning that underline the choice of representations and examples when teaching mathematical concepts. What this implies is that teachers make decisions which are based on sound pedagogical reasons. However, the current study shows that three of the six prospective teachers made content decisions, which were influenced by the structure of the examination or by the textbooks in use. I have referred to such decisions which are not influenced by pedagogy as pedagogy-free decisions.

These decisions were made at the planning stage although one would have expected the prospective teachers' decisions to be informed by their knowledge of mathematics pedagogy. If these teachers are going to teach for examinations when they graduate, they are likely to make more pedagogy free decisions. For example, in Zimbabwe there are school based subject panels which are made up of the subject specialists whose role is to determine how specific subjects could best be presented to learners. Besides, the school-based subject panelists monitor the quality of teaching at the respective schools. In most cases, the school based panelists measure the performance of the teachers in terms of percentage of pupils who would have passed the external examinations. Pupils are interested in passing the examination and they constantly sent messages to the teachers that they want to be shown how to solve different problems which are likely to be examined. The scenario described above entails that mathematics teachers are forced to drill learners in preparations for examinations

If these prospective teachers are going to teach for procedural understanding of the concept of a function pupils might just barely pass their 'A' level examinations. However there may be long-term drawbacks. The learners may become more procedural and might not develop a process conception of a function, let alone the procept conception of a function. Failure to reify the concept of a function implies that the learners are likely to suffer from cognitive overload leading to eventual failure in further mathematical studies in which the concept of a function is assumed to be a unifying idea. The reason for subsequent failure by learners in further mathematical studies is explained by Tall's (2001) notion of proceptual divide which

allude that learners who rely on procedures are doing much more difficulty mathematics than those who rely on precepts.

6.6 Conclusions

The results of the study show that:

1. The prospective teachers have compartmentalized their knowledge of the definitions and their concept images of a function. Although they could state versions of the set-theoretic definitions of a function, they did not use their definitions in deciding whether a mathematical object was a function or not. Rather they relied on their concept images of a function to distinguish functions from non-functions sometimes resulting in inconsistent behaviors on the part of the prospective teachers.
2. Prospective teachers' conceptions and images of a function as a teaching-learning object influenced what they considered important for the learner and affected their use of the modes of representing the concept to learners.
3. The process conception of a function was more prevalent among the prospective teachers than other conceptions of a function. This was the case irrespective of the representational form used to represent a function. Besides, the prospective teachers found translating from the numerical to the graphical representation of a function was more challenging than translating from the graphical to the numerical representation.
4. Prospective teachers made pedagogical decisions which were informed by their knowledge of the Nature of Mathematics, knowledge on how learners learn mathematics and the contexts in which functions are can be used as tools for modeling real world situations. However some decisions were influenced by the demands of the examination structure and the textbooks being used in schools.

CHAPTER 7

DISCUSSIONS, CONCLUSIONS AND RECOMMENDATIONS.

7.1 Introduction

This study was an investigation of the nature of six Zimbabwean prospective teachers knowledge for teaching the concept of a function. The six undergraduate prospective teachers were majoring in mathematics with the intention of completing a programme leading to certification as secondary mathematics teachers. At the time of the study they were in their third and final year. The study was meant to answer the following research questions:

- What are the prospective teachers' understandings of the definition of a function?
- What is the breadth and depth of prospective 'A' level mathematics teachers' understanding of the concept of a function?
- What is the influence of prospective teachers' conceptions and images of a function on their pedagogical content knowledge?
- What are the prospective teachers' warrants in the process of formulating representations, examples, definitions of a function as they plan to teach the concept to an 'A' level class?

7.1.2 Prospective teachers understanding of the definition of a function

Task-based interviews were used to assess the prospective teachers' understanding of the definition of a function. Using the different conceptions of a function (a function as a procedure, a process, an object, a procept or pseudo-structural) and the associated concept images (a rule, a formula, a dependence relation or an equation) outlined in the theoretical framework developed in Chapter 3, it was possible to characterise prospective teachers' understanding of the definitions of a function, the extent to which they could use their definitions and their understanding of the properties of a function.

The results of the study revealed that the prospective teachers defined a function either as a dependence relation or correspondence (Chipo and Edith), a formula (Fari) an operation (Alice) or a rule (Ben and Daniel). In their formulation of the definition of a function the prospective teachers explicitly stated the univalence property of a function i.e., that for each element in the domain there exist a unique image in the range. However along the same lines with other studies (Vinner, 1991, Even, 1992 and Tall, 1999), the prospective teachers did not use this important property of a function as a criteria for checking whether a given mathematical situation was a function or not. Rather the prospective teachers relied on their concept images of a function to determine the functionality of a mathematical object. For example, Ben and Daniel said that if there is no clearly defined formula for getting the output for a given input then there is no function. According to Bakar and Tall (1992) students gain their impression of what a function is from its use in the curriculum, implanting deep-seated ideas which may be at variance with the formal definition. Therefore one possible reason why prospective teachers in this study developed such a conception is that the examples used to illustrate and work with functions in their calculus course are exclusively functions whose rule of correspondences is given by a formula.

Five of the six prospective teachers had not yet generalized their function schema. As a result, just like the 18th and 19th century mathematicians these prospective teachers still think that functions are defined on sets of real numbers. According to Dubinsky (1991) an individual is said to have generalized the notion of a function if he or she has mastered the arbitrariness property of the concept of a function. The arbitrary nature of the concept refers to the nature of the inputs and outputs. The arbitrariness property of a function implies that the functional relationship does not necessarily have to be defined on numbers only but can also be defined on other mathematical objects such as vectors or other functions as long as these mathematical notions have been reified by the individual into objects.

Also some of the prospective teachers (Daniel, Edith and Fari) had a tendency of identifying a function with its representation such as an equation or an algebraic expression. The conception of a function is referred to by Sfard (1991) as a pseudo-structural conception of a function. Euler seemed to have had a similar conception of a function since he too referred to functions as 'analytical expressions' (Malik, 1980).

7.1.3 Prospective teachers' breadth and depth of prospective 'A' level mathematics teachers' understanding of the concept of a function

The second research question focused on prospective "A" Level mathematics teachers' understanding of the concept of a function in terms of its breadth and depth. To understand prospective teachers' breadth and depth of understanding of the concept, an analytical model developed by De Marois and Tall (1996) in which the representational forms of the concept of a function (numeric, symbolic, tabular, verbal, algebraic, graphical) are represented as sectors of a dish in which movement towards the centre is seen as representing levels of compressing the concept through the pre procedure, procedure, process, object and procept layers. Task-based interviews were used to measure the breadth and depth of prospective "A" Level mathematics teachers' understanding of the concept of a function. The breadth of understanding the concept of a function was assessed through their ability to translate among the various representational forms while their depth of understanding was assessed by determining their level of compression through the pre-procedure, procedure, process and object layers.

The findings of the study were that:

1. The process conception of a function was more prevalent among the participants than the other conceptions of a function. This was the case irrespective of the representational form used to represent a function
2. Translating from the numerical to the graphical representation of a function was more challenging to the prospective teachers than translating from the graphical to the numerical representation

Schwarz and Hershkowitz (1999) attribute the difficulties in translating from one representational form to another to the fact that different representations of a function have different properties for mathematical work with functions. In other words, two different representations of a function exhibit different properties or features of a function. For example, information about a function given in graphical form is necessarily always partial because of the choice of the view window and limited precision of the graphical tools. Similar remarks apply to the numeric representations. On the other hand one can deduce false properties about the function from its graphical representation, e.g., if a function has the zero limit at infinity its derivative have the same zero limit. . Similarly, in a numeric table

one can see only a few values and one could infer that the function is linear or has an extreme value even when this is not true. Continuity cannot be induced in a numeric table as it is on a graph.

