Computing Mislin genera of certain groups with non-abelian torsion radicals

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Declaration

I declare that **Computing Mislin genera of certain groups with non-abelian torsion radicals** is my work, that it has not been submitted before for any degree or examination at any other university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Victor George Hess

Signed ....

January 2004

Abstract

In this mini-thesis we present some generalities of non-cancellation and localization and we compute non-cancellation groups. We consider groups belonging to the class $\mathcal{X}_0$ of all finitely generated groups that have finite commutator subgroups. For a $\mathcal{X}_0$-group H, we study the *non − cancellation set*, $\chi(H)$, which is defined to be the set of all isomorphism classes of groups K such that $H \times \mathbb{Z} \simeq K \times \mathbb{Z}$. In particular, we prove some basic facts such as that for a group G which is either finite or finitely generated abelian, we have $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$ only if $G \cong H$.

For a finitely generated nilpotent group N, the Mislin genus, $\mathcal{G}(N)$, is defined to be the set of all isomorphism classes of finitely generated nilpotent groups M such that for every prime p, the groups M and N have isomorphic p-localizations. It was shown by Warfield that if N is a nilpotent $\mathcal{X}_0$-group, then $\chi(N) = \mathcal{G}(N)$. Various calculations of such Hilton-Mislin genus groups can be found in the literature, for example, in an article of Hilton and Scevenels. Most of these calculations are for a special subclass of nilpotent $\mathcal{X}_0$-groups, in particular, groups with abelian torsion radicals.

For a $\mathcal{X}_0$-group H on $\chi(H)$ Witbooi defined a group structure in terms of embeddings of K into H, for groups K of which the isomorphism classes belong to $\chi(H)$. If H is nilpotent, then the group $\chi(H)$ we obtain coincides with the genus group $\mathcal{G}(H)$ defined by Hilton and Mislin.

In particular we make a new calculation, computing the non-cancellation group

$\chi(H^k)$ of a Cartesian power of a certain type of group H. The group we consider is of the form $T \rtimes \mathbb{Z}$, a semidirect product with T a finite non-commutative nilpotent group on three generators.

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KEYWORDS

automorphism

finitely generated group

group action

localization

Mislin genus

metacyclic group

torsion subgroup

nilpotent group

finite commutator subgroup

non-cancellation

semi-direct product

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Table of Contents

Chapter 1

INTRODUCTION

The study of localization of groups goes back as far as at least the middle of the twentieth century with, for instance, papers by Malcev and Lazard, see the book [10] of Hilton, Mislin and Roitberg. The study of non-cancellation for direct products of groups seems to have been around even earlier than that, for instance, see the paper of Remak [18]. By non-cancellation we mean the phenomenon that for a given pair of groups G and F, it may be possible to find a group H such that $H \times F \simeq G \times F$, while H is not isomorphic to G. In this mini-thesis we study such non-cancellation phenomena in the particular case that F is the infinite cyclic group ($\mathbb{Z}$) and G is finitely generated (inter alia).

When localizing groups, it is possible to find a pair of non-isomorphic groups which end up having isomorphic localizations. This leads to the notion of genus (and there are different variations of this concept), which in turn is useful in classifying groups into isomorphism classes. There is a very interesting interplay between non-cancellation and genus, see for instance the theorem [25, Theorem 3.5] of Warfield.

In this mini-thesis our focus is mainly on the class $\mathcal{X}_0$ of finitely generated groups having finite commutator subgroups. For such a $\mathcal{X}_0$-group G, the non-cancellation set $\chi(G)$ is defined to be the set of all isomorphism classes of groups H such that $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$. Closely related to the non-cancellation set is the localization genus. There are different variations of the notion of genus associated to localization of groups. The one with which we deal is referred to as the Mislin genus. Other than the Mislin genus there are for instance the restricted genus as in O'Sullivan's paper [16], the extended genus as in the paper [5] by Casacuberta and Hilton.

The Mislin genus $\mathcal{G}(N)$ of a finitely generated nilpotent group N is defined by Mislin [15] to be the set of all isomorphism classes of finitely generated nilpotent groups M such that for every prime p, the group M and N have isomorphic p-localizations. Hilton and Mislin [9] defined an abelian group structure on the set $\mathcal{G}(N)$. Warfield [25, Theorem 3.5] has shown that if N is a finitely generated nilpotent group with finite commutator subgroup, then $\chi(N) = \mathcal{G}(N)$, thus linking localization and non-cancellation sets. The latter relationship inspired a sequence of papers of Witbooi, which culminates in the article [27]. In the latter article the non-cancellation set $\chi(M)$ of a $\mathcal{X}_0$-group, M, is furnished with a group structure that coincides with the Hilton-Mislin genus group $\mathcal{G}(M)$ if M is nilpotent. In the paper of O'Sullivan [16] it is shown that for a $\mathcal{X}_0$-group the non-cancellation set coincides with the *restricted genus*. Computation of the group $\chi(M)$ is enhanced by the presence of certain homomorphisms between non-cancellation groups.

Most of the calculations of $\mathcal{X}_0$-groups so far have been for metacyclic groups and for direct products of such groups. Exceptions are the calculations in the papers of Mislin [15] (the calculation is completed in [9]), Hilton [7], Hilton-Witbooi [13] and

Witbooi [29]. In this mini-thesis we calculate non-cancellation groups of $\mathcal{X}_0$-groups which have non-abelian torsion subgroups.

In Chapter 2 we observe some basic properties of $\mathcal{X}_0$-groups. We note for instance, that such groups have finite torsion radicals and that their centres are of finite index. We also show that if G is either a finitely generated abelian group or a finite group, then for a group H we can have $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$ only if $H \simeq G$. In other words we show that if G is either a finite group or a finitely generated abelian group, then $\chi(G)$ consists of only one member, i.e., $\chi(G)$ is *trivial*. The concept of semidirect product also features very strongly in this mini-thesis. For the purpose of establishing notation, we include the definition of semidirect product.

In Chapter 3 we define the concept of P-local groups, for a set of primes P. We also introduce the notion of P-isomorphism between groups, and the notion of a P-localizing homomorphism. Examples illustrating these concepts are given on the level of finite groups and abelian groups. We prove some interesting fundamentals on P-local groups, such as for instance that a finite P-group is P-local. We use in some cases, the Five-Lemma, of which we shall include the relevant formulation and proof.

We present, in Chapter 4, an example of infinite non-abelian groups N and M, and a localizing homomorphism $N \longrightarrow M$. The group N is a metacyclic nilpotent group of the form $T \rtimes \mathbb{Z}$, a semi-direct product with T finite cyclic. Groups of this kind, and their direct squares reveal much of what can be expected when studying the localization genus and non-cancellation as we shall see in later chapters (but with non-cancellation as the central theme).

Chapter 5 is devoted to a discussion of the Hilton-Mislin group structure, as from [9]. This presentation is technical and is included as a forerunner to the non-cancellation group and for comparison. Detailed calculations of the genus of nilpotent groups of the form $\mathbb{Z}_n \rtimes \mathbb{Z}$ can be found in [7]. We briefly present an example from [7].

The description of the non-cancellation *group* of a $\mathcal{X}_0$-group G, as in [27], is presented in Chapter 6. Furthermore, we give a brief description of the non-cancellation group for the particular case when the short exact sequence consisting of the group, its commutator subgroup and the abelianization is split. In this particular case the non-cancellation group has a simpler description. The method for computing $\chi(G)$ for such semi-direct product groups are developed in [29], which generalizes the work in [23] by Scevenels and Witbooi. We also discuss induced morphisms between non-cancellation groups. These morphisms are quite useful when computing non-cancellation groups. For $\mathcal{X}_0$-groups G and H we have an epimorphism $\chi(G) \longrightarrow \chi(H)$, for instance when there is an epimorphism $\phi : G \longrightarrow H$ with $\ker\phi$ being a characteristic subgroup of the torsion radical of G, see [27]. Other instances of induced morphisms are discussed in [27], [13] and [31]. We shall specify those which are important for our computations.

In the final chapter, Chapter 7, we compute non-cancellation groups of certain groups having non-abelian torsion radicals. In [29] there are computations of non-cancellation groups of $\mathcal{X}_0$-groups having non-abelian torsion radicals, and the torsion radicals are wreath products. In our discussion here we consider torsion subgroups which are of a different type. In particular, the torsion radicals which we consider here are not split extensions. All the work in Chapter 7 are independent efforts of the author. Whatever is known of the finite groups $T_{n,m}$ defined in item 7.1, is scattered in the literature, and was developed here independently. In particular the last Theorem 7.11 is an original contribution by the author.

For some results it was difficult to find proofs in the literature and the author had to write proofs himself. The more significant examples of such occurrences are Propositions 2.10, 2.11, 2.13, 3.3 and even 3.6. Of course, the specific example of Chapter 4 was suggested by the supervisor and the detail provided by the author.

Chapter 2

BASIC PROPERTIES OF FINITELY GENERATED GROUPS WITH FINITE COMMUTATOR SUBGROUPS

In this chapter we provide some definitions pertaining to groups belonging to the class $\mathcal{X}_0$ (Definition 2.1 below), and we prove some basic properties of these groups. We give an alternative description, Proposition 2.7, of $\mathcal{X}_0$-groups in terms of short exact sequences. We also prove certain closure properties of $\mathcal{X}_0$ in Proposition 2.8, for instance, that $\mathcal{X}_0$ is closed with respect to taking subgroups. We conclude this chapter by proving that $\chi(G)$ is trivial (i.e, consists of only one member) if G is either a finite group, Theorem 2.11, or a finitely generated abelian group, Theorem 2.13. For the nilpotent case, these results are mentioned in Mislin's paper [15].

Definition 2.1 : We denote the class of all finitely generated groups that have finite commutator subgroups by $\mathcal{X}_0$, as in [27].

Definition 2.2 : (See [27] for instance) For any $\mathcal{X}_0$-group G, the non-cancellation set $\chi(G)$ is the set of all isomorphism classes $[H]$ of groups H such that $H \times \mathbb{Z} \simeq G \times \mathbb{Z}$.

In an abelian group the set of all elements of finite order forms a subgroup, see [19, p.90] for instance. For groups in general this is not the case. This can be proved, however, for $\mathcal{X}_0$-groups as we see below.

