PROBLEM-SOLVING IN GEOMETRY IN COLLABORATIVE SMALL GROUP SETTINGS: How learners appropriate mathematical tools while working in small groups.

PHADIELA COOPER

A thesis submitted in partial fulfilment of the requirements for the degree of Magister Educationis in the Department of Education, University of the Western Cape.

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Supervisors: Prof. M. Mbekwa (University of the Western Cape)

Prof. C. Julie (University of the Western Cape)

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Phadiela Cooper

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Abstract

Problem-solving in Mathematics is an important skill. The poor performance of South African learners in international tests such as the Trends in International Mathematics and Science Study (TIMSS) and in schools in general indicates that emphasis should be placed on problem-solving in the teaching and learning of Mathematics. The new national senior certificate curriculum in South Africa encourages group work amongst learners. The thesis proposes that learning is enhanced in a small-group setting, since learners actively engage with the problems. Furthermore, Euclidean Geometry is perceived by learners to be a 'difficult' section of Mathematics. However, Geometry is important since the skills acquired while doing Geometry can be applied to various fields of study.

This research focused on Geometry problem-solving in collaborative small-group settings. An inductive approach was taken that focused on what learners were doing while they were doing problem-solving in geometry in collaborative groups. Problem-solving is viewed as a situated and contextually-determined activity. The research focused on how learners appropriated tools (physical as well as intellectual) and how they interacted with one other and the subject matter. The socio-cultural perspective was the theoretical framework underpinning the study. In this perspective, learning is seen as a social process in which learners actively participate and contribute with ideas and arguments. In addition, learning is seen as a situated activity.

The research was carried out in the form of a case study that focused on three groups of three learners each, from a secondary school in Khayelitsha, a township approximately 30 km outside Cape Town, South Africa. The small groups were monitored and observed in a school setting and special attention was given to their interaction within their group, given their social and cultural context. The ethnographic approach to data gathering, which allows for the routine, everyday, taken-for-granted aspects of school and classroom life, was used.

Data were collected by means of audio and video recordings, interviews with learners and teacher observations. The data analysis included analysis of field notes, audio and video transcripts and learners' written work. The data were analysed in terms of Pickering's theory that all scientific practice is a "dialectic of resistance and accommodation" and that this constitutes a "mangle of practice" (Pickering, 1995).

This study found that during the conceptual practice of collaborative groups, instances of bridging, transcription and filling, as *per* Pickering's theory, could be identified. The cyclical nature of these processes also came to the fore, thus highlighting the classification of scientific practice as posthumanist. The appropriation of physical tools (calculator, mathematical instruments), intellectual tools (inscriptions, existing mathematical knowledge) and cultural tools (language, the manner in which learners addressed one another) by learners helped us to understand the process of problem-solving in small groups.

P. Cooper

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Declaration

I declare that *Problem-solving in geometry in collaborative small group settings: How learners appropriate mathematical tools while working in small groups* is my own work, that it has not been submitted for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged by complete references.

Phadiela Cooper Date: November 2011

Signed:

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CHAPTER 1

INTRODUCTION

1.1 Need for problem-solving in Mathematics

The poor performance of South African learners in international tests, such as the Trends In Mathematics and Science Study (TIMSS), and in national Mathematics examinations in general suggests that greater emphasis should be placed on problem-solving. In addition, the modern workplace increasingly requires workers with problem-solving skills. The school would be the appropriate place for these problem-solving skills to be learnt. Doing Mathematics provide ample opportunities for learners to acquire these problem-solving skills. Geometry, in particular, is well-suited to learning and enhancing problem-solving skills since it allows learners to experience the whole problem-solving process as explained by Polya (1957).

In this research I focused on Geometry problem-solving by secondary school learners and how this is facilitated in a small-group setting. The focus was on how learners interacted with Geometry material and how they worked collaboratively in small groups while solving problems. The aim was to understand what learners were doing when they were doing problem-solving in their particular context, i.e. how they appropriated the cultural tools at their disposal.

One of the critical outcomes on which the National Curriculum Statements was built states that the learners are required to "work effectively with others as members of a team, group, organization and community" (Department Of Education, 2003, p. 2). Educators in South Africa have been moving more towards using group work in classrooms while they act as facilitators. Thus, educators would be at hand to help, guide and assist learners whenever they encountered problems during group work. This research investigated how learners worked collaboratively in small groups while solving Geometry problems without the assistance of an educator.

In this chapter the background of the study is discussed, followed by a motivation on why the study was embarked on. The research questions are also stated and are followed by an overview of the thesis.

1.2 Background of the study

I have been involved in teaching Mathematics at secondary school level for 26 years and have seen how learners struggle with the solving of Geometry problems. Most of the learners whom I have encountered over the years have performed badly in Geometry.

Until the end of 2007, the final Mathematics mark for learners in grade 12 was obtained by adding the marks for paper 1, paper 2 and the continuous assessment mark. Paper 1 tested Algebra and Calculus while paper 2 tested Analytical Geometry, Euclidean Geometry and Trigonometry. A low mark in the Geometry section meant a low mark for paper 2, which in turn negatively affected the final mark. The continuous assessment mark consisted of marks obtained for assignments, tests, projects and investigations done throughout the year under controlled conditions.

With the implementation of the National Senior Certificate examinations in 2008, the final marks were obtained in a similar manner. However, Euclidean geometry was no longer a compulsory section of the new curriculum and was examined in paper 3, an optional paper that tested data handling, probability and Euclidean Geometry. Initially, it was envisaged that the contents of this paper would by 2011 either be incorporated into the other two papers or it would have formed a compulsory third paper. Neither course was followed. The present position is that Euclidean Geometry will be incorporated in the Curriculum and Assessment Policy Statement (CAPS) (Department of Basic Education, 2010) that would be introduced in 2012 at the grade 10 level and will be examined in paper 2. Thus, the first cohort of grade 12 learners will be examined in Geometry in 2014.