7.1.4 Influence of prospective teachers' conceptions and images of a function on their pedagogical content knowledge.

The other purpose of the study was to investigate whether the prospective teachers' conceptions and images of a function was going to influence how they intended to teach the concept to an 'A' level class. Task-based interviews were used to capture prospective teachers conceptions and images of a function. The second source of data to answer the same research question was the reflective interviews data based on the prospective teachers' reactions to hypothetical lesson plans.

The findings corroborate the previous researchers' (Sanchez and Llinares, 2003) findings that teachers' conceptions of the nature of the mathematical content influences the way they would approach the teaching of the mathematics. For example, all the prospective teachers' examples of a function were all given in algebraic form. As a result of this they gave a priority role to algebraic representations and computational activities over the problems of interpreting graphs. Fari and Edith incorporated the use of the algebraic equation in modeling real situations. Daniel and Ben emphasized the algebraic representation as defining a rule of obtaining the output for a given input. This influenced these prospective teachers' organization of the content and the type of problems chosen in the teaching sequence. For all the six prospective teachers their conceptions and images of a function as a teaching-learning object influenced what they considered important for the learner and affected their use of the modes of representations in teaching the concept.

Even (1998) considered that flexibility in moving from one representation to another is intertwined with flexibility in using different approaches in teaching functions. In the case of the prospective teachers in this study they were not very flexible in translating from the numerical to the graphical representation of a function. As a result this inflexibility in using different representations was evident in the way they intended to teach the concept

7.1.5 Prospective teachers' warrants in the process of formulating representations, examples, definitions of a function as they plan to teach the concept to an 'A' level class.

In the last interview the prospective teachers were asked to plan a hypothetical teaching sequence for the concept of a function. The hypothetical lesson plans were analysed with the intention of identifying aspects (e.g. examples and representations used, conception of a function emphasised) of the prospective teachers' intended actions that seemed to have been informed by the prospective teachers' knowledge of mathematics or pedagogy. In the interviews the prospective teachers were expected to provide reasons for their decisions.

Some of the reasons put forward by the prospective teachers for their actions include:

- Conceptualize the teaching of the function in order to motivate the learners to learn
- Use the knowledge learners bring to the learning of the function because that knowledge base is the key to the learners' subsequent understanding of the concept of a function.
- Use the graphical representation because they will use it later
- Give the pupils these tasks because they are popular in the 'A' level examinations

While the prospective teachers draw on their knowledge of the learners and the hierarchical nature of mathematical concepts when deciding what and how to teach, the examinations and the textbooks in use also seem to influence the aspects of the function being emphasized.

7.2 Conclusions

In the theoretical framework of the study it was pointed out that the syntactic structure of pedagogical content knowledge consisted of the warrants for judging the worth of products and processes of pedagogical reasoning that underline the choice of representations and examples when teaching mathematical concepts.

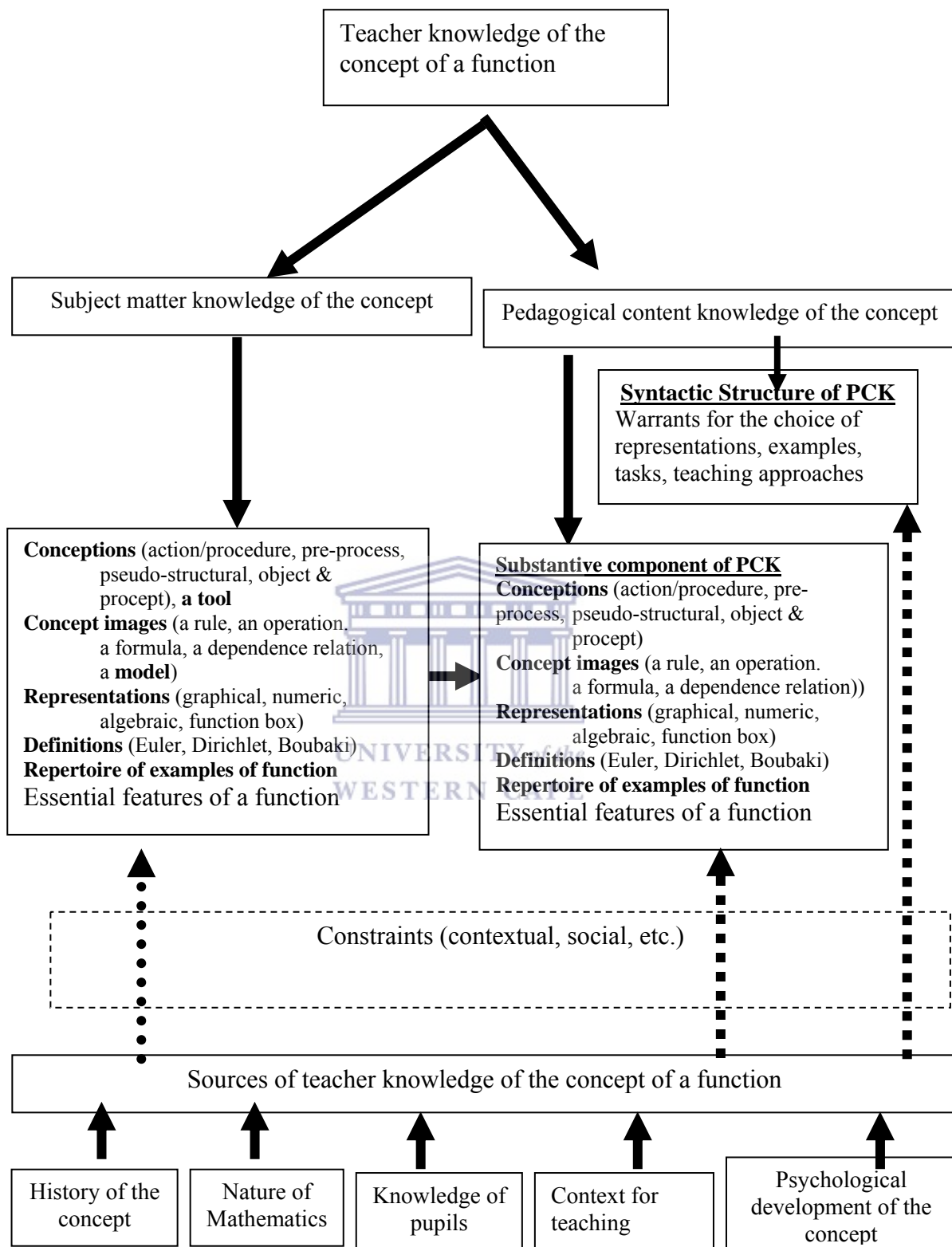
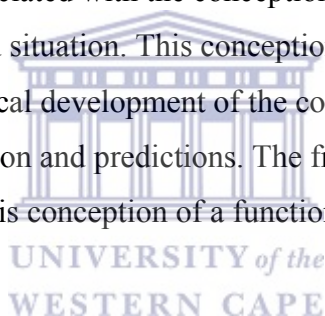


Figure 7.1 Revised schematic summary of the theoretical framework of teachers' knowledge of the concept of a function

However, the current study shows that three of the six prospective teachers made content decisions, which were influenced by the structure of the examination or by the textbooks in use. The examination structure which was found by the prospective teachers to be over-examining specific skills involving the concept of a function was seen as if it were providing clues as to which skills were to be emphasized in their teaching. What this implies is that these particular prospective teachers' knowledge for teaching is not being informed by the different knowledge bases outlined in the theoretical framework. Such contextual constraints act as barriers to these sources of knowledge. The existence of such barriers has necessitated an adjustment to the theoretical framework as shown in figure 7.1 above.

7.2.1 The notion of a function as a model

One conception of a function, which came up during the reflective interviews, was that of a function as a tool for describing and making sense of real world functional relationships. The concept image of a function associated with the conception of a function as a tool is that of a function as a model of real world situation. This conception is not peculiar to the prospective teachers under study. The historical development of the concept of a function shows that functions were tools for description and predictions. The framework in Figure 6.1 has been adjusted accordingly to reflect this conception of a function.