Proposition 2.3 : *Let G be a $\mathcal{X}_0$-group. The subset of all elements of finite order in G is a finite subgroup of G.*

Proof: Consider the canonical epimorphism $\sigma : G \longrightarrow G/[G,G]$. Since the factor group $G/[G,G]$ is a finitely generated abelian group, the elements of finite order in the abelian group $G/[G,G]$ form a finite subgroup F, and $F \triangleleft G/[G,G]$. Therefore the preimage $E = \sigma^{-1}(F)$ is a normal subgroup of G and is finite since $|E| = |F| \cdot |[G,G]|$. Since σ is surjective the set H, of all elements of finite order in G, is contained in E. So $H = E$. Thus H is a finite subgroup of G. $\square$

Definition 2.4 : For a $\mathcal{X}_0$-group G, the subgroup generated by all torsion elements in G is called the *torsion radical* of G. It will be written as T_G and is a finite group. At times we shall refer to the torsion radical as the torsion subgroup.

Definition 2.5 : An automorphism of a group G is an isomorphism $\varphi : G \longrightarrow G$. A subgroup H of G is called *characteristic* in G, if $\varphi(H) \subseteq H$ for every automorphism φ of G.

We also note that the set of all automorphisms of a group G is a group with respect to the operation of composition of functions. This group is written as $\mathrm{Aut}(G)$.

Proposition 2.6 : *For a $\mathcal{X}_0$-group G the torsion radical T_G of G is a characteristic subgroup of G.*

Proof : By Proposition 2.3, T_G is a finite subgroup of G. Consider any automorphism $\rho : G \longrightarrow G$. Then for any $t \in T_G$, we have $\rho(t) \in T_G$, since T_G contains all the elements of finite order in G. Thus $\rho(T_G) \subseteq T_G$. So, by definition 2.5, T_G is a characteristic subgroup of G. □

We provide an alternative description of $\mathcal{X}_0$-groups in the proposition below.

Proposition 2.7 : *A group G is a $\mathcal{X}_0$-group if and only if there is a short exact sequence*

$$1 \longrightarrow T \xrightarrow{\alpha} G \xrightarrow{\beta} F \longrightarrow 1$$

with T finite and F free abelian and of finite rank.

Proof : Suppose that the group G is a $\mathcal{X}_0$-group. Then, since $[G,G]$ is finite, $[G,G]$ is a subgroup of T_G. Thus $F = G/T_G$ is abelian. If $x \in G$ and the element $x.T_G$ of G/T_G is of finite order r, so $x^r \in T_G$ and then x^r has finite order, say s. Therefore x has order at most rs. So in fact $x \in T_G$. Thus F is torsion free. Finally, since G is finitely generated, G/T_G is finitely generated. Thus F is a free abelian group of finite rank.

Conversely, suppose that G is a group satisfying the exact sequence condition. Since F is abelian we have $[G,G] \leq \alpha(\mathrm{T})$. Thus $[G,G]$ is finite. Since T and F are finitely generated, it follows that G is finitely generated. Therefore $G \in \mathcal{X}_0$. $\square$

The following results reveal some closure properties of the class $\mathcal{X}_0$, for instance, closure with respect to taking subgroups and formation of direct products.

Proposition 2.8 : *(a) If G and K are $\mathcal{X}_0$-groups, then $G \times K$ is an $\mathcal{X}_0$-group.*
(b) If H is any subgroup of a $\mathcal{X}_0$-group G, then H is a $\mathcal{X}_0$-group.
(c) If L is a group for which $L \times \mathbb{Z} \simeq G \times \mathbb{Z}$ and G is a $\mathcal{X}_0$-group, then L is a $\mathcal{X}_0$-group.

Proof : (a) Let G and K be $\mathcal{X}_0$-groups. Then $[G \times K, G \times K] = [G,G] \times [K,K]$, and thus $[G \times K, G \times K]$ is finite.
If $G = \langle X \rangle$ and $K = \langle Y \rangle$, then let $X_1 = X \cup \{1\}$ and $Y_1 = Y \cup \{1\}$, 1 being the respective identity elements of the groups G and K.
Then $G \times K = \langle (x,y) : x \in X_1, y \in Y_1 \rangle$.
Thus, since G and K are finitely generated, it follows that $G \times K$ is finitely generated.
This proves the statement (a).

(b) If $H \leq G$, then $[H,H]$ is finite since $[H,H] < [G,G]$. The set T_H of all elements of H which have finite order generates a subgroup S of T_G. But also then, $S \subseteq H$ and $S \subseteq T_G$. Therefore $S = T_H$ and in fact $T_H \triangleleft H$.
The inclusion map $H \subseteq G$ induces a homomorphism $\phi : H \longrightarrow G/T_G$. Obviously, $\phi(T_H) = 1 \in G/T_G$. Thus ϕ induces a homomorphism $\psi : H/T_H \longrightarrow G/T_G$. If $x \in H$ and $\psi(x.T_G) = T_G$, then $x = \phi(x) \in T_G$. So $x \in H \cap T_G = T_H$. Thus ψ is a monomorphism. Since G/T_G is a finite rank free abelian group it follows that H/T_H is a finite rank free abelian group. Therefore, by Proposition 2.7, it follows that H

is a $\mathcal{X}_0$-group.

Thus (b) is proved.

(c) By (a), $G \times \mathbb{Z}$ is a $\mathcal{X}_0$-group. Thus $L \times \mathbb{Z}$ is a $\mathcal{X}_0$-group.

By (b) it follows that L is a $\mathcal{X}_0$-group. $\square$

We shall require the *Five Lemma* in a special form. We give the appropriate version, and we deduce the proof from the version proved in the book of [21, Rotman].

Proposition 2.9: *Suppose that we have a commutative diagram of groups and group homomorphisms as below, in which the rows are short exact sequences.*

$$\begin{tikzcd}
1 \arrow[r] & H_0 \arrow[r] \arrow[d, "\alpha_0"'] & H \arrow[r] \arrow[d, "\alpha"'] & H_1 \arrow[r] \arrow[d, "\alpha_1"'] & 1 \\
1 \arrow[r] & K_0 \arrow[r] & K \arrow[r] & K_1 \arrow[r] & 1
\end{tikzcd}$$

If any two of the three homomorphisms α_0, α and α_1 are isomorphisms, then the third one is also an isomorphism.

Proof: If α_0 and α_1 are isomorphisms then by [21, Corollary 10.14] it follows that α is an isomorphism. There are two more cases to be proved. These two cases follow from [21, Corollary 10.14] if we augment the diagram above by inserting vertically the homomorphism $1 \longrightarrow 1$ between the object to the left of α_0 (or to the right of α_1) and extra homomorphisms in each of the two rows at the left end (or at the right, respectively). $\square$

The following proposition is required to prove the fact that for a finite group G,

$\chi(G)$ is trivial (Theorem 2.11). For the moment we prove the following special case. Note that Proposition 2.10 says that if G is the trivial group, then $\chi(G)$ is trivial.

Proposition 2.10 : *Let A be any group. If $h : \mathbb{Z} \longrightarrow A \times \mathbb{Z}$ is a group isomorphism, for some group A, then A is necessary trivial.*

Proof : Suppose that A is a group. We regard the groups A and $A \times \mathbb{Z}$ as multiplicative groups. Let $h(1) = (x, n)$ for some $x \in A$ and some $n \in \mathbb{Z}$. Then for any $m \in \mathbb{Z}$ we have $h(m) = h(1)^m = (x, n)^m = (x^m, mn)$. In order for h to be an isomorphism, there must exist $m_0 \in \mathbb{Z}$ such that $h(m_0) = (1, 1)$, where $(1, 1)$ is the identity element of the group $A \times \mathbb{Z}$. Then we have $h(m_0) = (x^{m_0}, m_0 n) = (1, 1)$. This implies that $|m_0| = 1$ and $|n| = 1$ (in order that $m_0 n = 1$), and thus also that $x^1 = 1$ or $x^{-1} = 1$. Thus $x = 1$. Consequently A cannot have any element other than 1. $\square$

We shall prove that if G is either a finite group or a finitely generated abelian group, then $\chi(G)$ is trivial.

Theorem 2.11 : *If G is a finite group, and H is any group then*
$H \times \mathbb{Z} \cong G \times \mathbb{Z} \Longleftrightarrow H \cong G.$

Proof: An isomorphism $\theta : G \times \mathbb{Z} \longrightarrow H \times \mathbb{Z}$ maps G into H, since $\mathbb{Z}$ is torsion free. But G is a characteristic subgroup of $G \times \mathbb{Z}$ and θ is an isomorphism, hence it follows that $\theta(G) \lhd H \times \mathbb{Z}$, and eventually that $\theta(G) \lhd H$. But then the isomorphism θ induces a homomorphism $h : \mathbb{Z} \longrightarrow [H/\theta(G)] \times \mathbb{Z}$. From the Five Lemma, proposition 2.9, it follows that h is an isomorphism. This means, by Proposition 2.10, that $H/\theta(G) = 1$, and the proof is complete. $\square$

Now in order to prove our final theorem we require the following remarkable result of Hirshon.

Proposition 2.12: [14, Lemma 1], *If G and H are groups such that*

$$(G \times \mathbb{Z}) \times \mathbb{Z} \cong H \times \mathbb{Z},$$

Then $G \times \mathbb{Z} \cong H$.

Now we can prove

Theorem 2.13: *If G is a finitely generated abelian group, then $\chi(G)$ is trivial.*

Proof: Let G be any finitely generated abelian group. Then by the Fundamental Theorem of Finitely Generated Abelian groups [21, Theorem 2.29], $G = G_0 \times G_1$ where G_0 is a finite subgroup of G and G_1 is a free abelian group of finite rank. Suppose now that H is a group such that $H \times \mathbb{Z} \cong G \times \mathbb{Z}$. The proof will be completed by showing that H is necessarily isomorphic to G.
Now if $G_1 = 0$, then $G_0 = G$, so G is finite, and then by Theorem 2.11 it follows that $H \cong G$.
We consider the case $G_1 \neq 0$. Then $G_1 \cong G_2 \times \mathbb{Z}$ for some group G_2. But then we have

$$H \times \mathbb{Z} \cong G \times \mathbb{Z} \cong G_0 \times G_1 \times \mathbb{Z} \cong G_0 \times G_2 \times \mathbb{Z} \times \mathbb{Z}.$$

From Proposition 2.12 it follows now that $H \cong G_0 \times G_2 \times \mathbb{Z}$, Therefore $H \cong G$. $\square$

Many of the groups we shall deal with are semidirect products. For convenience we give here the definition of the concept of semidirect product.