The September examinations (trial examinations) tend to give an indication of how learners will perform in the final examinations. The performance of grade 12 learners in the Geometry questions of the internal September Mathematics higher grade examinations of 2005, 2006 and 2007 at the school where the research was conducted is reflected in the following statistics. There were 27 candidates who wrote Mathematics higher grade in 2005, 31 candidates in 2006 and 39 in 2007. Owing to the fact that Geometry was not examined in the two compulsory papers after 2007, only data up to 2007 were used.

Table 1: Average Results of Euclidean Geometry of grade 12 learners at Sunrise High school

	2005	2006	2007	Average for 3 years
Average Percentage (Geometry)	16,3%	15,9%	20%	17,4%
Average percentage Whole Examination	46%	48,5%	43,6%	46,3%

These statistics come from a school that was awarded a certificate for the highest percentage increase in Mathematics passes in the final examinations in 2007 in the Western Cape. The following number of distinctions (80% and above) for Mathematics higher grade were achieved in the final external examinations by learners at the school: 2005 – 4 out of 27 learners, 2006 – 6 out of 31 learners, 2007 – 3 out of 39 learners. From the averages it can be seen that learners at this school perform much better in other areas of Mathematics than Geometry. These results are merely an indication of the trend at this school over the past few years with regard to Geometry. The position of the majority of schools in the townships was much worse.

In my many years' experience as an external marker of the grade 12 Mathematics second paper, which examined Analytical Geometry, Trigonometry and Euclidean Geometry, I observed that a large number of learners performed poorly in the Geometry section of the paper. This is what prompted my interest in investigating how Geometry was tackled by learners and how the learning of Geometry could be improved. After reading literature on learning in small groups and how, through discussion, argument and thinking out loud, learners gain more in small groups than they would as individuals, I decided to research how the learning of Geometry proceeds in small-group situations. My focus was on problem-solving in Geometry by secondary school learners in a small-group setting. Of interest would be how learners used tools and inscriptions while they interacted collaboratively in small groups within their given social context, how learning took place and what kind of learning took place. This would be situated theoretically within a socio-cultural perspective.

1.3 Rationale for the study

Problem-solving has been given prominence in the new curriculum in South Africa. The National Curriculum Statement (NCS) was implemented at different times for the different grades. It was implemented in grades 1, 2 and 3 in 2004; in grades 4, 5 and 6 in 2005; in grades 7 and 10 in 2006; in grades 8 and 11 in 2007 and in grades 9 and 12 in 2008. The research was conducted during the period in which the NCS was implemented and therefore the discussion mainly focuses on the key envisaged outcomes of the NCS. It has since been amended to improve its implementation and it is now named the Curriculum and Assessment Policy (CAPS) which will be implemented in grades R to 3 and grade 10 in 2012, in grades 4 to 11 in 2013 and in grade 12 in 2014. The first critical outcome in the National Curriculum Statement requires that learners should be able to identify and solve problems and make decisions using critical and creative thinking (Department of Education, 2003). This statement is repeated *verbatim* in the new CAPS document (DBE, 2010). Throughout the National Curriculum Statement document there are instances where problem-solving in Mathematics is highlighted:

- As part of the definition of Mathematics: "Mathematical problem-solving enables us to understand the world and make use of that understanding in our daily lives" (p. 9).
- As part of the purpose: Mathematics enables learners to: "use mathematical process skills to identify, pose and solve problems creatively and critically" (p. 9).
- As part of the scope: Learners should work towards being able to: "solve non-routine, unseen problems using mathematical principles and processes" (p. 10).
- Under educational and career links: "Mathematics is being used increasingly as a tool for solving problems related to modern society" (p. 11).
- Under learning outcome 1: "...solve problems related to arithmetic, geometric and other sequences and series, including contextual problems, related to hire purchase, bond repayments and annuities…" (p. 12).
- Under learning outcome 2: "The emphasis is on the objective of solving problems and not on the mastery of isolated skills …for their own sake" (p. 13).

- Under learning outcome 3: "…solve problems involving geometric figures and geometric solids" (p. 13).

This emphasis on problem-solving can be interpreted as the fact that problem-solving strategies should be shown or demonstrated to learners, so that they are able to handle problems that are of a type that they would not have encountered before. Increasingly, these types of problems are examined in final examination papers. Below are two examples of these problems from recent papers.

NCS March 2009 Supplementary examination: Paper 1

Question 2

Consider the series:

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots$$

2.1 Express each of the following sums as a fraction of the form $\frac{a}{b}$:

2.1.1 The sum of the first two terms of the series (1)

2.1.2 The sum of the first three terms of the series (1)

2.1.3 The sum of the first four terms of the series (1)

2.2 Make a conjecture about the sum of the first n terms of the given series. (2)

2.3 Use your conjecture to predict the value of the following:

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots+\frac{1}{2008\times 2009}$$ (1)

NCS November 2010: Paper 1

1.3 Calculate the integer that is the closest approximation to:

$$\frac{5^{2007}+5^{2010}}{5^{2008}+5^{2009}}$$

Show ALL workings. (3)

Figure 1: Problem-solving questions from past papers

The new Curriculum and Assessment Policy Statement (CAPS) states that "[m]athematical problem-solving enables us to understand the world (physical, social and economical) around us, and, most of all, to teach us to think creatively" (CAPS, 2010, p. 6). I think a study of problem-solving in general and of Geometry problem-solving in particular is important because the skills acquired in the study of Geometry can be used in numerous other fields. For example, spatial perception skills may come in useful when doing courses in Engineering and Technical Drawing, and for occupations like computer programming, architecture, cartography, draughtsmanship, etc. The logical skills acquired in reaching conclusions may be a life skill used in any situation. Meserve (1973) argues that Geometry is a gateway to Mathematics and that Geometry could be applied as an approach to numerous topics throughout all branches of Mathematics. Geometry problems, in addition, were ideal for use in small groups because they allowed learners to experience the entire problem-solving process (Carlsen, 2008).

Since Euclidean Geometry has, since 2008, not been examined in the two compulsory papers and only in the optional third paper, most teachers are currently not teaching Euclidean Geometry. It is important that studies are undertaken during this period while Euclidean Geometry is not a compulsory topic in the curriculum, to investigate possible methods to improve the learning and teaching of Geometry. The danger in not doing this would be a perpetuation of the results of the past, once Euclidean Geometry is reintroduced in the curriculum in 2012.