7.3 Implications for mathematics teacher education in Zimbabwe

Mathematics teacher educators in Zimbabwe should seriously consider the possibilities of

1. Using task –based interviews to understand prospective teachers' understandings of mathematics
2. Developing a new mathematics teacher education programme

7.3.1 Use of task –based interviews to understand prospective teachers' understandings mathematics

The common assessment practice in mathematics teacher education courses in Zimbabwe is through written tests. These tests are designed to assess students' abilities to execute routine skills or their abilities to demonstrate a pre-identifiable set of understandings. However, the current study has shown that students' answers to a seemingly recall question like '*What in*

your opinion is a function?' do not always reveal their true level of understanding. The prospective teachers were able to regurgitate the correct formulations of the definition of a function but I could not figure out their understandings of the concept of a function from their responses to the question. It was only after probing using task-based interviews that I was able to understand the prospective teachers' cognitions of the concept. Task-based interviews could be used by mathematics teacher educators who are interested in understanding more deeply the nature of their students' understandings of selected mathematical concepts and ideas and to assess the depth and breadth of students' mathematical understandings.

Task-based interviews may reveal not only what the interviewee is thinking about a piece of mathematics but also why his or her thinking is reasonable to himself or herself.

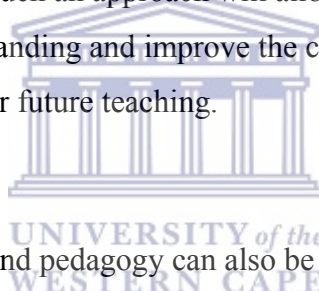
Mathematics teacher educators in Zimbabwe are encouraged to make use of task-based interviews since they reveal a variety of aspects of students' mathematical understandings that are not visible through many other methods of assessment. As mathematics instructors learn more about their students' mathematical understandings through interviews, several benefits arise. First, the experience of designing and conducting task-based interviews can help mathematics teacher educators to listen with a new attention and ability to focus on the student's personal interpretations and ways of thinking. Second, a mathematics teacher educator who is aware of possible pitfalls in students' reasoning can construct examples that are likely to pose cognitive conflicts for students as they struggle to refine the ways they are thinking about particular aspects of mathematics. These cognitive conflicts are helpful in inducing a more useful and robust way to think about the concept in question. However, key to success in using task-based interviews is a deep-seated belief in the mathematics teacher educator's mind that each student's understanding is unique and that this understanding is best revealed through open-ended questions and related probes.

7.3.2. The need to develop a new mathematics teacher education programme

The structure of the current mathematics education programme is such that the content courses and the mathematics education courses are taught in two different departments in the university with more time being devoted to content courses than mathematics education courses. In Bernstein's (1996) terms, there is undue emphasis on the vertical development of mathematics at the expense of its horizontal development. In designing a new mathematics

teacher education programme, serious consideration should be given to the idea of integrating mathematics content and pedagogy, with a significant component of that integration consisting of activities that encourage teachers to reflect on their own views of mathematics and mathematics teaching while actively exploring important mathematical concepts and processes that they will be required to teach. One possible approach that can be used to integrate mathematics content and pedagogy is by using teaching scenarios

This study has shown that use of teaching scenarios can engage prospective teachers in activities in which they interpret and make mathematical and pedagogical judgements about students' questions, solutions, problems and insights. Ball et al (2004) describe the above mathematical practices as elements of the specialised mathematical problems teachers solve as they teach mathematics in schools. Llinares (2004) remarked that analysis of teaching scenarios during teacher training could be considered as an effective strategy for developing a knowledge base for teaching. Such an approach will allow teachers to make important connections in their own understanding and improve the chances that such an integrated approach will be reflected in their future teaching.



The integration of mathematics and pedagogy can also be achieved if mathematics content is taken as a context for the study of pedagogical issues (e.g., discussion about different ways of representing specific mathematical topics, their strengths and limitations linked to aspects of the concept emphasised). In this context prospective teachers can discuss and evaluate the multiple representations linked to different approaches of teaching a specific concept. Also the psychology, sociology and philosophy courses offered to prospective teachers should be fine tuned so that they relate to the psychology, sociology and philosophy of mathematics education

7.4 Implications for teaching the concept of a function in teacher education programmes.

In this section highlights some of conditions, which should be fulfilled when teaching the concept of a function

7.4.1 Use of multiple representations

The study showed that the prospective teachers tended to fixate on one particular representation, the algebraic representation and yet .an emerging theoretical view on mathematical learning that has been growing in significance is that multiple representations of concepts can help students to develop deeper, more flexible understanding of mathematical concepts (Skemp. 1987). Therefore at tertiary level it is important for mathematics teacher educators to provide prospective mathematics teachers with a broad spectrum of ways of teaching functions, speaking about functions (e.g., mappings, transformations, etc.) and representing functions in order to prevent exclusive identification by students of any one of these representations of functions. Students should be given an opportunity to acquire certain flexibility in using these modes of expressing and representing functions. Confrey (1991) proposed an epistemology of multiple representations. According to his theory it is through the interweaving of one's actions and representations that one construct mathematical meaning. The algebraic, numeric, statement of contextual problems and spoken language taken together should improve the learners' understanding of the concept of a function although each of these representations views a function from a particular perspective that capture some aspects of the function well leaving other aspects less clear.



7.4.2 Use of definitions in concept formation

Currently the practice at tertiary level is to introduce new concepts such as the concept of a function to students through definitions in the hope that students would construct the concept through deduction. However the prospective teachers in this study did not use their definitions when deciding whether a given mathematical object was an example or non-example of a function. The idea of giving a verbal definition of a function as a list of criteria and expecting students to construct the concept from the definition is a reversal of how the concept evolved. In order to ensure that mathematics is accessible to as many students as possible, mathematics educators have to provide students with many examples that form the desired concept image not only at the beginning but also throughout the whole period of learning.

7.4.3 Promoting the operational conception before the structural conception

The claim about the developmental precedence of operational conception over structural conception implies that certain kinds of instructional actions, however natural and legitimate in the eyes of the teachers, should in fact be carefully avoided. Two didactic principles can be formulated regarding the things that should not be done.

First, new concepts such as the concept of a function should not be introduced in structural terms. Dubinsky's (1991) model of concept formation implies that it would be of little or no avail to throw unfamiliar abstract objects upon the students without giving them time and means to prepare them for the structural conception by building a sound operational base. In the case of the concept of a function the process conception of a function should be developed in the learners before they are expected to view a function as an abstract mathematical object.

Second, a structural conception of a function should not be required as long as the students can do without it. Since the structural conception of a function must precede an attempt to perform a higher level manipulation, such as combining functions which form a Hilbert Space teachers must not expect "A" level students to master the structural conception of a functions since at that level students do not perform higher level manipulations on functions. Also before a real need arises for regarding the concept of a function as an object, students may lack the motivation for putting up with the new intangible mathematical object called a function. As long as Analysis concepts appear nowhere in the "A" level Calculus course the student can do quite well in the Calculus courses with an operational conception of function alone. An operational conception, namely viewing a function as a process, is sufficient for dealing with differentiation and integration. Although the prospective teachers are taught the structural modern definition of a function in their Analysis course at undergraduate level, there is need to inform them that introducing the concept of a function set-theoretically as a particular kind of relation to an "A" level class is little justified from both didactical and epistemological points of view.

7.5 Implications for further studies

At the time of the study the prospective teachers had not taught in schools except for the sixteen-week teaching practicum. Now that these teachers have been deployed in schools

other researchers might be interested in investigating how these prospective teachers' think about the concept of a function, how they teach and the pedagogical reasons behind their teaching approaches. Research of this nature will hopefully result in the development of a theoretical framework for teaching the concept of a function that is derived from practice.

Mathematics teacher educators are being encouraged to generate a bank of teaching scenarios for use in their lecture rooms thereby bringing the realities of the classrooms into the lecture rooms. At the same time mathematics teacher educators can investigate the possibilities of using task-based interviews for the purpose of grading the prospective teachers.