In particular we shall often refer to groups of the form $\mathbb{Z}_n \rtimes_\nu \mathbb{Z}$ which we now define.

Definition 2.14: Given groups H and K and a homomorphism $\omega : K \longrightarrow \mathrm{Aut}(H)$, we define $G = H \rtimes_\omega K$ to be the following group.
As a set the group G, as mentioned above, coincides with the cartesian product $H \times K$, but the group operations are as follows:
For the ordered pairs $(h_1, k_1), (h_2, k_2) \in H \times K$,

$$(h_1, k_1)(h_2, k_2) = (h_1 + \omega(k_1)(h_2), k_1 + k_2).$$

We recall the following notation, used in [31] for instance. If (n, u) is a relatively prime pair of natural numbers, then the group

$$\langle x, y \mid x^n = 1, yxy^{-1} = x^u \rangle$$

is denoted by $G(n; u)$. Let $M = G(n; u)$. We note that M happens to be (isomorphic to) the semidirect product $\mathbb{Z}_n \rtimes_\omega \mathbb{Z}$ where $\omega : \mathbb{Z} \longrightarrow \mathrm{Aut}(\mathbb{Z}_n)$ is the action for which $\omega(1)$ is the automorphism $t \longmapsto ut$ of $\mathbb{Z}_n$.

The description of nilpotency for groups of the type $M = \mathbb{Z}_n \rtimes_\nu \mathbb{Z}$ is found in a result of Hilton, and we quote it.

Lemma 2.15 [7, Lemma 1.1]. For a relatively prime pair of natural numbers

n and u, the group $N_0 = \langle x, y \mid x_n = 1, yxy^{-1} = x^u \rangle$ is nilpotent if and only if $p \mid (u-1)$ for every prime divisor p of n.

These groups have been studied extensively with respect to their finite quotients, localization and non-cancellation phenomena in for example [7], [1], [2], and elsewhere. Such groups shall also feature prominently in this mini-thesis.

Chapter 3

LOCAL GROUPS: BASICS

We now introduce some concepts and basic theory on localization of groups. Understanding localization is important in itself, but its relevance in this mini-thesis is due to its relationship with non-cancellation. (This relationship is made more explicit in Chapter 4). The localization of an abelian group is relatively simple, and we show how to attain this by forming tensor products (as of $\mathbb{Z}$-modules). We briefly discuss the problem arising in localization of non-nilpotent groups - the smallest finite non-abelian group already demonstrates the problem. We give, in detail, some results on finite groups. We also give some general results such as of short exact sequences in which two groups local imply the third to be local (under certain conditions).

In this chapter we deal with the subclass of all infinite nilpotent groups in $\mathcal{X}_0$ denoted by $\mathcal{N}_0$. The theory of the localization of nilpotent groups are discussed in the textbook [10] of Hilton, Mislin and Roitberg. We start with a string of definitions from the textbook, [10].

Definition 3.1 : Let P be a set of primes and let P' be the set of natural numbers which are relatively prime to the elements of P.

(a) A group G is said to be P-*local* if for each $n \in P'$, the function $g \longmapsto g^n$ of G into itself is a bijection.

(b) Let $h : G \longrightarrow H$ be a homomorphism of groups. The homomorphism h is said to be P-*injective* if for each $g \in \ker h$, g is of finite order n, for some $n \in P'$.

(c) The homomorphism h is said to be P-*surjective* if for every $x \in H$, there exists an integer $n \in P'$ such that $x^n \in h(G)$.

(d) The homomorphism h is said to be P-*bijective* if it is both P-injective and P-surjective.

(e) The homomorphism h is said to be a P-*localizing homomorphism* if H is P-local and h is P-bijective.

(f) A finite group T is said to be a P-*group* if for every element $x \in T$ and every prime divisor p of the order of x, we have $p \in P$. (See, for instance [20, (Definition 3.41, p.56)].

When P consists of a single prime p, i.e. $P = \{p\}$, then as is common, we slightly abuse notation by writing p-local instead of $\{p\}$-local, etc.

We illustrate below an example of a P-localizing homomorphism for abelian groups.

Example 3.2: For any abelian group A, the group $A \otimes \mathbb{Z}_{(p)}$ is p-local, where

$\mathbb{Z}_{(p)}$ is the subring of $\mathbb{Q}$ generated by the subset $\{\frac{1}{q} : q \text{ is prime}, q \neq p\}$, and the homomorphism $A \longrightarrow A \otimes \mathbb{Z}_{(p)}$, defined by $a \longmapsto a \otimes 1$, is a p-isomorphism. $\square$

In the next chapter we shall exhibit a localizing homomorphism between two infinite non-abelian groups.

If we 3-localize, for example, the symmetric group S_3, then we need to kill (i.e factor out) all the elements of order 2, otherwise the identity would not have a unique square root (i.e the function $g \longmapsto g^2$ would not be injective). But then we kill the entire group. We also kill the elements of order 3. The problem with this phenomenon is that, in particular, the Sylow 2-subgroup of S_3 is not normal.
For a nilpotent finite group N such destruction of elements of p-power order does not happen when p-localizing because, of course, the group N splits as a direct product of its Sylow subgroups [19, item 5.2.4 on p.126]. This is the reason why nilpotent groups have been enjoying more attention in localization theory.

In [10] it is shown how we obtain a p-localizing *functor* for the class of nilpotent groups. In this mini-thesis we shall not dwell on categorical aspects of localization. The interested reader can consult the survey article [3] of Casacuberta that discusses a more general version of the notion of localization.

The following result for nilpotent groups is proved in the book [10, Theorem 2.3] of Hilton, Mislin and Roitberg for nilpotent groups.

Proposition 3.3 : *Let G be any group in which every element has finite order. If r is any natural number such that the order of every element of G is relatively* *prime to r, then the function $\rho : G \longrightarrow G$ defined by $x \longmapsto x^r$ is an injective function.*

Proof : Suppose $x^r = y^r$, for some $x, y \in G$. By the coprimality condition it follows that $\langle x^r \rangle = \langle x \rangle$ and $\langle y^r \rangle = \langle y \rangle$. But then $\langle x \rangle = \langle y \rangle$. Thus $y = x^a$ and $x = y^b$, for some $a, b \in \mathbb{Z}$. In particular, the subgroup $\langle x, y \rangle$ of G is commutative. But then $(xy^{-1})^r = x^r y^{-r} = 1$, since $x^r = y^r$. Thus the order of xy^{-1} is a factor of r. Since the order of each element is relatively prime to r, it follows that xy^{-1} has order 1. Thus $x = y$. □

Proposition 3.4 : *If G is a finite group of order relatively prime to r, then the function $\rho : G \longrightarrow G$ defined by $x \longmapsto x^r$ is a bijection.*

Proof : By Proposition 3.3 above, ρ is injective. Since G is finite it follows that every injective function of G into G is a bijection. □

In particular, we have the following proposition which provides us with examples of P-local groups.

Proposition 3.5 : *For any set P of primes, any finite P-group is P-local.*

Proof : Let G be a finite P-group. If n is any natural number relatively prime to the elements of P, then by Proposition 3.4 above, the function $\rho : G \longrightarrow G$ defined by $x \longmapsto x^n$ is a bijection. So G is P-local. □

The following result can be compared with [10, Corollary 2.5 on p.19].

Proposition 3.6 : *Let G be any group for which there is a surjective homomorphism $h : G \longrightarrow B$ such that $A = \ker h \leq Z(G)$, $Z(G)$ being the centre of G. Let P be any set of primes.*
If B and A are P-local, then G is P-local.

Proof : Consider any P'-number r. Let $\alpha : G \longrightarrow G$ be the function defined by $x \longmapsto x^r$. Then α induces functions $\alpha_0 : A \longrightarrow A$ and $\alpha_1 : B \longrightarrow B$.
We show now that α is injective.
Suppose that $x, y \in G$ and $\alpha x = \alpha y$.
Then

$$h(x^r) = h(y^r)$$

$$\text{therefore,} \quad (hx)^r = (hy)^r.$$

Since α_1 is injective, (by assumption B is P-local), it follows that $h(x) = h(y)$. This means that $y = ax$ for some $a \in A$. Now $x^r = y^r = (ax)^r = a^r x^r$ since $a \in Z(G)$. Therefore $a^r = 1$. Since A is P-local it therefore follows that $a = 1$.
therefore $\quad x = y$, so α is injective.

We show that α is surjective.
Consider any $y \in G$. Then $h(y) = x^r$ for some $x \in B$ (Since B is P-local, it has r-th roots). Let $z = h^{-1}(x)$
Then $h(z^r) = x^r = h(y)$
Therefore $\quad y = bz^r$ for some $b \in A$
Since A is P-local, $b = c^r$ for some $c \in A$.
Since $c \in Z(G)$, $c^r z^r = (cz)^r$
therefore $\quad y = (cz)^r$.

Thus, for any element $y \in G$ there exist an element $c \in G$ such that $y = \alpha(cz)$.
Therefore α is surjective.
Therefore α is bijective, and so, G is P-local. $\square$

The proposition above is of course very important to identify local groups. It will be applied in the next chapter where we exhibit a non-trivial localizing homomorphism of a metabelian nilpotent group.

Chapter 4

A LOCALIZING HOMOMORPHISM

In the previous chapter we treated the fundamentals of localization of groups. In this chapter we shall exhibit a localizing homomorphism

$$\lambda : N \longrightarrow M$$

for which the groups N and M are infinite non-abelian groups. The group N will be a metacyclic nilpotent group on two generators. Such metacyclic groups have been studied extensively to form an understanding of the Mislin genus (and also of non-cancellation and related phenomena), for instance by Hilton [7] and other authors.

Here, the group N will be of the form $N = \mathbb{Z}_m \rtimes_\nu \mathbb{Z}$, for some $m \in \mathbb{N}$, where ν is an action of $\mathbb{Z}$ on $\mathbb{Z}_m$, i.e., ν is a homomorphism $\nu : \mathbb{Z} \longrightarrow \mathrm{Aut}(\mathbb{Z}_m)$.

Proposition 4.1 : *Fix any relatively prime pair of natural numbers, m and k.*

(a) For each $z \in \mathbb{Z}$, let $\nu(z)$ be the function $\nu : \mathbb{Z} \longrightarrow \mathbb{Z}_m$ defined by

$\nu(z) : \overline{n} \longrightarrow k^z.\overline{n}$.