1.4 Research questions

In order to investigate the issue of what learners are doing while engaging in Geometry problem-solving in a small group setting, the following research questions were formulated:

- How do learners appropriate tools (physical tools like calculators, computer software and intellectual tools, like inscriptions) while doing geometry problem-solving in a small group setting?
- How do learners interact with one another and with the subject matter while doing Geometry problem-solving in a small group setting?
- What is the structure of the problem-solving process while doing problem-solving in a small group setting?

The constructs used in this study – appropriation, problems, problem-solving, structure of the problem-solving process, collaborative group work, tools, artefacts, inscriptions, meta-cognition, accommodation and resistance – will be defined and clarified in chapter 2.

The thesis consists of five chapters. In this chapter the introduction, background and rationale of the study were provided. In addition, the research questions were presented, as well as an overview of the study.

The second chapter looks at an overview of the literature regarding problem-solving and group work. An understanding of what constitutes a problem, as well as the processes involved in problem-solving is highlighted. The theoretical framework that underpins the research is the socio-cultural theory. The writings of Vygotsky, Schoenfeld, Bjuland and Carlsen are amongst those discussed in this section.

The following chapter explains the methodology used in this research. The method is the case study method, where three groups of learners are observed and video-taped while they are engaged in Geometry problem-solving using tools such as dynamic Geometry software. The social background of the school and the participants, as well as the ethical issues that were considered in the research, is discussed. Data are collected by means of audio and video recordings, teacher observation as well as from interviews with learners.

Chapter 4 gives a detailed account of the learners' attempts to solve the problems and an exposition of how their conceptual practice corresponds to the analytic framework developed from the work of Pickering (1995), which suggests that all scientific practice takes on a particular form of interplay between resistance and accommodation.

Chapter 5 consists of the findings of this case study, which are also discussed with reference to the literature and to the South African context. The learners' use of tools and inscriptions, where this influenced their thinking and understanding, is also analyzed. Issues involving the situational character of the problem-solving process are discussed in terms of the theoretical framework presented in the study.

The discussion relates the research questions to the findings. Suggestions and implications for further research around this topic are also given in this section.

CHAPTER 2

OVERVIEW OF THE LITERATURE

2.1 Introduction

In order to answer the research questions, the interaction of small groups engaged in Geometry problem-solving is underpinned by socio-cultural theory that sees learning as embedded in social and cultural contexts and is best understood as a form of participation in those contexts. Therefore, it is important that all constructs and terminology that are used in this thesis are defined and explained. The theoretical constructs that are used within this framework are tools, artefacts, inscriptions, mediation, meta-cognition and appropriation. This chapter also looks at what characterizes a mathematical problem and what the process of problem-solving entails. The difference between cooperative and collaborative group work is explained and thus the classification of the group work in this thesis as collaborative. Within studies of collaborative group work research differs in terms of their unit of analysis, whether it is the individual within the group, the group itself, the activity or the context. This will depend on the theoretical frame of reference. Therefore, previous research done with respect to collaborative small groups is reviewed and classified in terms of socio-constructivist, socio-cultural, shared cognition or situated perspectives. The analytical framework based on the work of Pickering, which will be used to analyse the problem-solving process, is also discussed.

The work of Bjuland (2002) served as a source of inspiration for this thesis. His study focused on the reasoning processes and the heuristic strategies used by learners while engaging in Geometry problem-solving in collaborative small groups. Therefore, there are similarities between his study and the present one and extensive reference to his work has been made.

2.2 Problems and problem-solving

Different people have different perceptions on what constitutes a problem in Mathematics, as well as to what constitutes problem-solving. Some educators and researchers would regard learners doing basic algebra problems as engaged in problem-solving, while others would regard doing highly specialized mathematical problems as problem-solving. This research acknowledges the many interpretations.

There is also a difference in how problems are used as learning and teaching tools. It is thus important to clarify what is meant by problems and problem-solving in this thesis.

After 1994, with the advent of a new democracy in South Africa, the changes in the curriculum in South Africa intended to improve the quality of education for all learners. This curriculum was learner-centred and outcomes-based, with the intention of producing learners with skills such as critical thinking and logical thought. There was great emphasis on problem-solving in the new Mathematics curriculum.

Schoenfeld (1993) makes the following definition of a mathematical problem:

> For any student, a mathematical problem is a task (a) in which the learner is interested and engaged and for which he wished to obtain a resolution, and (b) for which the learner does not have a readily accessible mathematical means by which to achieve that resolution" (quoted in Carlsen , 2008, p. 28; Bjuland, 2002, p. 9).

This definition implies that a problem is a relationship between the person and the situation and that what is considered a problem for one person may not be a problem for another person. It is the person who is engaging with the problem that may experience it as a problem. Therefore, what is a problem is not entirely dependent on the task formulation. For the purpose of this study, Geometry problems that satisfy the criteria based on Schoenfeld's definition for the particular group of learners were chosen.

In all the literature that was consulted for this chapter regarding problem-solving, reference was made to the method advocated by Polya (Bjuland, 2002; Kilpatrick, 1985; Carlsen, 2008; Lester, 1985). Polya (1957) regards problem-solving as a practical skill that has to be practiced through observing and imitating other problem-solvers in order for it to be learnt. Polya explains his conception of a problem by contrasting it with a task. With a problem one might not know what to do nor how to do it, while with a task it would be clearly defined what was to be done and how to do it. The problem-solving approach advocated by Polya (1957) makes use of a list of questions that aimed to guide the learner through the following processes: understanding the problem; devising a plan; carrying out the plan; looking back.

Some of the questions for each of the steps in problem-solving were as follows:

- Understanding the problem: *What is the unknown? What are the data? What is the condition?*
- Devising a plan: *Do you know a related problem? Could you use it?* (A well-known quotation illustrating this step is: "[i]f you cannot solve the proposed problem, try to solve first some related problem" (Polya, 1978, p. 31)).
- Carrying out the plan: *Can you see clearly that the step is correct? Can you prove that it is correct?*
- Looking back: *Can you check the result? Can you check the argument?*

These steps constitute the entire process of problem-solving. Thus, this thesis has followed the description of problem-solving where it is defined by referring to the entire process of dealing with a problem in attempting to solve it. Bjuland (2002, p. 9) states that "Mathematical problem-solving then becomes the cognitive, meta-cognitive, socio-cultural and affective process of figuring out how to solve a mathematical problem when one does not already know how to solve it."