Three of the prospective teachers indicated that they were going to introduce functions as tools for modeling relationships between variable quantities which depends on one another. The approach is likely to foster a conception of a function as a tool in learners. The learners are likely to perceive functions as models of some real world functional relationships with the algebraic formulae as the only possible form of representing functions. This approach to the teaching of the function of function is consistent with Sierpiska's (1992) assertion that the meaning of a concept lies in the problems and questions that give birth to it. The approach is also consistent with the historical development of the concept of a function whereby, initially, functions were considered as tools for description and predictions. Assuming that the meaning of the concept of a function lies in the real world problems which can be described, studied and understood by using mathematical functional models and that by so doing learners would initially develop a conception of a function as a tool, there is need to carry out some action research to determine whether learners can reify the function as a tool conception to a function as an object.

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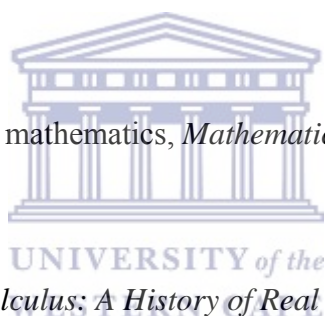
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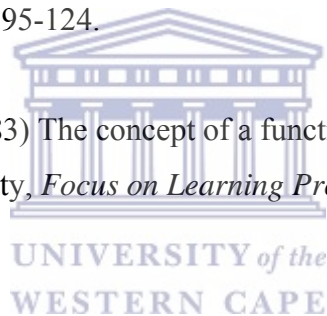
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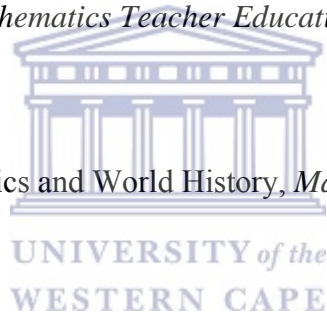
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APPENDIX A:

INTERVIEW GUIDE: TEACHER KNOWLEDGE OF THE CONCEPT OF A FUNCTION

1. Can you give a definition of a function?
2. A student says that he/she does not understand that definition. Can you give an alternative version that might help the student to understand?
3. Give an example of a function?
4. What did you assume the student didn't understand?
5. Why is there, in the definition of a function, the requirement of having only one image for each element in the domain?
6. How are functions and equations related to each other?
7. Can all functions be represented by equations? Why do you say so?
8. Do all equations represent functions? Why do you say so?
9. Can all functions be represented by an algebraic expression? A formula?
10. Which of the following algebraic expressions represents the same functional relationships

$$(a) f(n) = \sum_{k=1}^n (2k-1) \quad k, n \in \mathbb{N} \dots (b) f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

$$(c) f(n) = n^2 \quad n \in \mathbb{N} \qquad (d) f(x) = \max \{ -x, x \}$$

$$(e) \text{ Recursively defined relation } g(x+1) = g(x) + 2x + 1 \quad \text{where } g(0) = 0 \text{ and } x \in \mathbb{N}$$

- 11 Which of the following propositions describe functions? (x and y are real numbers)

$$(a) y = \begin{cases} 2x + 5 & \text{if } x \text{ is even} \\ 1 - 3x & \text{if } x \text{ is odd} \end{cases}$$

$$(b) y = 4$$

- 12 Which of the following sentences is, in your opinion, a better description of the concept of a function?

(i) A function is a computational process which produces a unique value of one variable (y) for a given input value of another variable (x).

(ii) Function is a kind of (possibly infinite) table in which to each value of one variable corresponds a certain value of another variable.

13 True or False?

(i) Every function expresses a certain regularity (the values of x and y cannot be matched in a completely arbitrary manner).

(ii) Every function can be expressed by a certain computational formula (e.g., $y = 2x + 1$ or $y = 3\sin(\pi + x)$)

14 Determine whether the following are functions or not. If they are functions state the range and domain of the function.

(a) $f(x, y) = x + y \quad x, y \in \mathfrak{R}$

(b) $f(x, y) = 2x + 1 \quad x, y \in \mathfrak{R}$

15 One of the differential operators is what you know as the derivative. Instead of writing \dot{y} or $\frac{dy}{dx}$ we just use a D . You may recall this sort of notation from your differential equations

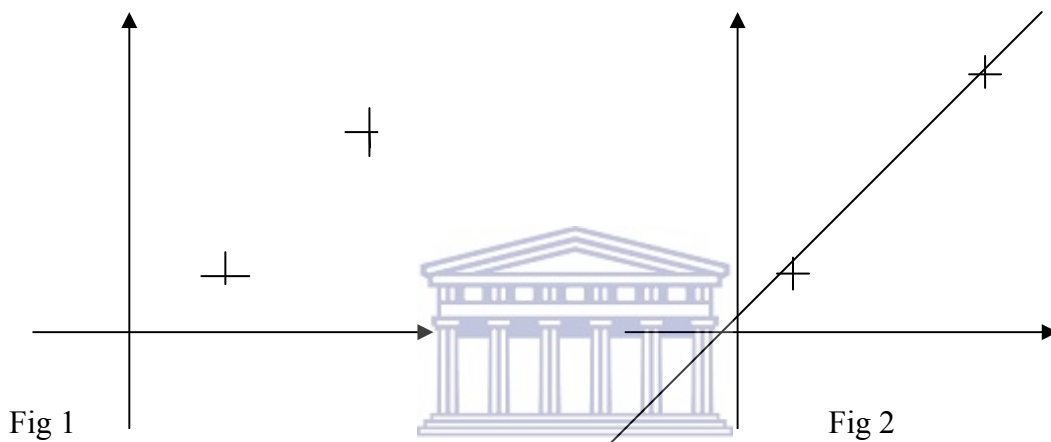
course. For example, $D(x^2) = 2x$ or $D(\cos x) = -\sin x$. Is D a function in any way?



APPENDIX B

INTERVIEWS BASED ON TEACHING SCENARIOS

1. A student is asked to give an example of a graph of a function that passes through the two marked points (see figure 1). The student gives the answer as in figure 2. When asked if there is another answer the student says 'NO'

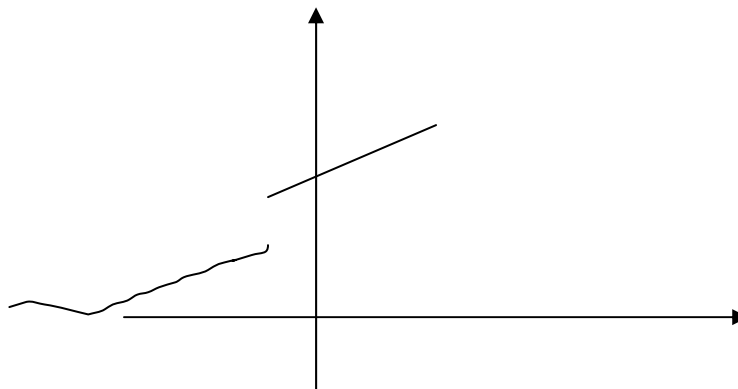


If you think the student is right – explain why

If you think the student is wrong- how many functions which satisfy the conditions can you find?

- 2 A student marked all the following as non-functions.

(i)



(ii) $f(x) = 4$

(iii) A correspondence that associates -1 with each negative number, $+1$ with each positive number and 3 with zero.

iv) $f(x) = \{(1,10);(2,20);(3,31)\}$

For each case decide whether the student was right or wrong. Give reasons for each one of your answers

Do you think there is a formula to describe the graph in (1)?

Can you graph (iv)?

Many students said (iv) is just a set of points and not a function since 3 should be associated with 30 . What do you think?



APPENDIX C.