Then $\nu(z) \in \mathrm{Aut}(\mathbb{Z}_m)$ *for each* $z \in \mathbb{Z}$.

(b) The function $\mathbb{Z} \longrightarrow \mathrm{Aut}(\mathbb{Z}_m)$, *obtained by letting* $\nu(z)$ *be as in* (a) *above for each* $z \in \mathbb{Z}$, *is a group homomorphism* $\nu : \mathbb{Z} \longrightarrow \mathrm{Aut}(\mathbb{Z}_m)$.

Proof : (a) *We first show that for each* $z \in \mathbb{Z}$, $\nu(z) \in \mathrm{Aut}(\mathbb{Z}_m)$.

$$\begin{aligned} \nu(z)(\overline{n}_1 + \overline{n}_2) &= k^z(\overline{n}_1 + \overline{n}_2) \\ &= k^z(\overline{n}_1) + k^z(\overline{n}_2) \\ &= \nu(z)(\overline{n}_1) + \nu(z)(\overline{n}_2), \end{aligned}$$

so $\nu(z)$ is a homomorphism. Also $\nu(z)$ is injective, because

$$\begin{aligned} \nu(z)(\overline{n}_1) &= \nu(z)(\overline{n}_2) \\ \Longleftrightarrow k^z(\overline{n}_1) &= k^z(\overline{n}_2) \\ \Longleftrightarrow \overline{n}_1 &= (k^z)^{-1}k^z\overline{n}_2 \\ \Longleftrightarrow \overline{n}_1 &= \overline{n}_2. \end{aligned}$$

(since k is a unit in the ring $\mathbb{Z}_m$).

Since $\mathbb{Z}_m$ is a finite set every injective self-map is a bijection. And since multiplication by a unit in a ring R is an automorphism on R^+, it follows that $\nu(z) \in \mathrm{Aut}(\mathbb{Z}_m)$ $\forall z \in \mathbb{Z}$.

(b) Now we show that ν is a homomorphism.

If $z_1,z_2 \in \mathbb{Z}$, then for any $x \in \mathbb{Z}_m$

$$\begin{aligned}(\nu(z_1).\nu(z_2))(x) &= \nu(z_1)[(\nu(z_2)(x)] \\ &= k^{z_1}(k^{z_2}x) \\ &= (k^{z_1} \cdot k^{z_2})x \\ &= (k^{z_1+z_2})x \\ &= \nu(z_1+z_2)x.\end{aligned}$$

So the map $\nu(z)$ is a homomorphism. □

We are now in a position to construct an example of a P-local group referred to earlier. We refer to definition 2.15 for semidirect product. For our construction we use a specific group of the form $N = Z_{9261} \rtimes_\nu Z$, and construct from it a 7-local group $M = \mathbb{Z}_{343} \rtimes_\omega \mathbb{Z}_{(7)}$. Here $\mathbb{Z}_{(7)}$ is the 7-localization of $\mathbb{Z}$, as stated in Example 3.2. Note that the prime factorization of 9261 is $9261 = 3^3 \cdot 7^3$.

If $z = \frac{n}{m}$ with $m \in \mathbb{N}$, $n \in \mathbb{Z}$ and $(n,m) = 1$, then the automorphism $\nu(z)$ is as follows.
Pick any number k such that $k^m \equiv 148^n$. Such a number k does exist and its residue class mod 7 is unique since 148 is relatively prime to 7. And then $\nu(z) : \overline{n} \longrightarrow k\overline{n}$.
We abuse notation and write:

$$\nu(z) : \overline{n} \longrightarrow 148^z \overline{n}$$

Of course $\nu(0)$ is taken to be the identity automorphism.

Proposition 4.2 : *Let $N = \mathbb{Z}_{9261} \rtimes_\nu \mathbb{Z}$, where ν is the action of $\mathbb{Z}$ on $\mathbb{Z}_{9261}$, defined by $\nu(z) : \overline{n} \longrightarrow 148^z \cdot \overline{n}$.*

Then N is a two-generator nilpotent group.

Proof : We apply [7, Lemma 1.1] which is quoted as Lemma 2.15. Since 148 is relatively prime to 9261, the group N as described is well-defined. Moreover, we note that $u = 148 = 3.7^2 + 1$ and $m = 9261 = 3^3.7^3$. Thus for every prime divisor p of m, $p \mid u - 1$. Therefore by [7, Lemma 1.1] it follows that N is nilpotent. $\square$

Proposition 4.3 : *Let $M = \mathbb{Z}_{343} \rtimes_\omega \mathbb{Z}_{(7)}$, where the action $\omega : \mathbb{Z}_{(7)} \longrightarrow \mathrm{Aut}\mathbb{Z}_{343}$ is defined by $\omega(z) : x \longrightarrow 148^z \cdot x$.*
Then M is a 7-local group.

Proof : Let $\beta : \mathbb{Z}_{(7)} \longrightarrow \mathbb{Z}_7$ be defined by $\frac{x}{n} \longmapsto \overline{n}^{-1}\overline{x}$, where we assume that $x, n \in \mathbb{Z}$ and $7 \nmid n$. We show that β is well-defined.
We note that for any $c \in \mathbb{Z}$ which is such that c is not a multiple of 7, in $\mathbb{Z}_7$ we have

$$\overline{cn}^{-1} \cdot \overline{cx} = \overline{c} \cdot \overline{n}^{-1} \cdot \overline{c} \cdot \overline{x} = \overline{n}^{-1} \cdot \overline{x}.$$

This shows that β is well-defined.

Also, β is a homomorphism since for any $\frac{x}{n}, \frac{y}{m} \in \mathbb{Z}_{(7)}$ we have

$$\begin{aligned} \beta(\frac{x}{n}) \cdot \beta(\frac{y}{m}) &= (n^{-1}x) \cdot (m^{-1}y) \\ &= (n^{-1}m^{-1}xy) \\ &= (nm)^{-1}xy \\ &= \beta(\frac{xy}{nm}) \end{aligned}$$

$$= \beta(\frac{x}{n} \cdot \frac{y}{m}).$$

Clearly β is surjective since $\beta(1) \neq 0$ and $\mathbb{Z}_7$ is a cyclic group.

Let $\gamma : M \longrightarrow Q = \mathbb{Z}_{343} \rtimes_\omega \mathbb{Z}_7$ be the function induced by β. We mean to say, $\gamma : (a, \frac{x}{n}) \longmapsto (a, \beta(\frac{x}{n}))$. Then γ is a homomorphism, since

$$\begin{aligned} & \gamma[(a, \frac{x}{n})(b, \frac{y}{m})] \\ = & \ (a + \nu(\frac{x}{n})b, \frac{x}{n} + \frac{y}{m}) \\ = & \ (a + 148^{\frac{x}{n}} b, \beta(\frac{x}{n} + \frac{y}{m})) \end{aligned}$$

$$\begin{aligned} & \gamma[(a, \frac{x}{n})]\gamma[(b, \frac{y}{m})] \\ = & \ (a, \beta(\frac{x}{n}))(b, \beta(\frac{y}{m})) \\ = & \ (a + \omega(\beta(\frac{x}{n}))b, \beta(\frac{x}{n} + \frac{y}{m})) \\ = & \ (a + \nu(\frac{x}{n}))b, \beta(\frac{x}{n} + \frac{y}{m})), \ \text{because } \omega \circ \beta = \nu \\ = & \ (a + 148^{\frac{x}{n}} b, \beta(\frac{x}{n} + \frac{y}{m})) \end{aligned}$$

Now $\ker \gamma = \{(0, r) \in M : r = 7s \text{ for some } s \in \mathbb{Z}_{(7)}\}$. Note that the function $\zeta : \mathbb{Z}_{(7)} \longrightarrow \ker \gamma$ defined by $\zeta : s \longmapsto 7s$ is a group homomorphism. In fact, ζ is an isomorphism. Therefore, the finite group Q is a 7-group and consequently 7-local (see Proposition 3.4). Thus since we have a short exact sequence

$$\ker\gamma \longrightarrow M \longrightarrow Q$$

in which $\ker \gamma$ and Q are 7-local. Note that $148^7 \equiv 1 \bmod 343$, and therefore $\ker \gamma < Z(M)$. Thus by Proposition 3.6 it follows that M is 7-local. □

We are now ready to give an example of a non-trivial localizing homomorphism $\lambda : N \longrightarrow M$ of non-abelian groups.

Example 4.4 Let N and M be the groups defined in Proposition 4.2 and Proposition 4.3 respectively. We shall define a 7-localizing homomorphism $\lambda : N \longrightarrow M$.

Let $\lambda_1 : \mathbb{Z}_{9261} \longrightarrow \mathbb{Z}_{343}$ be the epimorphism that converts a residue class mod 9261 to a residue class 343. Such an epimorphism does exist since 9261 is a multiple of 343, and has kernel K is isomorphic to $\mathbb{Z}_{27}$.
Let $\lambda_2 : \mathbb{Z} \longrightarrow \mathbb{Z}_{(7)}$ be the inclusion map, whose cokernel C is an abelian torsion group having no elements of order 7.
Let $\lambda : N \longrightarrow M$ be the function defined by $\lambda : (x, n) \longmapsto (\lambda_1(x), \lambda_2(n))$. □

In the following proposition we prove that λ 7-localizes N.

Proposition 4.5 : *The function $\lambda : N \longrightarrow M$ of Example 4.4 is a 7-localizing group homomorphism.*

Proof : The function λ is a homomorphism, since for any $(x, n), (y, m) \in N$, we have

$$\begin{aligned} \lambda[(x,n)(y,m)] &= \lambda(x + \nu(n)y), n + m) \\ &= \lambda(x + 148^n y, n + m) \\ &= (\lambda_1(x + 148^n y), \lambda_2(n + m)) \end{aligned}$$

$$
\begin{aligned}
\lambda(x,n)\lambda(y,m) &= (\lambda_1(x),\lambda_2(n))(\lambda_1(y),\lambda_2(m)) \\
&= (\lambda_1(x)+\omega(\lambda_2(n)\lambda_1(y),\lambda_2(n)+\lambda_2(m)) \\
&= (\lambda_1(x)+\nu(n)\lambda_1(y),\lambda_2(n)+\lambda_2(m)), \quad \text{because } \omega\circ\lambda_2=\nu \\
&= (\lambda_1(x)+148^n\lambda_1(y),\lambda_2(n)+\lambda_2(m)) \\
&= (\lambda_1(x+148^n y),\lambda_2(n+m))
\end{aligned}
$$

Also, we note that $\ker\lambda = K$, and so λ is 7-injective. Since λ_2 is 7-surjective it follows that also λ is 7-surjective. Thus λ is a 7-isomorphism. Finally, M is 7-local, and therefore λ is a 7-localizing homomorphism. □

For further details on the Mislin genera of groups similiar to N above we refer to Hilton's paper [7]. These groups will receive more attention in the coming chapters.