In a review of research on problem-solving Carlsen (2008) concludes that research on problem-solving (from 1990 – 1994 and beyond) implied an emphasis on viewing problem-solving as a situated and contextually determined activity. What have also become important in research are the rules of collaboration and communication while engaging in the activity of problem-solving. Bjuland (2002) concurs with Wyndham and Säljo (1997) that the general trend in the study of problem-solving was that the activity of problem-solving could not be explained and understood only by looking at mental structures of individuals. One should also locate individuals in practices situated in time and space. When learning is seen by researchers from this perspective, it is called situated (McCormick and Murphy, 2008). Bjuland (2002) found that most research from the 1990s onwards had used a more ethnographically-inspired approach and a situated perspective on mathematical reasoning where the focus was on what pupils in fact did when they solved problems. Individual actions and learning were understood as being embedded in cultural practices and took place in a social and physical world. This supplemented the relationship definition by extending it to include situatedness and contexts. Learning was viewed as a process of participation in cultural activity.

Competent problem-solvers, when solving a mathematical problem, would gauge its scope and their chances of success. They would look at various strategies and decide on an initial plan, select appropriate tactics to implement the plan, and monitor and evaluate the progress and outcome of the chosen tactics and plan. They would modify or replace the tactics, abandon or replace the initial plan if these were deemed necessary. This demonstrated that much more was involved in problem-solving than just possessing the necessary skills, algorithms, facts and strategies (Lester, 1985). Lester looked at research on problem-solving and emphasized that attention should be paid to the guiding forces of problem-solving, which he calls managerial skills. The focus should not only be on skills and procedures but should include aspects like meta-cognitive skills, which acted as guiding forces in problem-solving. Research on problem-solving had also paid too little attention to the total environment in which problem-solving takes place (Lester, 1985; Bjuland, 2002). Lester (1985) also points out that the Polya model of problem-solving did not include meta-cognition and did not consider the interaction between meta-cognition and cognition, which he felt was important in problem-solving. Problem-solving ability develops slowly and at different rates for different learners. Therefore, Lester suggests that:

> [t]he growth of problem-solving ability, then, should be studied longitudinally, the teacher should be considered a part of the environment and attempts should be made to catch processes as they are developing rather that looking for their presence at the end of a predetermined period of time (Lester, 1985, p. 43).

The role of powerful tools, like calculators and computers, should be recognized in problem-solving. Learners have access to these tools and long and tedious calculations were done by these tools. Thus, the activity of problem-solving could not be understood only by reference to mental structures. Learners interacted with powerful tools and other learners to solve problems in situated activities (Carlsen, 2008). In this study one of the tools that was used by learners in Geometry problem-solving was a dynamic Geometry system, *Geometer's sketchpad*.

Kilpatrick (1985) in his account of research on teaching mathematical problem-solving spanning a period of 25 years, notes that researchers were increasingly using group problem-solving sessions by which to do research and advocating them as a vehicle for instruction.

He states that by discussing the ideas in small groups, where they could be refined and defended, learners were helped in clarifying concepts and rehearsing procedures in ways that were difficult to do alone. He was in favour of a microscopic look at problem-solving, so that the potential usefulness of a careful scheme for analysing problem-solving behaviour should not be overlooked. He concedes that there was no final vision of what problem-solving was and how to teach it, but teachers and academics were much more keenly aware of the complexity of both (Kilpatrick, 1985). The features that successful problem-solving programmes had in common were that a good amount of instructional time was spent on problem-solving, teachers encouraged the learners to adopt an active stance towards problem-solving and provided a congenial setting in which problem-solving could occur. The three factors which were instrumental in successful problem-solving were:

- Organised knowledge about the problem domain.
- Techniques for representing and transforming the problem.
- Reflection and meta-cognitive practices while doing problem-solving.

Lessons to be learned from Kilpatrick's work were that a teacher needs to understand that there are various sorts of problems, that problems could be used to serve various instructional goals, and that a problem or technique that worked in one instructional setting might not work in another (Kilpatrick, 1985).

Taplin (2011) motivates why teaching through problem-solving should be undertaken in schools. The problem-solving approach to teaching Mathematics is characterized by learners engaging in creating, conjecturing, explaining, testing and verifying.

These activities help learners to understand mathematical ideas and processes. The reasons why problem-solving should be prominent in Mathematics teaching includes motivational aspects, justification of Mathematics by simulating real life contexts (rather than treating Mathematics as an end in itself), skills and functions that are part of everyday life were learnt, adaptation to changes in careers and other aspects of peoples' lives were helped and experience was gained of the power of Mathematics in the world around one (Taplin, 2011).

There are different teaching approaches using problems. One can distinguish between problem-based approaches and problem-solving approaches to teaching and learning. In problem-based approaches the problem is designed so that learners can identify and search for the knowledge needed to approach the problem.

In a problem-solving approach it is assumed that learners already possess the knowledge required to approach the problem before they start on the problem. In problem-based learning the learning comes from the work on the problem (Ross, 1997). The main idea for learning in problem-based learning is "…that the starting point for learning should be a problem, a query or a puzzle that the learner wishes to solve" (Boud and Felleti, 1997. p. 1). Kilpatrick (1985) also considers a mathematical problem from two perspectives: in general where the process of problem-solving was looked at or specifically focused on the roles that problems played in Mathematics teaching. This study focused on a problem-solving approach and analysed the process of problem-solving by learners working in small groups.

2.3 Collaborative group work

Group work can take on a variety of forms and occurs when learners work together on tasks. Cohen (1994) defines cooperative learning "as learners working together in a group small enough so that everyone can participate on a collective task that has been clearly assigned" (p. 3). This definition is broad since it includes collaborative learning, cooperative learning and group work (Cohen, 1994). "Cooperative learning is generally understood to be learning that takes place in an environment where learners in small groups share ideas and work collaboratively to complete academic tasks" (quoted in Carlsen, 2008, p. 22). In this study cooperative learning is distinguished from collaborative learning. Cooperative learning can take on many forms, but the basic idea is that the task is divided into smaller units and each participant is responsible for a unit. In collaborative work, there is no division of labour and the whole group collectively tries to solve the problem. Learners in this study worked collaboratively in small groups to solve the set problems.