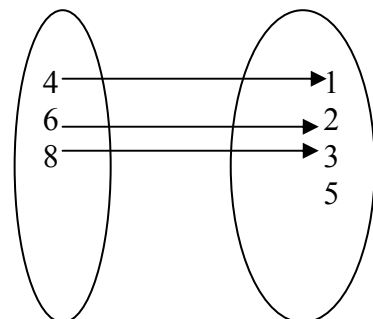
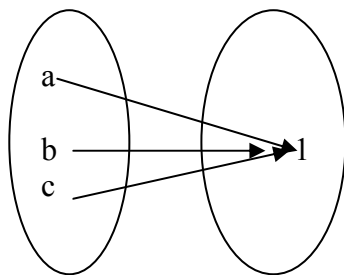
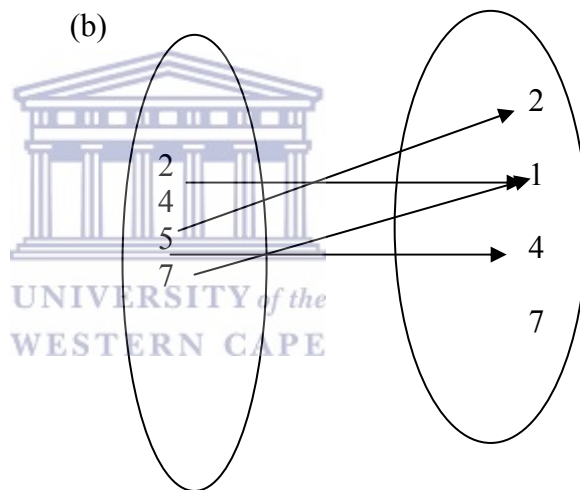
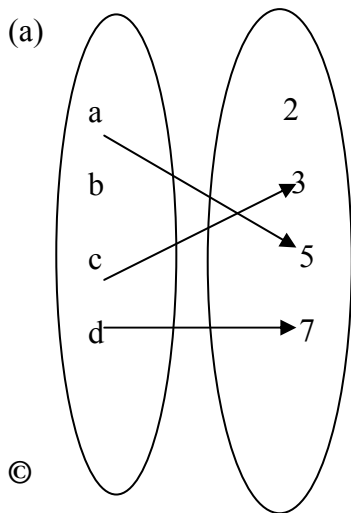
TASK-BASED INTERVIEWS

Different representations of a function

Determine whether each of the below is a function or not. Give reasons for your answers

What misconceptions are likely to develop in pupils when a function is first introduced using each of the functional representations below

Set Diagrams



Formulae

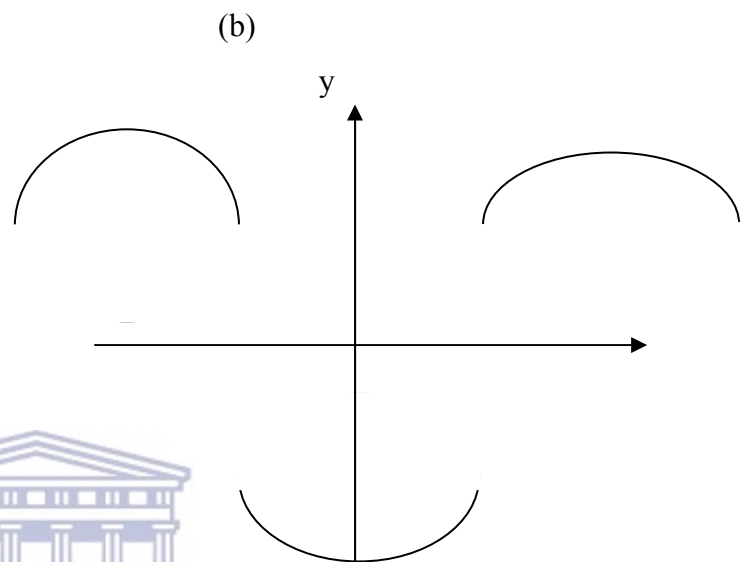
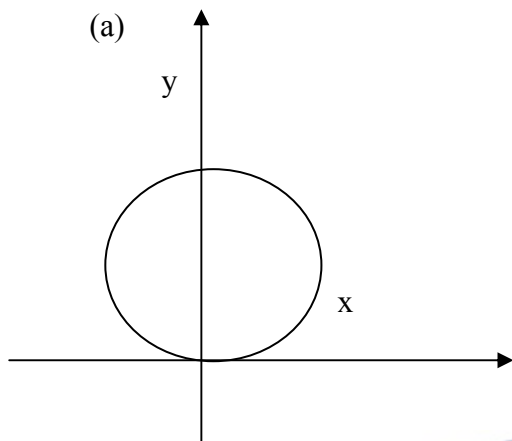
(a) $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = \sqrt{x}$

(b) $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = x^{-1}$

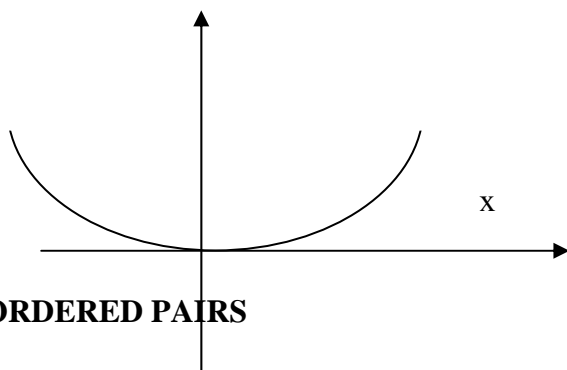
(c) $f_n \rightarrow n^{-1} \quad n \in \mathbb{N}$

(d) $f : \mathbb{R} \rightarrow \mathbb{R}$ where $f(x) = 4$

Graphs



(c)