Chapter 5

THE HILTON-MISLIN GROUP

As a forerunner to the more generally defined non-cancellation group of a $\mathcal{X}_0$-group (discussed in Chapter 6), in this chapter we briefly describe the Hilton-Mislin genus group of a nilpotent $\mathcal{X}_0$-group. We briefly describe the first examples of nilpotent groups with non-trivial genus groups. The groups in the given examples are metacyclic nilpotent $\mathcal{X}_0$-groups, similiar to the group N of the Example 4.4. We conclude the Chapter by quoting the important theorem of Warfield [25, Theorem 3.5] which connects genus, non-cancellation and finite images.

Hilton and Mislin [9] defined an abelian group structure on the genus set $\mathcal{G}(N)$. This group structure facilitates the computation of the genus. Indeed many such computations have appeared since, for instance in [11]. Subsequent theorems which identify induced homomorphisms between genus groups, for instance in [28], further simplify and systematize the computation of genus sets. We utilize such methods in Chapter 7 when we perform some original calculations.
A brief description of this group structure is given below, a synopsis from [9].

5.1 The Hilton-Mislin genus group

Let N be a finitely generated nilpotent group of which the commutator subgroup $[N, N]$ is finite.

Let $Z(N)$ be the center of N, and $T_{Z(N)}$ the torsion subgroup of the center.

Let $n = \mid T_{Z(N)} \mid$.

Define $Z_F(N) = \{x^n : x \in Z(N)\}$. The group $Z_F(N)$ is called the free centre of N. Recall that for the nilpotent $\mathcal{X}_0$-group N, the Hirsch length of N is the rank of the torsion free quotient of the abelianization of N and is finite (see [19] for a more general definition of Hirsch length).

Then $Z_F(N) < Z(N)$ and $Z_F(N) \simeq \mathbb{Z}^h$, where h is the Hirsch length of N.

Let $t = \exp\ (N_{ab}/Z_F(N_{ab}))$, where N_{ab} is the abelianization of the group N.

Then N fits into the short exact sequence:

$$\mathbb{Z}^h \longrightarrow N \longrightarrow N/Z_F(N)$$

If we let f be any endomorphism of $\mathbb{Z}^h$ for which the determinant $\det(f)$ is relatively prime to t, then there exists a group N_f which fits into the following commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}^h & \longrightarrow & N & \longrightarrow & N/Z_F(N) \\ f\downarrow & & \downarrow & & f_1\downarrow \\ \mathbb{Z}^h & \longrightarrow & N_f & \longrightarrow & N/Z_F(N) \end{array}$$

where f_1 is an isomorphism and then $N_f \in \mathcal{G}(N)$. Moreover, if g is another endomorphism of $\mathbb{Z}^h$ such that $\det(g) \cong \pm \det(f) \bmod (t)$, then it can be proved (and

in the next chapter we shall be more explicit) that $N_g \simeq N_f$.
In this way we obtain a surjective function $\delta : \mathbb{Z}_t^*/\{1,-1\} \longrightarrow \mathcal{G}(N)$ which induces a group structure on $\mathcal{G}(N)$ with N as the neutral element.

It is not hard to find examples of metacyclic groups with nontrivial Mislin genera, and we refer to groups of the type $G(n;u)$, which we define below. Such groups have been studied extensively with respect to genus, non-cancellation and finite quotients, e.g [7], [1], [2] and [26]. We now describe such groups.

5.2 Computations

We essentially summarize the computation by Hilton [7] of genera of metacyclic groups. For $n, u \in \mathbb{N}$, with gcd(n,u)= 1, then G(n;u) is the group as described in [6]. $G(n;u) = \langle a, b \mid a^n = 1, b^{-1}ab = a^u \rangle$. Also, $G(n;u)$ happens to be isomorphic to the semidirect product $\mathbb{Z}_n \rtimes_\omega \mathbb{Z}$ where $\omega : \mathbb{Z} \longrightarrow \mathrm{Aut}(\mathbb{Z}_n)$ is the action for which $\omega(1)$ is the automorphism $t \longmapsto ut$ of $\mathbb{Z}_n$. Moreover, if d is the multiplicative order of $\overline{u}$ in $\mathbb{Z}_n^*$, then $G(n;u) \cong G(n;v) \Longleftrightarrow u \cong v$ mod d or $uv \cong 1$ mod d.
The group $G(n;u)$ is nilpotent if and only if every prime divisor of n is a divisor of $u-1$. If H is any finitely generated nilpotent group such that $[H] \in \mathcal{G}(G(n;u))$, then it is shown that $H \simeq G(n;u^a)$ for some a relatively prime to d. Also, $G(n;u) \simeq G(n;v)$ when $uv \equiv 1$ mod n. Thus $\mathcal{G}(G(n;u)) \simeq \mathbb{Z}_d^*/\{1,-1\}$.

The following is a specific example which illustrates the calculation of the genus group $\mathcal{G}(G(n;u))$ of a group of the type $G(n;u)$ as described above. Sample computations can be found in Hilton's paper [7]. For convenience we include a similar example.

Example 5.3 : Choose $N = G(20449; 144)$.

The prime factorization of $n = 20449 = 11^2.13^2$ and $u = 11.13 + 1$.

Then, by [7], $G(20449; 144)$ is nilpotent, since 11 and 13 is a divisor of 143. Also $G(20449; 144) \cong \mathbb{Z}_{20449} \rtimes_\omega \mathbb{Z}$, where $\omega : \mathbb{Z} \longrightarrow \mathrm{Aut}(\mathbb{Z}_{20449})$ is the action for which $\omega(1)$ is the automorphism $z \longmapsto 144 \cdot z$ of $\mathbb{Z}_{20449}$.

The the multiplicative order of u in $\mathbb{Z}^*_{20449}$ is 15.

So $\mathcal{G}(G(20449; 144)) \cong \mathbb{Z}^*_{15}/\{1, -1\}$. □

For a group G, by $\mathcal{F}(G)$ we denote the set of all isomorphism classes of finite quotient groups of G.

The following theorem of R. Warfield allows us to draw comparisons between the genus set $\mathcal{G}(N)$ and the non-cancellation sets $\chi(N)$ and the finite quotient groups $\mathcal{F}(N)$ of a group N.

Theorem 5.4 : [25, Theorem 3.5]. *Let N and M be finitely generated nilpotent groups having finite commutator subgroups. Then the following conditions are equivalent*

(a) $N \times \mathbb{Z} \simeq M \times \mathbb{Z}$

(b) For every prime p, the p-localizations of N and M are isomorphic.

(c) $\mathcal{F}(N) = \mathcal{F}(M)$.

Through this theorem of Warfield, it is shown that for a nilpotent $\mathcal{X}_0$-group N, we have the Mislin genus coinciding with the non-cancellation set of N. This motivated the generalization of the Hilton-Mislin group structure in later works, such as

[27], and the work of O'Sullivan [16]. The more general analogue of the construction of this chapter is given in the next chapter, the non-cancellation group.

Chapter 6

NON-CANCELLATION GROUPS

In this chapter we consider the theory and methods that are employed to detect and describe non-cancellation in the class $\mathcal{X}_0$ of finitely generated groups having finite commutator subgroups. More particularly, we look at the theory needed, and we impose a group structure on the non-cancellation set $\chi(G)$ of a $\mathcal{X}_0$-group. After considering the case of general $\mathcal{X}_0$-groups we shall specialize to the class $\mathcal{K}$ of groups which are of the form $T \rtimes F$, where F is a finite rank free abelian group acting on a finite group T. For the class $\mathcal{K}$ the computation of the non-cancellation group is simpler and more tractable. In the paper [9] of Hilton and Mislin, and in O'Sullivan's paper, [17], there are genus computations for groups which are not in the class $\mathcal{K}$, but otherwise all the calculations of non-cancellation groups are for $\mathcal{K}$-groups. Moreover, in most cases the $\mathcal{K}$-groups have abelian torsion subgroups. Exceptions where we have computations of $\chi(G)$ for $\mathcal{K}$-groups G with T_G non-abelian, can be found in [8], [29] and in [13]. In the final chapter of this mini-thesis we compute $\chi(G)$ for G similar to those of [13]. In this chapter we prepare the theory to support the forthcoming calculations. We also look at the non-cancellation groups of direct products of certain non-nilpotent groups.

Throughout this chapter propositions and theorems will almost in every case be merely stated, mainly from [27], [29] and [28]. The group structure on $\chi(G)$ is in terms of certain subgroups of G, more particularly, in terms of the indices of such subgroups. In this way we obtain a very nice surjective map $\mathbb{Z}_n^* \longrightarrow \chi(G)$, for some $n \in \mathbb{N}$, which induces the group structure on $\chi(G)$. We now proceed with the detail.

6.1 The group $\chi(G)$

From Proposition 2.11 in Chapter 2 we know that if G is a finite group, then $\chi(G)$ is trivial, i.e., has only one member, and is then trivially a group. Thus, we need only consider the case of infinite $\mathcal{X}_0$-groups G when describing the group structure on $\chi(G)$.

We associate with every $\mathcal{X}_0$-group G an integer n_G which we define as follows:

Let n_1 be the exponent of T_G, let n_2 be the exponent of $\mathrm{Aut}(T_G)$ and let n_3 be the exponent of the torsion subgroup of the centre $Z(G)$ of G, and we define $n_G = n_1 n_2 n_3$.