Brodie and Pournara (2005) argue that the way group work is used in classrooms depended on whether group work is perceived as a means or an end to learning. If group work was seen as an end then it would be done to ensure that learners work effectively as members of a team. This could be done, for example, to prepare learners for the work environment where they would have to have the skills to work in a team. If group work were seen as a means to Mathematics learning, then group work was seen as an effective medium to promote Mathematics learning. This study perceives group work as a means to learning and focuses on what learners do while they do problem-solving in collaborative groups.

Most of the research done on collaborative group work has found that group work renders positive results in terms of raised levels of achievement. It is averred that during group work learners achieve better understanding of the subject matter than they would when working individually (Biggs, 1973; Kilpatrick, 1985; Noddings, 1985; Gokhale, 1995; Bjuland, 2002; Carlsen, 2008). Within small groups the dialogue represents an opportunity to hear the reasoning processes of group members. It is an opportunity to study "the *externalized* internal dialogue" of problem solvers (Silver, 1985).

The unit of analysis ranges from the individual, in that researchers want to investigate how the cognitive system of one individual is transformed by interaction with another, to the group where researchers want to understand how the cognitive systems interact and work together so that a shared understanding of the problem can be achieved. According to Dillenbourg, Baker, Blaye and O'Malley (1996), research concerning group work could be classified in terms of three theoretical perspectives: socio-constructivist, socio-cultural and shared or distributed cognitive approaches. They traced how the research in collaborative group work had evolved from focusing on individual achievement to focusing on shared understanding of the group. What was common in these perspectives was the recognition that learning was a social process within a particular social context and that the social and individual aspects of development are intertwined.

Within the socio-constructivist perspective, which is inspired by Piaget's theory, the research focuses on how the interaction in groups affected individual cognitive development. While interacting with others, "socio-cognitive conflict" occurs, i.e. conflict between different answers seen from different viewpoints, and this conflict is the main feature of the mediating process and provides the platform to arrive at a more advanced "decentred" solution (Dillenbourg *et al.*, 1996, p. 192).

The research methods used within the socio-constructivist perspective would be an experimental group versus a control group and the subjects would be of the same age. Webb (1997) argues that, from a socio-constructivist perspective, assessment could measure how well learners could perform after they had had an opportunity to learn from others in the context of collaboration. Here learner competence was not only what a learner could do without assistance but also included what a learner was able to learn from a collaborative group experience.

Gokhale (1995) found in research done at college level that the learners who participated in collaborative learning had performed significantly better on the critical thinking (based on the last three categories of Bloom's taxonomy, synthesis, analysis and evaluation) test items than the learners who had studied individually. This approach encouraged learners to help one another and was also called a co-operative approach to group work (Brodie & Pournara, 2005).

Within the socio-cultural perspective, which is inspired by the work of Vygotsky, the basic unit of analysis is social activity from which individual mental performance grows. The research method would be an in-depth analysis of the social interaction. The social interaction would take place between an adult and a child or among peers. Bjuland (2002) analyzed how reasoning was verbalized in collaborative small groups doing problem-solving. He was not concerned with the cognitive development of individuals within a group, but rather saw learning as a social process and knowledge as socially constructed. Hogan's (1999) interest was to understand the social and cognitive processes that came to the fore when a group of grade 8 learners engaged in an open-ended scientific task in collaborative groups. She focused on the different cognitive roles that emerge naturally as they reasoned together (as opposed to roles being assigned by educators), as well as the group dynamics that give rise to them. She was also interested in the achievement of the group as a unit and not on the achievements of individuals. Therefore, she used the group as a unit of analysis. This study is an example of how understanding evolves collaboratively and of how thinking is shared to create a common knowledge product.

Noddings (1985) has the view that children internalized what the group did and that these internalizations then appeared as individual cognitive ability. Also, the conversation in the group should manifest as more problem-oriented inner talk in the individuals after engaging in group work.

Within the shared cognitive approach the environment, which includes a physical as well as a social context, played an important role in group collaboration and was deeply intertwined with the "situated cognition theory" (Dillenbourg et al, 1996, p. 194). Emphasis was placed on the social context, the social communities in which the group participated, the cultural practices in which they participated and the role these played in cognitive development and shared understanding. The unit of analysis within this perspective was the group and the outcomes from the interactions were regarded as a group product (Dillenbourg et al., 1996).

The research methodology employed within the shared cognitive approach would, similar to the socio-cultural approach, involve an in-depth analysis of the social interaction within the group, with the focus on the processes involved in the interaction.

Wenger (1998) defined participation "to describe the social experience of living in the world in terms of membership in social communities and active involvement in social enterprises" (p. 34). Members of social communities engaged in participation. This participation in social communities shaped the experience of the participant and shaped the communities as well. Central to every practice was giving form to experiences in the form of abstractions, tools, symbols, stories, terms and concepts. This process of giving form to experiences was what Wenger call reification, which could take on a variety of forms. Participation and reification were complementary and formed a unity in their duality. The collaborative problem-solving done in groups was regarded as participation in specific scientific practice situated in a specific context.

Vidakovic and Martin (2004) add another classification, the co-constructivist approach. Within this approach learning was seen as the interplay of the individual's cognitive development and the group's development. Because of personal contributions, individuals were said to co-construct the collective culture. According to Valsiner, as cited by Vidakovic and Martin (2004), internalisation occurs when individuals constructed their personal meaning from collective cultures. At the same time, they contribute to the reconstruction of the collective culture through externalisation.

Brodie and Pournara (2005) have similar classifications for approaches to group work to those of Dillenbourg, *et al.*: co-operative, collaborative, socio-cultural, socio-political and situated. The socio-political approach focused on the power relations within the group with respect to race, gender, class, language and mathematical competence. It was felt that these factors should be considered and overlap all approaches to group work.