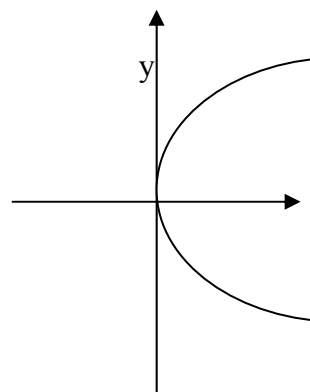


ORDERED PAIRS

(a) $\{(2,4);(3,9);(4,16);(5,20)\}$

(b) $\{(1,3);(2,5);(3,2);(7,-1);(9,1)\}$

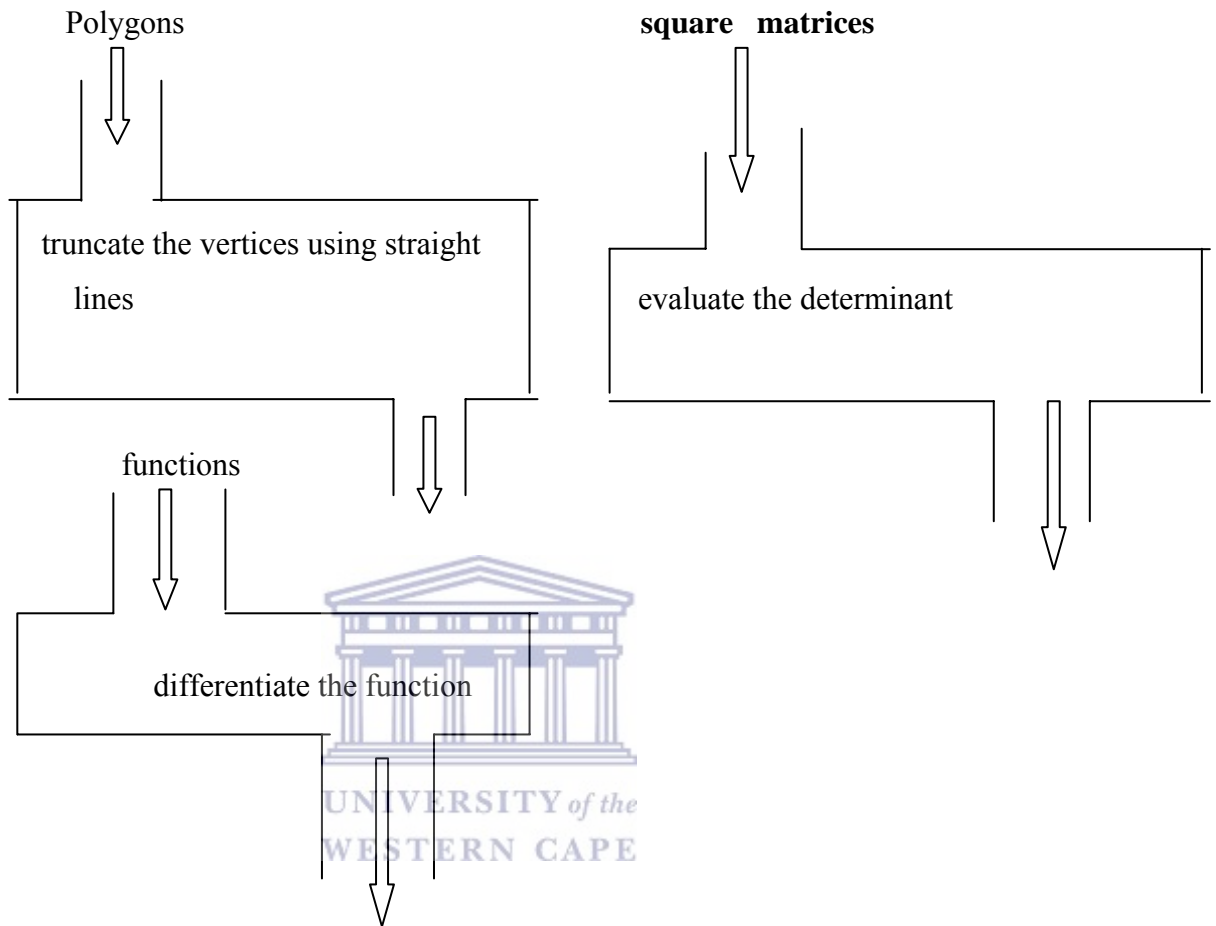
(d)



(c) $\{(1,1);(2,1);(4,1);(20,1)\}$

(d) $\{1,2,3,9\} \rightarrow \mathcal{R}$

Function Machines



Links among functional representations

Given the equation $f(x) = 4$ create the following functional representations in any order of your choice: a table, a graph, a function box, a set diagram.

What information do you lose/ have incidentally added on as you changed from one representational form to another

What assumptions did you make as you changed from one representational form to another

2 Given the table

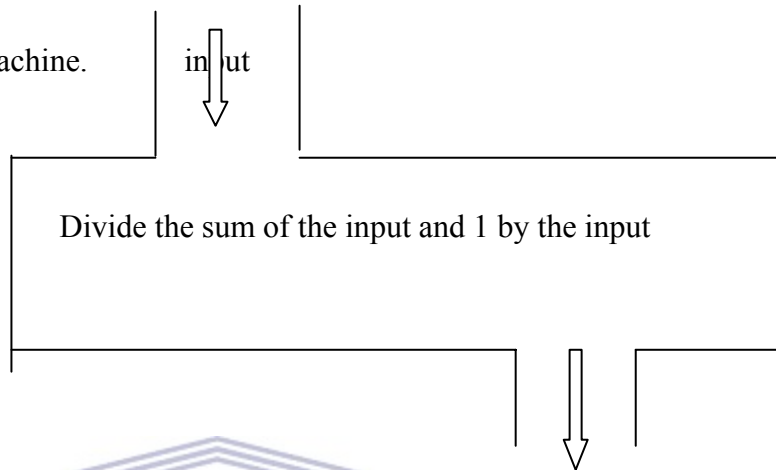
x	-2	-1	0	1
y	-4	-2	0	2

create the following functional representations in any order of your choice: an equation, a graph, a function box, a set diagram.

What information do you lose/ have incidentally added on as you changed from one representational form to another?

What assumptions did you make as you changed from one representational form to another?

3 Given the function machine.



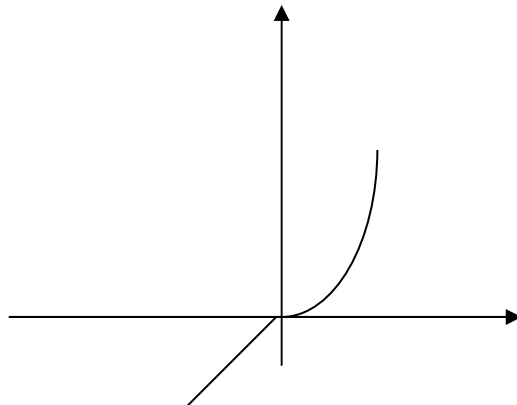
create the following functional representations in any order of your choice: an equation, a graph, a table, and a set diagram.

What information do you lose/ have incidentally added on as you changed from one representational form to another?

What assumptions did you make as you changed from one representational form to another?

What assumption(s) have to be imposed on the nature of the inputs if the function machine is to represent a function?

4 Given the graph



create the following functional representations in any order of your choice: an equation, a table, a function box, and a set diagram.

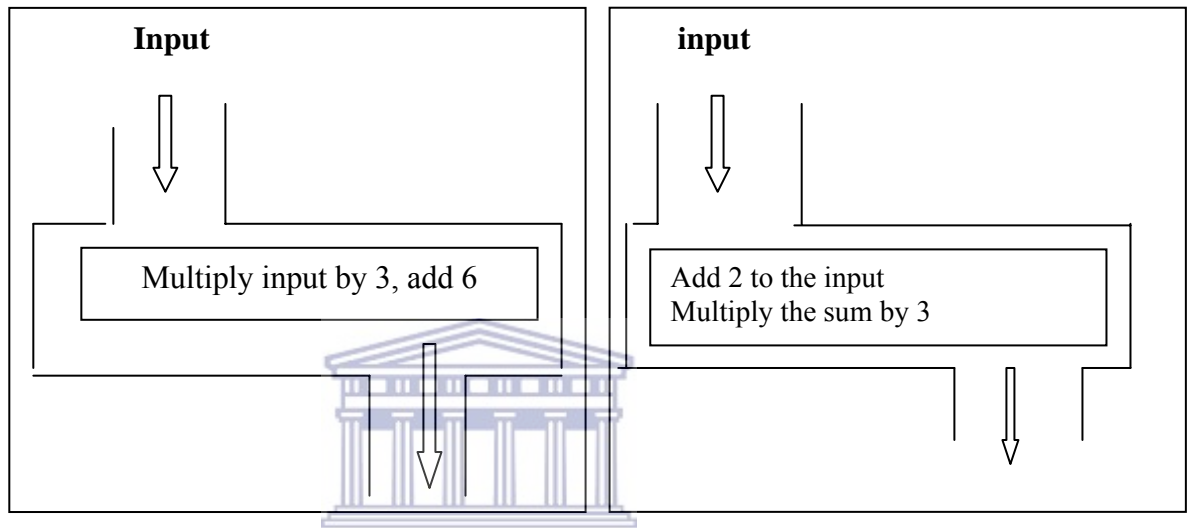
What information do you lose/ have incidentally added on as you change from one representational form to another?

What assumptions did you make as you changed from one representational form to another?