Let X be the set of all natural numbers which are relatively prime to n_G. If G is infinite then for every $x \in X$ there is a subgroup H_x of G such that the index $[G : H_x]$ of H in G is exactly x (see [27, Theorem 2.5]). By [27, Theorem 4.1] we have that for such a subgroup $H_x \times \mathbb{Z} \simeq G \times \mathbb{Z}$, by [27, Theorem 4.3]. Such a subgroup H_x is unique up to isomorphism. Thus we obtain a well-defined function $\mu : X \longrightarrow \chi(G)$ by picking, for each x, $\mu(x) = [H_x]$, of a subgroup H_x of G such that $[G : H_x] = x$. By [27, Theorem 4.2], μ is surjective. Moreover, [27, Theorem 4.3] implies that μ induces a well-defined function

$$\theta : \mathbb{Z}_n^*/\{1,-1\} \longrightarrow \chi(G)$$

where $n = n_G$, and θ is surjective. Finally, by [27, Theorem 5.1] we have that for every $\Gamma \in \chi(G)$, $\theta^{-1}(\Gamma)$ is a coset of $\Gamma \in \chi(G)$. Thus θ induces a bijection between $\chi(G)$ and some quotient group of $\mathbb{Z}_n^*$, and we regard $\chi(G)$ as a group via this induced bijection which is of course regarded to be an isomorphism. Moreover, for a nilpotent $\mathcal{X}_0$-group G, the groups $\chi(G)$ and $\mathcal{G}(G)$ coincides. Examples will be shown after we have discussed the special case of $\mathcal{K}$-groups.

6.2 Notation: (a) For a positive integer k, let $\mathcal{K}^{(k)}$ be the class of all groups G which are semidirect products of the form $G = T \rtimes_\omega \mathbb{Z}^k$, where T is a finite group and $\omega : \mathbb{Z}^k \longrightarrow \mathrm{Aut}(T)$ is an action of $\mathbb{Z}^k$ on T. Of course, in this case $T = T_G$. By $\mathcal{K}$ we mean the union of the classes $\mathcal{K}^{(k)}$.

(b) The subclass of all nilpotent groups in $\mathcal{K}$ is denoted by $\mathcal{K}_{nil}$.

6.3 Definition

The *Prüfer rank* of an abelian group B is the least of the cardinalities of the generating subsets of group B, and we shall write this as $\mathrm{rank}(B)$ (see Robinson's book [19, p.96] for instance).

For the remainder of this chapter our groups will be of the form $G = T \rtimes_\omega \mathbb{Z}^k$, where T is a finite group and $\omega : \mathbb{Z}^k \longrightarrow \mathrm{Aut}(T)$ is the action of $\mathbb{Z}^k$ on T.

6.4 The group $\chi(T \rtimes_\omega \mathbb{Z}^k)$

We describe a simpler means of determining $\chi(G)$ where G is assumed to be a specific $\mathcal{K}$-group $G = T \rtimes_\omega \mathbb{Z}^k$, for some $k \in \mathbb{N}$ and some homomorphism $\omega : \mathbb{Z}^k \longrightarrow \mathrm{Aut}(T)$. Now let

$$S = \{t \in \mathbb{N} : \mathrm{rank}(t\ \mathrm{Im}\omega) < \mathrm{rank}(\mathrm{Im}\omega)\},$$

where for the moment we regard the group $\mathrm{Im}\,\omega$, which is of course commutative, as an additive group and $t \cdot \mathrm{Im}\,\omega$ means the set of all t-fold compositions $\alpha \circ \alpha \circ \cdots \circ \alpha$ of elements $\alpha \in \mathrm{Im}\ \omega$. If $S = \phi$, i.e., if S is empty then G is abelian and $\chi(G)$ is trivial. Thus we shall further assume that S is non-empty. Then S has a least number, which we shall denote by d. The number d can be obtained in the following way. For a finite abelian group B ($B = \mathrm{Im}\ \omega$ in this case) there is an essentially unique way of writing B as a direct sum of cyclic subgroups $B = B_1 \times B_2 \times \cdots \times B_r$, with r as small as possible and such that $|B_i|$ is a factor of $|B_{i-1}|$ $\forall\ i \geq 2$ (see the book of Stewart and Tall [24]). Given such a decomposition for $B = \mathrm{Im}\omega$, then $d = |B_r|$.

In [27] it is proved that there is an epimorphism

$$\mathbb{Z}_d^*/\{1, -1\} \longrightarrow \chi(G).$$

Note that if p is a prime then a group of the type $G(p; u)$ is nilpotent if it is abelian, since for nilpotence we must have $u \equiv 1 \bmod p$ (see [7]). So it is easy now to find a non-nilpotent group of the type $G(p; u)$ and calculate the non-cancellation group $\chi(G(p; u))$.

Example 6.5 Let $G = G(23; 2)$. Then the order of $\overline{2}$ in $\mathbb{Z}_{23}^*$ is 11. So G is of the form $G = \mathbb{Z}_{23} \rtimes_\omega \mathbb{Z}$ where $\mathrm{Im}\omega$ is a cyclic group of order 11. Thus from [23, Theorem 3.8] we have

$$\chi(G) = \mathbb{Z}_{11}^*/\{1,-1\}.$$

Computation of the non-cancellation groups is facilitated by existence of certain group homomorphisms

$$\chi(G) \longrightarrow \chi(H)$$

between $\mathcal{X}_0$-groups which arises from suitable homomorphisms between the groups G and H themselves. Such induced homomorphisms are discussed in [27], [13], [29] and [28]. We cite here those which are required for our computation in Chapter 7.

Theorem 6.6 : *([13, Theorem 4]), Let H be a $\mathcal{X}_0$-group, and let $n = n(H)$. Let F be a finite subgroup of H with the property that, given any embedding $\phi : H \longrightarrow H$ such that $[H : \phi(H)]$ is relatively prime to n, then $\phi(F) = F$. Then for subgroups K of H with $[H : K]$ relatively prime to n, the association $K \longmapsto K/F$ defines an epimorphism $\eta : \chi(H) \longrightarrow \chi(H/F)$.*

Remark: We note in particular that if in Theorem 6.6 the subgroup F is a characteristic subgroup of the torsion radical T_H of H, then the function $\eta : \chi(H) \longrightarrow \chi(H/F)$ is an epimorphism. This is because an embedding such as descibed in the theorem, induces an automorphism of T_H, because $\mid T_H \mid$ is relatively prime to $[H : \phi(H)]$.

For $\mathcal{X}_0$-groups G and H and for groups K belonging to $\chi(G)$, the rule $K \longmapsto K \times H$ induces a well-defined function

$$\phi : \chi(G) \longrightarrow \chi(G \times H).$$

In fact we have the following theorem.

Theorem 6.7 : *([27, Theorem 6.2]), Let G and H be $\mathcal{X}_0$-groups and suppose that G is infinite. Then the function*

$$\phi : \chi(G) \longrightarrow \chi(G \times H)$$

is a surjective homomorphism of groups.

Theorem 6.8 : *([28, Theorem 4.1]), Let G be a $\mathcal{X}_0$-group and F is a characteristic subgroup of the torsion subgroup of $G \times F$. Then the induced epimorphism $\phi : \chi(G) \longrightarrow \chi(G \times F)$ is injective.*

We note in particular that while we always have an epimorphism

$$\chi(G) \longrightarrow \chi(G \times G)$$

for every $\mathcal{X}_0$-group, such an epimorphism is not always an isomorphism. Even for a nilpotent $\mathcal{K}^{(1)}$-group N it is possible that the group $\mathcal{G}(N \times N)$ may fail to be isomorphic to $\mathcal{G}(N)$. This problem was solved in the paper [12] by Hilton and Schuck. In greater generality for G of the form $G(n; u)$ the comparison between $\chi(G)$ and $\chi(G \times G)$ appears in [31]. One case we want to formulate in this regard is the following.

Theorem 6.9 : *[23, Theorem 3.9], Let $G = T \rtimes_\omega \mathbb{Z}$ be a group in $\mathcal{K}$ such that the torsion subgroup T is cyclic of prime power order. Then for a fixed positive integer n, the function which sends $H \in \chi(G)$ to $G^n \times H$ sets up an isomorphism $\chi(G) \simeq \chi(G^{n+1})$.*

The following theorem enables us to immediately produce examples. The theorem generalizes the main result of [4].

Theorem 6.10 : [*23, Theorem* 3.8], *Let T be a finite abelian group and suppose $G = T \rtimes_{\omega} \mathbb{Z}^k$ is in $\mathcal{K}$ such that the image* $\mathrm{Im}\omega$ *of the action belongs to the center of* $\mathrm{Aut}(T)$. *Then*

(a) if $k \geq \mathrm{rank}(\mathrm{Im}\omega)$, *then* $\chi(G) = 0$

(b) if $k = \mathrm{rank}(\mathrm{Im}\omega)$, *then* $\chi(G) \cong \mathbb{Z}_d^*/\{1, -1\}$,

where d is the least positive integer n such that the rank of $m \cdot \mathrm{Im}\omega$ is strictly less than the rank of $\mathrm{Im}\omega$.

The relationship between localization and non-cancellation, beyond the scope of Warfield's theorem, is discussed in O'Sullivan's paper [16]. Calculations of non-cancellation groups appear in [6], [16] and [29].

Chapter 7

A CLASS OF GROUPS WITH NON-ABELIAN TORSION

Most of the computation of non-cancellation groups so far, for example in [6], [27] and [16], has been for groups G for which the torsion radicals T_G are abelian. The groups arising in [13] and in [29] are basically the only cases of groups with non-abelian torsion subgroups for which the non-cancellation set has been calculated. In this section we compute the non-cancellation groups of a class of groups similar to those in [13]. We consider here a $\mathcal{X}_0$-group H that has a torsion radical which is a non-commutative, three-generator nilpotent (finite) group, T.

The first part of this chapter is an exercise to analyze the structure of certain finite groups $T_{n,m}$. These finite groups will serve as the torsion radicals of a group H, for which we calculate $\chi(H)$ and even, in cases, $\chi(H \times H)$. All of this work is independent. While the analysis of the group T may be a relatively routine exercise, eventually the computation of $\chi(H \times H)$ is a new contribution.

7.1 The groups $T = T_{n,m}$

For any pair of natural numbers n, m let $T_{n,m}$ be the group defined as follows.

$$T_{n,m} = \quad \langle x, y, z \mid x^m = 1 = y^m, [x, y] = z^n, \quad z^{nm} = 1, \quad [x, z] = 1 = [y, z] \rangle$$

We shall study a class of groups H of which the torsion radical T will be the group $T = T_{n,m}$, for some $n, m \in N$. The groups H will be defined in item 7.5. For now we want to make a detailed study of these candidate torsion groups. The following two propositions are on the structure of the groups $T_{n,m}$.