The situated approach focused on research done on group work in which the learner in the Mathematics classroom was becoming a better participant in mathematical discourse. According to them, this approach brought together the ends and means of group work. "Mathematical discussion, conversation and interaction is a goal or end of the learning process, and also the means for achieving this end" (Brodie & Pournara, 2005, p. 47).

However, they cautioned that concepts for analyzing Mathematics-learning consistent with this perspective had not been worked out.

Webb (1997) explored how assessment should be structured to incorporate assessment of group work, as well as the learning that took place in group work. She concluded that the design of the assessment should take the purpose of the assessment into account.

Certain characteristics of group work were conducive to problem-solving. According to Noddings (1985), the three features of small group interaction that promote problem-solving were:

- Learners encountered challenge and disbelief from their peers which may lead them to re-examine their own thinking.
- The group would supply background information that individuals might not have.
- Learners developed orderly approaches to problem-solving by taking charge of their own learning and that of their peers.

Schoenfeld (1985) puts forward the following four justifications for collaborative problem-solving in groups with the teacher acting as a consultant. Firstly, it gave the teacher an opportunity for direct intervention as the learners solved problems as opposed to after the fact. Secondly, when learners worked with a small group of their peers it provoked discussions about plausible choices. When different learners had different strategies to solve a problem, it necessitated a discussion on the merits of those strategies if one approach were to be settled on. When a learner worked a problem alone, the first plausible option would most likely be chosen and the discussion on the merits of different approaches that should take place internally, would not take place. Furthermore, working on problems with other learners helped learners to realise that fellow learners also had to struggle to learn and this could be reassuring, especially when learners were insecure about their abilities. Finally, it was an opportunity to engage in collaborative problem-solving.

There are three major ways of organizing collaborative working groups: expert-novice collaboration, peer collaboration and group collaboration. Expert-novice collaboration is the situation where a child or a learner is able to solve a problem in cooperation with a more competent adult, but which the learner would have difficulties in working out alone. This differs from peer tutoring since there is a more mature expert that stimulates the learning process. In peer tutoring a knowledgeable peer instructs the other peer who is a novice.

Peer collaboration is when two relatively non-experts work together to solve challenging problems, tackling a problem that neither could do on his/her own. How this differs from peer tutoring is that the two start roughly at the same level. Group collaboration is when three or more learners engage in problem-solving without the intervention of a teacher. In this study the focus will be on group collaboration. According to Carlsen, group collaboration is found to be beneficial for the learners, because they could give each other immediate feedback and they had a common mathematical language through which they communicated (Carlsen, 2008).

Bjuland (2002) concurs that learners have the opportunity to initiate the questions and ideas themselves when there is no teacher intervention. This study investigated, *inter alia*, bridging, which would be instances where learners initiated attempts to find solutions.

The research problem addresses, *inter alia*, how learners interact with each other and the subject matter. The small group interactions in this study were regarded as collaborative. Learners jointly tried to solve the problem as a group. Group work was thus a means to an end. Group work was classified according to their theoretical orientation and the unit of analysis. The collaborative group work in this study was based on the socio-cultural perspective and the group was the unit of analysis. The analysis focused on the participation of the group in cultural discourse, in this case their participation in mathematical discourse, and would investigate what learners, as members of the community of mathematicians, do when they do problem-solving.

2.4 Constructs informing the study

The focus of this study was on the strategies that learners develop (through argument and debate) and used to deal with a problem while interacting in small groups and how they would use the tools and artefacts in their learning, in their own cultural context. Therefore, the socio-cultural perspective, which took all of this into account, was appropriate to use for this study. The socio-cultural perspective saw learning as a social process, in which learners actively participated and contributed with ideas and arguments. The Mathematics classroom was a social situation jointly constructed by the participants, in which the learners and the teacher interpreted one another's actions and intentions in the light of their own agendas (Kilpatrick, 1985). Noddings (1985) concurs that Vygotsky's socio-cultural theory is possibly the most useful theoretical framework if one wanted to study learning in small groups.

Individual mental functions were internalized from relations among children in groups, and that reflection was induced by the need for each child to defend his views against challenges brought by other children. The conclusion was that Mathematics educators should encourage small- group work in Mathematics.

Within the socio-cultural perspective, terms such as appropriation, tools, inscriptions and artefacts would be used. Cultural tools would include intellectual tools and physical tools that learners would use. Examples of intellectual tools would be language and sign systems and examples of physical tools would be calculators and computers. Inscriptions were regarded as specific types of cultural tools and were used to support verbal arguments, for example, drawings, graphs and diagrams.

According to Carlsen (2008) learning and knowing took place first of all through interaction between people and, secondly, this knowing became part of the individual's thinking actions. The results of this knowing would then appear as artefacts and procedures in society. Collective thinking was thus the context in which individuals operated and the means through which individuals familiarized themselves with tools and actions. People's actions were mediated by tools and these tools were inextricably tied to the context in which they were used.

Schwebel (1986) explains how the way people construct their understanding of the physical and social world had transformed from the formula S-O-R to S-H-O-R. The first formula meant that a stimulus (S) from the environment was mediated by the individual (O) before the individual gave a response (R). By "mediated" was meant that the individual responded to S in accordance with the meaning he or she gave to it. The modification added the human mediator (H) to the formula. According to the adapted formula, another person was added who helped interpret stimuli.

According to this formulation, successful cognitive development depended not only on the stimuli of the environment, but also on the quantity and quality of mediation by the other person who helped the child observe, interpret what was seen, and organize new information. It was appropriate to ask whether it was possible to determine the extent to which a given child's cognitive functioning could be enhanced through mediation. This is explained by Vygotsky's "zone of proximal development" (Schwebel, 1986, p. 7).

Carlsen (2008) states that a socio-cultural perspective on learning considers three different but interrelated topics: the development and use of intellectual tools, the development and use of physical tools and thirdly, communication and the different ways humans develop collaborative strategies in different collective activities.