Proceptual divide

Do the following pairs of functional representation represent the same function or not? Give reasons for your answer

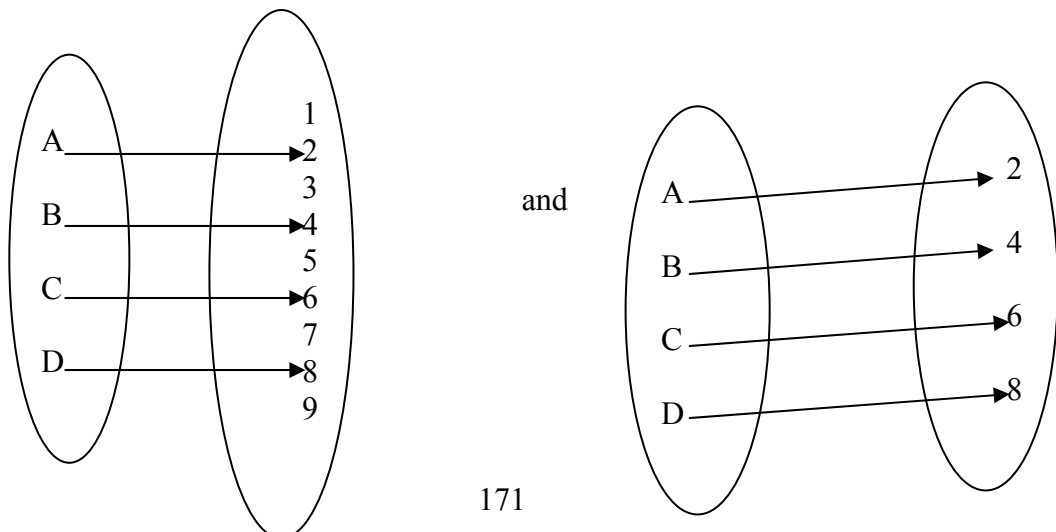
(a)



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- (b) $f(x) = 2x+8$ and $f(x) = 2(x+4)$
- (c) $f(x) = |x|$ and $f(x) = \sqrt{x^2}$
- (d) $\{(1,4);(5,20);(3,12)\}$ and $\{(2,8);(4,16);(6,16)\}$

(e)



(f) A continuous function f satisfies

$$f(1) = 3; f(1.1) = 3.1; f(1.2) = 3.3; f(1.3) = 3.6; f(1.4) = 4; f(1.5) = 4.5; \\ f(1.6) = 5; f(1.7) = 5.4; f(1.8) = 5.7; f(1.9) = 5.9; f(2) = 6$$

Represent the information graphically

Teachers' understandings of the definition of a function

Here are some textbook definitions of a function. Discuss the relationships between your definition of a function and each of the following textbook definitions of a function. What misconceptions are likely to develop in learners when they are first introduced to each of the definition?

1. A function of a variable is an expression that change in value when the variable changes in value. In general, any expression that involves a variable is a function of that variable (Wentworth,1898)
2. If two variables are so related that to each definite value assigned to one of the variables there correspond one or more definite values of the other variable, then the second (dependent) variable is said to be a function of the first (independent) variable. (Rosenbach & Whitman, 1939)
3. A function is a special kind of relation. A function is a relation in which no two ordered pairs have the same first coordinate (the domain) and different second coordinates (the range)
4. A function f from a set X to a set Y is a rule that assigns to each element x in X a unique element y in Y .
5. A function f is a set of ordered pairs (x,y) no two of which has the same first member. That is if $(x,y) \in F$. and $(x, y) \in F$ and $(x,z) \in F$ then $y = z$
6. A function consists of three objects: two empty sets and y and a rule f which assigns to each element x in X a single fully determined element y in Y
7. A function is any correspondence between two sets that assigns to every element in the first set exactly one element in the second set

Is a function a sequence? Explain

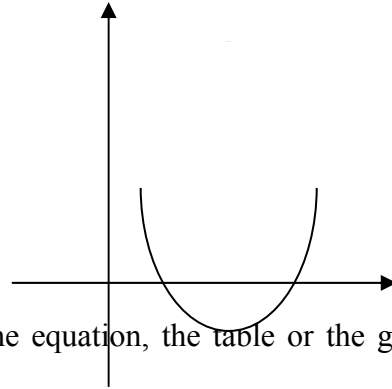
Is a sequence a function? Explain

How do you distinguish a function from non-functions? Does that apply when a function is represented by a function machine, a graph, a table, set of ordered pairs

An equation, a table and a graph are displayed below for the same function.

$$F(x) = x^2 - 3x - 10$$

x	y
-5	30
-4	18
-3	8
-2	0
-1	-6
0	-10
1	-12



What is the output if the input is -1 ? Did you use the equation, the table or the graph to answer the question? Is there any other approach?

What is the output if the input is 12 ? Did you use the equation, the table or the graph to answer the question? Is there any other approach?

What is the output if the input is 4 ? Did you use the equation, the table or the graph to answer the question? Is there any other approach?

What are the inputs if the output is 0 ? Did you use the equation, the table or the graph to answer the question? Is there any other approach?

APPENDIX D

PROSPECTIVE TEACHERS' HYPOTHETICAL LESSON PLANS

Name of student: Alice

Lesson Topic: The concept of a function

Assumed knowledge

Can draw graphs of polynomials using table of values

Can shift between set builder notation and roster method of naming a set

Lesson objectives: By the end of the lesson pupils should be able to:

Define a function

Identify the range and domain of a function

Distinguish between relations which are functions from relations which are non-functions

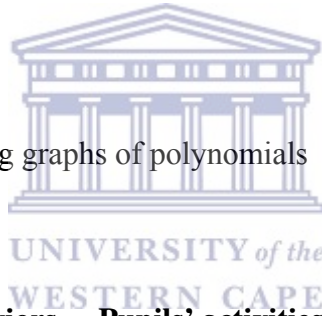
Introduction

Revisit stages required in drawing graphs of polynomials

Lesson development

Stage	Expected teacher behaviors	Pupils' activities
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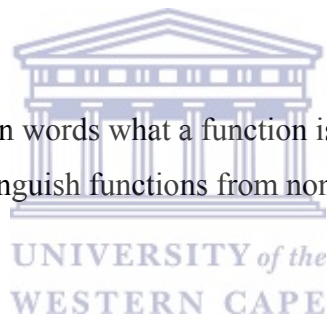
1	<p>Give pupils the definition of a function, highlighting the basic conditions. "A function f from a set X to a set Y is a formula that assigns to each element x in X a unique element y in Y.</p> <p>The set X is called the domain of f. The set of corresponding elements y in Y is called the range of f,"</p> <p>Give examples and non-examples of a function. Ask</p>	<p>Pupils distinguish functions from non-functions from a given set of relations given in algebraic, numeric and geometric forms</p>
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- pupils to distinguish functions from non-functions from a set of relations
- 2 Illustrate the different ways of representing a function, using $y = x^2$ (i.e. table of values, formulae, graphical)
 - Given a graph pupils determine a the table of values, the algebraic formulae,
 - Given a table of values pupils determine the graph, the algebraic formulae of the function
 - Given the algebraic formula pupils determine the table of value and the graph of the function
 - 3 Explain the terms domain, codomain and range and ask pupils to state the domain and range of given functions
 - Pupils state the domain and range of given functions

Closure

Ask pupils to explain in their own words what a function is
 Pupils to describe how they distinguish functions from non- functions



Student's Name: Alice

Lesson topic: Special functions

Assumed knowledge: Pupils can:

Define what a function is

Use the definition of a function to distinguish functions from non-functions

Learning aids: No concrete learning aids

Objectives: By the end of the lesson pupils should be able to identify the following special functions:

Constant functions

Absolute value function

Piece-wise defined functions

Introduction

Ask pupils to define a function and how they can distinguish functions from non-functions

Lesson development

Stage	Expected teacher behaviors	Pupils' activities
1	Ask pupil whether the function $y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ would still remain a function if $a_1 = a_2 = \dots = a^n = 0$	Pupils debate whether $y = a_0$ is a function or not. Pupils are asked to justify their responses.
2	Ask pupils whether a piece-wise defined relation like $y = \begin{cases} x^2 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$ is a function or not	Pupils debate whether a piece-wise defined relation is a function or not. Pupils are asked to justify their responses
3	Ask pupils to sketch the modulus functions $f(x) = x $, $f(x) = x + 1 $, $f(x) = x - 2 $ on the same Cartesian plane and then ask them to describe how the three functions are related to each other.	Pupils sketch the modulus functions and then describe the relationships between the three functions

Closure

Teacher tells the students that there are many special functions they will learn as they pursue further studies in mathematics and that the constant function and the absolute value functions are some the special functions. Pupils are then asked how they would determine whether a mathematical relation they are likely to encounter is a function or not.