Proposition 7.2 : *The order of T is $\mid T \mid = nm^3$.*

Proof : Let $Z = \langle z \rangle$, then $Z \lhd T$ and

$$\begin{aligned} T/Z &= \langle x, y, z \rangle / Z \\ &= \langle xZ, yZ \rangle, \end{aligned}$$

but $\langle x \rangle \simeq \mathbb{Z}_m$.

Since $[x, y] \in Z$ it follows that the group T/Z is abelian and $T/Z = \langle xZ, yZ \rangle$. In fact $T/Z \cong xZ \times yZ \equiv \mathbb{Z}_m \times \mathbb{Z}_m$.

$$\begin{aligned} \langle xZ, yZ \rangle &\simeq \mathbb{Z}_m \times \mathbb{Z}_m \\ \text{so} \quad \mid \langle xZ, yZ \rangle \mid &= m^2 \\ \text{hence} \quad \mid T \mid &= m^2 \mid Z \mid \\ &= m^2.mn \\ &= m^3 n. \end{aligned}$$

$\square$

Proposition 7.3 : *Every element $g \in T$ can be expressed as $g = x^{\alpha}y^{\beta}z^{\gamma}$, for some $\alpha, \beta, \gamma \in \mathbb{N}$. Moreover, given such a $g \in T$, then α is unique up to congruence modulo m, β is unique up to congruence modulo m, and γ is unique up to congruence modulo mn.*

Proof : Note that in Proposition 7.2 it was shown that $T/Z = \langle xZ \rangle \oplus \langle yZ \rangle$, and $\langle xZ \rangle \cong \langle yZ \rangle \cong \mathbb{Z}_m$.

Consider any element $g \in T$. Then we have for $gZ \in T/Z$ that

$$\begin{aligned} gZ &= (xZ)^a.(yZ)^b, \quad \text{for some } a, b \in \mathbb{N} \\ \therefore g.1 &= x^a z^c.y^b z^d, \quad \text{for some } c, d \in \mathbb{N} \\ \therefore g &= x^a y^b z^{c+d}, \quad \text{since } z \text{ commutes with } x \text{ and } y \end{aligned}$$

Suppose g can also be expressed as $x^{\alpha}y^{\beta}z^{\gamma}$, for some $\alpha,\beta,\gamma \in \mathbb{N}$. Then for some element gZ of T/Z, we get

$$\begin{aligned} (x^a y^b z^{c+d})Z &= (x^{\alpha}y^{\beta}z^{\gamma})Z \\ \iff \quad x^a y^b Z &= x^{\alpha}y^{\beta}Z \end{aligned}$$

Thus it follows that $\alpha \equiv a \pmod m$ and $\beta \equiv a \pmod m$. Hence we can express gZ uniquely as $gZ = x^{\alpha}y^{\beta}Z$ for $0 \leq \alpha, \beta < m$.

Now $g = x^a y^b z^e = x^{\alpha}y^{\beta}z^{\gamma}$ (where $e = c + d$).

But then $z^e = z^{\gamma}$

therefore $\gamma \equiv e \bmod mn$. □

We now identify some specific automorphisms of the group $T_{n,m}$.

Proposition 7.4 : *For fixed $u, v \in \mathbb{N}$, with $gcd(u, mn) = 1$ and $gcd(v, m) = 1$, define a function $\sigma : T \longrightarrow T$ by $\sigma : x^a y^b z^c \longrightarrow x^a y^{vb} z^{uc}$. Then the function σ is an automorphism if $m \mid u - v$.*

Proof : We show that σ is well-defined:

Let t_1 and t_2 be arbitrary elements of T, with $t_1 = x^a y^b z^c$ and $t_2 = x^f y^g z^h$ for some $0 \leq a, b, f, g \leq m$ and $0 \leq c, h \leq mn$.

$$\begin{array}{rrcl} \text{Suppose} & t_1 & = & t_2 \\ \text{Then} & x^a y^b z^c & = & x^f y^g z^h \\ \text{Thus,} & x^a y^b z^c \cdot (x^f y^g z^h)^{-1} & = & 1 \\ \text{so,} & x^a y^b z^c . z^{-h} y^{-g} x^{-f} & = & 1 \\ & x^a y^b z^{c-h} y^{-g} x^{-f} & = & 1 \\ & x^a y^{b-g} x^{-f} z^{c-h} & = & 1, \qquad (*) \end{array}$$

(since x and y commute with z)

Now, from the definition of T, we have

$$\begin{array}{rrcl} \text{Now} & [x, y] & = & z^n \\ \Rightarrow & x^{-1} y^{-1} x y & = & z^n \\ \Rightarrow & y^{-1} x y & = & x z^n \\ \Rightarrow & y^{-1} x^a y & = & x^a z^{na}, \end{array}$$

(since x and z commute).

Conjugating by y yields

$$\begin{aligned} y^{-2}x^a y^2 &= y^{-1}x^a z^{na} y \\ &= y^{-1}x^a y z^{na}, \qquad \text{since } y \text{ and } z \text{ commute.} \\ &= x^a z^{2na}. \end{aligned}$$

Thus, inductively, conjugating by y^b gives

$$\begin{aligned} y^{-b}x^a y^b &= x^a z^{nab} \\ \therefore \quad x^a y^b &= y^b x^a z^{nab} \\ \therefore \quad y^b x^a &= x^a y^b z^{-nab}. \end{aligned}$$

We return to the identity $(*)$:

$$\begin{aligned} \text{Thus, } 1 = \quad x^a y^{b-g} x^{-f} z^{c-h} &= x^{a-f} y^{b-g} z^{nf(b-g)}.z^{c-h} \\ &= x^{a-f} y^{b-g} z^{nf(b-g)+(c-h)} \end{aligned}$$

By Proposition 7.3 it follows that

$$\begin{aligned} a - f &\equiv 0 \quad \text{mod } m, \\ \text{and} \qquad b - g &\equiv 0 \quad \text{mod } m, \\ \text{i.e.} \qquad a &\equiv f \quad \text{mod } m \text{ and } \quad b \equiv g \quad (\text{mod})\ m \\ \text{Also} \qquad nf(b-g) + (c-h) &\equiv 0 \quad (\text{mod})\ nm \\ \text{but} \qquad b - g &\equiv 0 \quad (\text{mod})\ m, \\ \text{and so} \qquad c &\equiv h \quad (\text{mod})\ nm \end{aligned}$$

In particular we have

$vb \equiv vg \mod m$ and $uc \equiv uh \mod nm$.

$$\text{so} \quad x^a y^{vb} z^{uc} = x^f y^{vg} z^{uh}$$

so σ is well-defined.

We show that σ is a homomorphism : For arbitrary elements t_1 and t_2 in T, as above, we have

$\sigma(t_1 \cdot t_2) = \sigma(x^a y^b z^c . x^f y^g z^h)$.

We calculate the product $t_1 \cdot t_2$:

$$\begin{aligned} t_1 \cdot t_2 &= (x^a y^b z^c . x^f y^b z^h) \\ &= (x^a y^b x^f y^g z^{c+h}), \quad \text{since } x \text{ and } y \text{ commute with } z. \\ &= (x^a x^f y^b y^g z^{-nfb+c+h}) \\ &= (x^{a+f} y^{b+g} z^{c+h-nfb}). \end{aligned}$$

$$\begin{aligned} \text{Therefore,} \quad \sigma(t_1 \cdot t_2) &= \sigma(x^{a+f} y^{b+g} z^{c+h-nfb}) \\ &= x^{a+f} y^{v(b+g)} z^{u(c+h-nfb)}. \end{aligned}$$

$$\begin{aligned} \text{On the other hand,} \quad \sigma(t_1).\sigma(t_2) &= \sigma(x^a y^b z^c).\sigma(x^f y^g z^h) \\ &= x^a y^{vb} z^{uc} . x^f y^{vg} z^{uh} \\ &= x^a y^{vb} x^f y^{vg} z^{u(c+h)} \\ &= x^a x^f y^{vb} y^{vg} z^{-nfvb} z^{u(c+h)} \\ &= x^{a+f} y^{v(b+g)} z^{u(c+h)-nvfb}. \end{aligned}$$

Thus, σ is a homomorphism if and only if for all $a, b, c, f, g, h \in \mathbb{N}$ we have

$$unfb \equiv nvfb \pmod{mn}.$$

$$\begin{aligned}
\text{Now} \quad unfb &\equiv nvfb \pmod{mn} \\
\iff \quad unfb - vnfb &\equiv 0 \pmod{mn} \\
\iff \quad (u-v)nfb &\equiv 0 \pmod{mn} \\
\iff \quad (u-v)fb &\equiv 0 \pmod{m}.
\end{aligned}$$

Now the latter condition must hold in particular for $f = b = 1$.

Thus σ is a homomorphism if and only if $m \mid u - v$, and the latter is assumed.

We show that σ is injective:

$$\begin{aligned}
\text{Suppose that} \quad \sigma(x^a y^b z^c) &= \sigma(x^f y^g z^h). \\
\text{Then} \quad x^a y^{vb} z^{uc} &= x^f y^{vg} z^{uh}.
\end{aligned}$$

$$\begin{aligned}
\text{Consequently,} \quad vb &\equiv vg \pmod{m} \quad \textit{and} \quad uc \equiv uh \pmod{mn} \\
v(b-g) &\equiv 0 \pmod{m} \quad \textit{and} \quad u(c-h) \equiv 0 \pmod{mn} \\
b-g &\equiv 0 \pmod{m} \quad \textit{and} \quad (c-h) \equiv 0 \pmod{mn} \\
\text{and so,} \quad b &\equiv g \pmod{m} \quad \textit{and} \quad c \equiv h \pmod{mn}. \\
\text{Thus,} \quad x^a y^b z^c &= x^f y^g z^h.
\end{aligned}$$

So σ is injective. Since σ is injective and T is finite, it immediately follows that the self-map σ of T is surjective. □

Now that we have constructed some automorphisms of $T_{n,m}$ we can define the class of groups for which we want to compute the non-cancellation group.

7.5 The group $H = T_{n,m} \rtimes_\zeta \mathbb{Z}$

Let $m, n, u, v \in \mathbb{N}$ such that u is relatively prime to mn and v is relatively prime to m, and $n \mid u - v$.