The use of tools in learning activities is a vitally important issue within the socio-cultural perspective, in respect of how people act intellectually and practically, individually and in interaction with others. Tools fundamentally relate to how concepts and actions are appropriated, since the physical and intellectual resources that are used was said to mediate the world for us in different activities (Carlsen, 2008). A tool, before considering its users and uses, is regarded as an artefact. For example, in the learning of Mathematics examples of artefacts are the graphical calculator, the ruler, the abacus, and the compass. When learners used these artefacts in thinking and communication in social practice, and in various contexts, they turn into useful tools in the problem-solving process (Carlsen, 2008).

Rogoff (2003) cites the example of James Wertsch where if one is asked to multiply

$$\begin{array}{r} 343 \\ \times \underline{822} \end{array}$$

one would easily use existing algorithms and find the answer, but if the question was asked for one to multiply 343×822 without placing the numbers vertically, it would not be as easy for most of us to find the answer. The point is that the organisation of the problem, the way it is presented is a cultural tool which helps to solve the problem.

An inscription is a particular kind of cultural tool in which both reasoning elements and physical aspects co-exist. When appropriating mathematical tools and actions the individual relies on inscriptions such as algorithms and graphs, tables and drawings, hence inscriptions are significant elements of individuals' reasoning and thinking. Thus, inscriptions are considered to be tools through which thinking is externalized and made explicit. In school settings, learners' use of inscriptions in problem-solving is considered as aspects of their thinking. Inscriptions, together with language, are mediating resources the learners use to communicate and solve problems. Through these mediating tools, the learners objectify their knowing and establish a shared focus of attention in their problem-solving (Carlsen, 2008).

These inscriptions are called "representations" by Silver (1985, p. 261), Noddings (1985, p. 348), Schoenfeld (1985, p. 365), Kaput (1985, p. 381) and Suchman and Trigg (1993, p. 144).

Suchman and Trigg's ethnomethodological study of the social practice when two researchers designed a computer programme that would replicate some aspect of human behaviour, focused on their representational practice. Their diagrams drawn on whiteboards, examined in relation to the activities of the practitioners, highlighted the important function of inscriptions. The use of these representational devices and the activities of the scientist influenced each other mutually. They are dependent on each other for progress in the scientific practice.

According to Carlsen (2008), communication is a tool through which learning and development takes place, i.e., it is the connection between the external (interaction) and the internal (thinking). Language was part of and mediated human action. Use of language connects the child with the environment. Participants building and using each other's contribution in the learning process was important.

Code-switching occurs when more than one language is used in the same conversation. Code-switching can be seen as a way of using language to facilitate understanding. In her research on teaching in multilingual classrooms, where more than two languages were the main languages of learners, Adler (2001) investigated the dilemmas (dilemma of code-switching, dilemma of mediation, dilemma of transparency) facing teachers and highlighted the complexities of teaching in a multilingual classroom. The dilemma of code-switching was that when teachers teach in English the learners often did not understand the work, yet when teachers switched to the main language of the learners, they might not become competent in mathematical English. Learners needed to competent in mathematical English since the examinations are written in English and in addition, they needed the skills for courses in higher education mathematical courses and the workplace. The dilemma of mediation, which teachers who promote discussion in the classroom faced, was whether they should intervene during discussion or whether they should allow learners to explore mathematical concepts through discussion. The dilemma of transparency was that the focus on teaching explicit mathematical concepts may take too much time, yet is it important for learners to understand these concepts in order to gain access to mathematical discourse.

Code-switching often occurs when the language of learning and teaching was not the same as the main language of the learners. Adler noted that where there has been a movement towards more meaningful, communicative and investigative Mathematics in school, there were opportunities for learners to talk among themselves while engaged with mathematical tasks and exercises. She used as an example Setati's (1998(b)) study of grade 4 classroom interaction and the finding that calculational Mathematics discourse was prominent in classrooms where switching was restricted. However, when the teacher switched to the learners' main language, conceptual discourse became the focus of discussion. Thus, the discussions showed evidence of understanding, rather than the learners just knowing the steps involved in calculations. In addition, when learners in this teacher's classroom were interviewed about the Mathematics they had learnt, they could shift between talking about steps in calculations and talking about concepts (Adler, 2001). In the context of multilingual classroom settings, it was a significant disadvantage if teachers were not able to understand discussions amongst learners because of not understanding the language. They could not listen effectively to learner discussion on mathematical tasks when this discussion was held in a language beside English. Although this study did not focus on the occurrences of code-switching, it should be noted that code-switching was a regular occurrence when learners engaged in group work (not only in Mathematics). In this research, learners' discussion mostly took place in their home language (isiXhosa), although the language of teaching and learning is English, and they code-switched by using mathematical concepts and formulae in English.

Mediation means that thinking and understanding of the world are shaped by culture and its psychological and physical tools. The main point is that people act with mediating artefacts, and to study human thinking and learning one has to consider individuals as interacting with artefacts and the important role they play in problem-solving. Hence, mediation is the term used to describe how humans interact with cultural tools in action (Carlsen, 2008).

Appropriation is the process of "taking something that belongs to others and making it one's own" (Carlsen, 2008, p. 34). The manner in which resistance manifests itself in the appropriating cultural tools is important to explore (Carlsen, 2008). According to the Collins concise English dictionary (2001), to 'appropriate' means 'to put aside for a particular purpose'. Thus, to appropriate a tool is to use the tool for special purposes, for example, to use a calculator as a tool to learn Mathematics or simply do a calculation.

The appropriation process includes taking what someone else produces during joint activity for one's own use in subsequent productive activity while using new meaning for words, new perspectives, new goals and new actions. Hence, involvement in joint activity is crucial for appropriation to occur, since the process is about "achieving a shared focus of attention, shared meanings and transforming what is appropriated" (Carlsen, 2008, p. 35). The appropriation process is a reciprocal process whereby participants make sense of one another's actions according to their own conceptual framework and thus change the group's understanding of a problem (Dillenbourg et al, 1996).