Student's Name: Fari

Lesson Topic: The concept of a function

Assumed knowledge

Pupils can identify sets of variables in which one set of variables depend on the other set of variables. E.g. when money is invested at some interest rate the interest (I) which is the output depends on the length of time (t) (input) that the money is invested.

Learning Aids: function Boxes

Lesson Objectives: by the end of the lesson pupils should be able to:

Determine the value of the function for a given input

Determine the input given the rule of assignment and the output

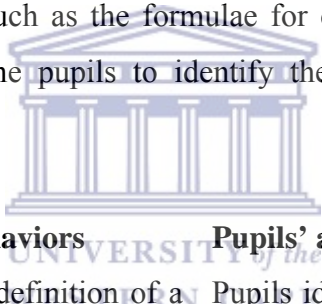
Distinguish between the independent and the dependent variable

Define a function as a dependence relation

Introduction

Revisit the different formulae such as the formulae for calculating the area of a circle or square and in each case ask the pupils to identify the dependent and the independent variables.

Lesson development



Stage	Expected teacher behaviors	Pupils' activities
1	Teacher refers to the definition of a function (Backhouse, page 88 i.e. “A <i>function is a dependence relation between variables – y depends on x,</i> ”) and explain the relevance of the qualifiers ‘to each’ and ‘exactly in the definition with reference to the formula $A = \pi r^2$	Pupils identify the independent and the dependent variables in the given functional relationships
2	Given a context e.g. value of a machine (V) and the depreciation rate, ask pupils to express the value of the machine (V) at the end of time (t) in years as a function of the time (t)	Pupils express one variable as a function of the other variable in a given context

- | | | |
|---|---|--|
| 3 | Ask pupils to evaluate:
The input when given the output of a given formula
The output when given the input of a given formula | Pupils evaluate:
The input when given the output of a given formula
The output when given the input of a given formula |
|---|---|--|

Closure

Pupils define a function and identify the independent and the dependent variables in a given mathematical contextual problem



Student's Name: Chipo

Topic The concept of a function

Assumed knowledge: pupils can

Draw the graphs of straight lines, parabolas, and hyperbolas

Given an equation in terms of y and x and the value of y or x pupils can calculate the corresponding values of x or y

Learning aids-----

Lesson objectives: pupils should be able to

Distinguish the everyday use of the word function and its technical use in mathematics

Define what a function is

Determine the range and domain of a given function

Introduction

Discuss the everyday use of the word function and its mathematical meaning

Lesson development



Stage	Expected teacher behaviours	Pupils' activities
1	Define a function as used in mathematics using illustrative examples in mathematics and science e.g. Cost of postage (C) is a function of the weight (w) of the letter Force (F) between two particles is a function of the distance (d) between them <i>"A function is a dependence relation between two variables which can be described by a formula or an equation"</i>	Pupils give examples of functional relationship in which the output for a given input is unique
2	Introduce the terms range and domain of a function and ask pupils to identify range and domain in their examples of functions which they had given in stage 1 above	Pupils identify the domain and range of the functions they would have formulated
3	Discuss the importance of the qualifiers 'for all'	Pupils determine the

and ‘there exist a unique’ in the definition of a domain and the range of function. Ask pupils to determine conditions given functions. Pupils to which should be imposed on given rational and justify their solutions equations involving square roots if they are to procedures represent functions

Closure

Pupils explain how they would differentiate functions from no-functions



Student' name Edith

Topic: The concept of a function

Assumed knowledge: Pupils can:

Draw a graph of a give equation

Calculate he value of a function for a given input

Learning aids: nil

Objectives: by the end of the lesson pupils should be able to:

Derive functional relationships from everyday practices

Define what a function is

Represent a function numerically, algebraically and geometrically

Lesson development

Stage Pupil activity

- 1 Ask pupils to determine the number of handshakes when there are 2, 3, 4, ...n people arriving at a party given that each person shake the other arrivals only once
- 2 Pupils tabulate the number of handshakes if there are 2, 3, 4, ...n people
- 3 Pupils draw the graph of the number of handshakes (S) as a function of the number of people (p)
- 4 Pupils express the number of handshakes (S) in terms of the number of people (n)
- 5 Define the term function as used in mathematics ‘A *function is any correspondence between two sets that assigns to every element in the first set exactly one element in the second set*’

Closure

Discus the implications of the words “*exactly one element in the second set*” and “*to every element in the first set*” in the definition of a function

Name of student: Ben

Topic: The concept of a function

Assumed knowledge: pupils can

Identify the independent and the dependent variables in a given context e.g. the yield and the amount of fertilizer used to produce the yield

Formulate equations in which one variable is a function of the other variable in a given context

Learning aids:

Lesson objectives: by the end of the lesson pupils should be able to

Define a function

Evaluate the output of a function when given the input

Determine the domain and range of a given function

Lesson development

Stage	Expected teacher behaviors	Pupils' activities
1	With reference to the amount of fertilizer used and the yield obtained the teacher define what a mathematical function is. "A function f from set X to a set Y is a rule that assigns to each element x in X a unique element y in Y ". Ask pupils to state functional relationships citing examples from commerce and science.	Pupils state functional relationships and in each case identify the dependent and the independent variables in the functional relationship.
2	Teacher introduces the terms domain and range of a function referring to the independent respectively. Teacher explains the implication of the words 'each element in the domain' and 'single' in the definition of a function using the following example $y = \sqrt{x^2 - 4}$ and	Pupils determine the domain and range of the following functions giving reasons for their solution method; $y = \frac{x^3 + x + 5}{x^2 - 3x - 4}$ $y = \frac{1}{x - 6}$
3	Teacher introduces the $f(x)$ notation and ask pupils to find the value of the functions	Pupils find the value of each of the following functions at the given values

for certain input values

$$(i) f(x) = x^2 + x : f(-2) : f(2x)$$

$$(ii) f(x) = \frac{x-5}{x^2+4} : f(3x); f(x+h)$$

- 4 Teacher introduces the difference quotient $\left[\frac{f(x+h) - f(x)}{h} \right]$ of the function $f(x)$ and ask pupils to determine the difference quotient of given functions

Pupils find $f(x+h)$ and $\frac{f(x+h) - f(x)}{h}$ of the following

$$f(x) = x^2$$

$$f(x) = \frac{1}{x}$$

Closure

Pupils define what a function is stating the main characteristics of a function



Name of student: Daniel

Topic: The concept of a function

Assumed knowledge: pupils can

Represent linear and quadratic functions graphically

Name a pair of variables in which one set of variables depend on the other set of variables

Learning aids:

Objectives: by the end of the lesson pupils should be able to:

Define what a function is

Distinguish one-to-one functions from many-to-one functions

Determine the inverse of a one-to-one function

Lesson development

Stage	Expected teacher behavior	Pupil activities
1	Teacher ask pupils to identify the independent and the dependent variables in the following set of relations $y = \sqrt{x}$ $y = x^2$ $y = \sin x$ $y = \sin^{-1} x$	Pupils identify the independent and the dependent variables in the given algebraic formulae Pupils also identify relations which have unique outputs for a given input
2	Refer to the definition of a function on page 29 in Backhouse-“ <i>A function consists of three objects: two non-empty sets and y and a rule f which assigns to each element x in X a single fully determined element y in Y,</i> ” and ask pupils to identify functions from non-functions from relations given in stage 1 above	Pupils use the definition to distinguish functions from non functions
3	Represent the functions $y = x^2$ and $y = y = 2x + 4$ using Venn diagrams. Use the Venn diagrams to illustrate the one-to-one and the many-to-one functions	Pupils find inverses of one-to-one functions
4	Class discussion on whether all functions have inverses	



Closure

Pupils define what a function is in their own words. Pupils describe how they can distinguish functions from non-functions when relations are given in graphical form