Let $T = T_{n,m}$ be the group as defined above in this Chapter.

Let ζ be the action of $\mathbb{Z}$ on T such that $\zeta(1)$ is defined by the formula:

$\zeta(1) : x^a y^b z^c \longmapsto x^a y^{vb} z^{uc}$. Now let $H(m,n,u,v)$ be the group

$$H(m,n,u,v) = T_{n,m} \rtimes_\zeta \mathbb{Z}.$$

Proposition 7.6 : *Let $m, n \in \mathbb{N}$ and let T be as defined in paragraph 7.1. Then the subgroup $F = \{t \in T : t^m = 1\}$ is a characteristic subgroup of T and $T/F \cong \mathbb{Z}_n$.*

Proof : Let $\alpha : T \longrightarrow T$ be an automorphism. Then $[\alpha(t)]^m = \alpha(t^m)$.

Therefore, for every $t \in F$, then $\alpha(t) \in F$.

Thus F is a characteristic subgroup of T.

Consider any $k, l \in \mathbb{N}$ and suppose that

$(zF)^k = (zF)^l$

Then $z^k F = z^l F$, and so $z^k = z^l f$ for some $f \in F$, and then $f = z^{k-l}$.

Now $z^{(k-l)m} = (z^{k-l})^m = f^m = 1$.

Since the order of z is mn, it follows that $n \mid k-l$.

Therefore the elements

$zF, z^2F, z^3F, \cdots, z^nF$ are all distinct, so $\mid T/F \mid \geq n$.

On the other hand $x, y, z^n \in F$, and by Proposition 7.3 it follows that $\mid F \mid \geq m^3$.

However by Proposition 7.2, $\mid T \mid = m^3 n$, therefore $\mid F \mid = m^3$ and $\mid T/F \mid = n$.

Thus $T/F \cong \mathbb{Z}_n$.

Proposition 7.7 : *Let H be the group as defined in 7.5 above, and let F be the characteristic subgroup of H defined in Proposition 7.6. Then*

(a) There is an epimorphism $\chi(H) \longrightarrow \chi(H/F)$.

(b) $\chi(H/F) \equiv \mathbb{Z}^*_{\widetilde{d}}/\{1,-1\}$ where $\widetilde{d}$ is the multiplicative order of u mod n.

(c) there is an epimorphism $\mathbb{Z}^*_s/\{1,-1\} \longrightarrow \chi(H)$, where s is the lcm of d and d', and where d is the multiplicative order of u mod mn and d' is the multiplicative order of v mod m.

Proof : (a) This follows by Theorem 6.6 (see the remark immediately after Theorem 6.6) since by Proposition 7.6 the subgroup F of T is characteristic.

(b) Note that $T_{(H/F)} \cong T_H/F \cong \mathbb{Z}_n$.

Thus our assertion follows by Theorem 6.10(b).

(c) For ζ (as in paragraph 7.5), the element $\zeta(1)$ of Aut(T) has order s. So $Im(\zeta) \cong \mathbb{Z}_s$, and therefore by [29, Theorem 2.6] we have the asserted epimorphism $\mathbb{Z}^*_s/\{1,-1\} \longrightarrow \chi(H)$. □

Corollary 7.8 : *If in the notation of Proposition 7.7 above, $d = \widetilde{d}$ and d is a multiple of d', then $\chi(H) \cong \chi(H/F)$.*

Proof : Under the conditions of the corollary, in view of Proposition 7.7 (c), we have $s = d$.

Thus from Proposition 7.7 (a) and (b) above, it follows that we have a sequence of epimorphisms

$$\mathbb{Z}_d^*/\{1,-1\} \longrightarrow \chi(H) \longrightarrow \chi(H/F) \longrightarrow \mathbb{Z}_{\widetilde{d}}^*/\{1,-1\}.$$

Since $d = \widetilde{d}$, the composition of these morphisms is an isomorphism. Since the groups in this sequence are finite, it follows that $\chi(H) \cong \chi(H/F)$. □

We give two examples, with $(m,n) = 1$ as well as with $(m,n) \neq 1$, where the corollary is applicable.

Example 7.9

Take $n = m = 11$ and $u = v = 3$. Then for these choices the group H is defined. We calculate d, $\widetilde{d}$ and d'. The powers of 3 are $3, 9, 27, 81, 243, \ldots$etc and we note that $243 = 2(11)^2 + 1$ i.e $3^5 \equiv 1 \ (mod\ 11^2)$. Thus $d = \widetilde{d} = d' = 5$ and so $\chi(H) \cong \chi(H/F) \cong \mathbb{Z}_5^*/\pm 1 \cong \mathbb{Z}_2$. □

Example 7.10

Take $n = 11$, $m = 7$, $u = v = 6$. The multiplicative order of $u\ (mod\ 7)$ is 2, i.e,

$$d' = 2.$$

A simple calculation shows that the multiplicative order of u mod 11 is 10, ie,

$$\widetilde{d} = 10$$

and the multiplicative order of u mod 77 is 10, i.e,

$$d = 10.$$

Thus $d = \widetilde{d} = 10$ and $d' \mid d$.
Therefore by Corollary 7.8 it follows that
$\chi(H) \cong \chi(H/F) \cong \mathbb{Z}_{10}^{*}/\{1, -1\} \cong \mathbb{Z}_2$. $\square$

Even for a group G of the form $G = \mathbb{Z}_n \rtimes_\mu \mathbb{Z}$ it is not always the case that $\chi(G)$ is isomorphic to $\chi(G^2)$. It was shown for such groups G in [31] that if $\chi(G) \cong \chi(G^2)$ then $\chi(G) \cong \chi(G^k)$ for all $k \in \mathbb{N}$ and the groups which satisfy the latter condition were identified in terms of n and u.

Our final result is on $\chi(H^k)$ with certain conditions.

Theorem 7.11 : *If H and (resp.) F are as in paragraph 7.5 and (resp.) Proposition 7.6 above and $\chi(H) \cong \chi(H/F)$ and $\chi(H/F) \cong \chi[(H/F)^2]$, then for every $k \in \mathbb{N}$, we have $\chi(H) \cong \chi(H^k) \cong \chi(H/F)^k \cong \chi(H/F)$, where the index k denotes the k-th direct power.*

Proof : We know that $H/F \cong G(n, u)$. We can assume that $k \geq 2$. It is shown in [31] that the conditions on n and u which imply $\chi[G(n,u)] \cong \chi([G(n,u)]^2)$, will immediately imply $\chi[G(n,u)] \cong \chi([G(n,u)]^k)$, for all $k \geq 2$. Thus we obtain an isomorphism

$$\delta : \chi([G(n;u)]^k) \longrightarrow \chi([G(n;u)]^2).$$

By Theorem 6.7 there is an epimorphism

$$\alpha : \chi(H) \longrightarrow \chi(H^k).$$

Similarly to the proof of Proposition 7.6 it can be proved that F^k is a characteristic subgroup of the torsion radical of H^k. Therefore by Theorem 6.6 we have an epimorphism

$$\beta : \chi(H^k) \longrightarrow \chi([H/F]^k).$$

Of course, since $H/F \cong G(n;u)$ we have an isomorphism

$$\gamma : \chi([H/F]^k) \longrightarrow \chi([G(n;u)]^k).$$

Thus we obtain a sequence of homomorphisms as below, and in particular, they are all epimorphisms.

$$\chi(H) \xrightarrow{\alpha} \chi(H^k) \xrightarrow{\beta} \chi([H/F]^k) \xrightarrow{\gamma} \chi([G(n,u)]^k) \xrightarrow{\delta} \chi([G(n,u)]^2)$$

Since by assumption the first and last group are isomorphic and are finite, it follows that each epimorphism in the above sequence is an isomorphism. This completes the proof. □

If $\chi(H/F)$ is not isomorphic to $\chi(H/F \times H/F)$ then it may be that $\chi(H)$ fails to be isomorphic to $\chi(H \times H)$. Such a computation will be rather a substantial one. On the other hand, cases in which we do get an isomorphism

$$\chi(H/F) \longrightarrow \chi(H/F \times H/F)$$

can be deduced from [31]. We quote such a theorem.

Let us suppose that n_a, n_b, u are pairwise relatively prime positive integers, and let $n = n_a n_b$. For $i = \{a, b\}$ let d_i be the multiplicative order of u modulo n_i and let γ be the greatest common divisor of d_a and d_b.

Theorem 7.12 : [*31, Theorem 3.1*]. *Let k be an integer bigger than* 1. *Suppose that $\chi([G(n_a;u)]^k) \simeq \chi(G(n_a;u))$, $\chi([G(n_b;u)]^k) \simeq \chi(G(n_b;u))$ and $\gamma > 2$. Then $\chi([G(n;u)]^k) \simeq \chi(G(n;u))$.*

The condition $\chi([G(n_a;u)]^k) \cong \chi(G(n_a;u))$ is known to hold when n_a is a prime power, see [23, Theorem 3.9].

Our final example illustrates Theorem 7.11. The calculation in the example, of the multiplicative orders modulo integers, was done in MAPLE and we attach the MAPLE worksheet.

Example 7.13

Take $n = 85 = 17.5$, $m = 17$, $u = 1022$, and $v = 2$.

Then we get $d = d' = \widetilde{d} = 8$. So by Corollary 7.8 we have $\chi(H) \simeq \chi(H/F)$.

Now we apply Theorem 7.12 to the group $G(85; 1022)$.
Taking $n_a = 17$ and $n_b = 5$ we get $d_a = 8$ and $d_b = 4$, so that $\gamma = 4 > 2$ and therefore, for any $k \in \mathbb{N}$, by Theorem 7.12 we have:

$$\chi(G(85;1022)) \cong \chi([G(85;1022)]^k).$$

Thus Theorem 7.11 applies, and we have

$$\chi(H) \cong \chi(H^k) \cong \chi(H/F) \cong \mathbb{Z}_8^*/\{1,-1\}.$$

MAPLE CALCULATION USED IN EXAMPLE 7.13

```
> msolve({1022^n=1},1445);
                                        {n = 8 _Z1~}
> msolve({1022^n=1},85);
                                        {n = 8 _Z1~}
> msolve({1022^n=1},17);
                                        {n = 8 _Z1~}
> msolve({1022^n=1},5);
                                        {n = 4 _Z1~}
> msolve({2^n=1},17);
                                        {n = 8 _Z1~}
>
```

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