Meta-cognitive knowledge concerning any learning situation develops when the learner is aware of how variables interact to influence outcomes of cognitive activities. This complex interaction includes person, task and strategy variables. In addition to the above variables the nature of the materials is also incorporated (Gordon and Braun, 1985). Meta-cognition represents, so to speak, the human ability to stand back from one's behaviour, to observe, monitor, evaluate, correct and otherwise control it. As children acquire meta-cognitive skills, they can increasingly assume more of the role of mediation themselves. The skills involved in meta-cognition involve prediction of the consequences of an action or event, reflecting on the results of one's own actions, monitoring one's ongoing activity, reality testing and a variety of other behaviours for coordinating and controlling deliberate attempts to learn and solve problems. Clearly these skills are essential for effective use of intellect in almost every kind of learning and problem-solving situation (Schwebel, 1986). The challenge to specialists in education is to determine precisely what changes in the interaction that children have with the curriculum and with the physical and social environments of the school will facilitate their cognitive development, especially by calling upon and strengthening the children's self-regulatory, meta-cognitive skills.

Meta-cognition is about being aware of one's own thinking and learning processes. Within groups, collaborative meta-cognitive activity happens when group members share their thoughts with other group members for inspection and also comment on other members' ideas (Goos, 2002). She further categorizes meta-cognitive failure, i.e. when group collaboration does not achieve a successful outcome, as "meta-cognitive blindness", "meta-cognitive vandalism" or "meta-cognitive mirage". Meta-cognitive blindness occurred when group members did not see that something was wrong and persisted with the wrong method or could not see an incorrect calculation.

Meta-cognitive vandalism happened when a problem was deliberately changed incorrectly to apply available knowledge. Meta-cognitive mirage occurred when difficulties were seen which do not exist, for example, when they have calculated a correct solution but mistakenly rejected it or when they did not pursue a plausible strategy for fear that it may be wrong.

Meta-cognition also refers to the regulation of one's cognitive processes (Gordon and Braun, 1985). Flavell (1985) quoted in the article by Gordon and Braun (1985) defines meta-cognition as follows:

> Meta-cognition refers to one's knowledge concerning one's own cognitive processes and products or anything related to them e.g. the learning relevant properties of information or data. For example, I am engaging in meta-cognition (meta-memory, meta-learning, meta-attention, meta-language or whatever) if I notice that I am having more trouble learning A than B; if it strikes me that I should double-check C before accepting it as fact; if it occurs to me that I had better scrutinize each and every alternative in any multiple choice type task situation before deciding which is the best one; if I sense that I had better make a note of D because I may forget it;.....Meta-cognition refers among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete goal or objective" (quoted in Gordon and Braun, 1985, p. 2).

Rogoff (2003) stated that cognitive development happens in shared endeavours with other people building on the cultural practices and traditions of communities. Thus research on cognitive development had changed to focus on how one learns to use the cognitive tools of one's cultural community. For example, the differences in performances in international Maths tests between US children and Japanese, Chinese and Korean children, who fare considerably better than US children, could be attributed to their language which represents numbers such as 12 in a base 10 system (ten-two) as opposed to a none-base 10 label, twelve.

Lave (1993) suggests that the context in which one lived while one is learning should be incorporated, i.e. situated activity. The idea is that the person, activity and situation as they were given in social practice should be viewed as a single, encompassing theoretical entity.

The constructs that framed the research questions were defined and explained in this section. Concepts such as appropriation, artefacts tools and meta-cognition are all linked to socio-cultural theory. This study reports on the concrete manifestations of these constructs.

Appropriation is demonstrated when learners doing collaborative group work, use ideas, sketches or suggestions from other group members and build on those ideas towards a solution to the problem at hand. Learners may also use knowledge and skills from other fields or subjects and use these to help find a solution to the problem. Before a tool is used for a specific purpose it is classified as an artefact. Cultural tools would include intellectual as well as physical tools. The physical tools used in this study would include calculators, mathematical instruments and dynamic Geometry software. Inscriptions are tools that contain physical as well as intellectual elements and that are used to stimulate the discussion and aid in the finding of a solution to the problem. The inscriptions would be any sketches or notes made in order to facilitate getting to a solution.

Tools and artefacts play a mediation role. Language is a tool in which thinking is made explicit and an important mediating factor in learning. The language of teaching and learning is English while the mother tongue of the learners involved in the study is isiXhosa. Learners mostly held their discussions in isiXhosa since they felt more comfortable expressing themselves in isiXhosa. The use of code-switching, i.e. when more than one language is used in the same conversation, is therefore important to highlight.

Meta-cognition is the ability to reflect on one's thinking processes. Collaborative meta-cognitive activity happens when group members reflect on one another's thinking processes. Attention will be given to the instances collaborative meta-cognitive activity where these have made an impact on the discussions.

2.5 The analytic framework

In addition to the socio-cultural theory of Vygotsky, the analysis of learners' problem-solving in collaborative small groups will be based on an approach proposed by Pickering (1995) who sees scientific practice as the interaction of material and human agency. Scientific practice, a term that includes Mathematics, Science, Technology and society, is examined as it occurs in real time while scientists are engaged in constructing artefacts of interest to the practice. For Pickering, it is important to study activities in the social setting at the actual time and site that the construction is taking place and not to give a retrospective account of what has taken place using a different frame of reference. He warns against the 'scientist's account' in which accepted scientific knowledge functions as an interpretive yardstick in reconstructing the history of its own production." (p. 3).

His approach is compatible with the socio-cultural theory in that both sees learning as situated within a specific cultural and historical context that needs to be considered. From my perspective there are no meaningful incompatibilities between Pickering and the essences of socio-cultural theory.

This approach also perceives science as moving beyond a purely representational function to a performative one. Within the representational idiom, science is seen as "an activity that seeks to represent nature, to produce knowledge that maps, mirrors, or corresponds to how the world really is." This view, according to Pickering, is inadequate since scientific representation does not adequately take into account the "philosophical problematics of realism and objectivity." (p. 5)

A view of science which includes the material, social and temporal dimensions of science and goes beyond science-as-knowledge is possible if we assume that

> the world is filled not, in the first instance, with facts and observations, but with *agency*... [and] is continually *doing things*, things that bear upon us not as observation statements upon disembodied intellects but as forces upon material beings (p. 6).

A performative image of science is therefore proposed "in which science is regarded as a field of powers, capacities, and performances, situated in machinic captures of material agency" (p. 7). The representational aspect of science is included in the performative image since science is not only about the production of machines. Within the performative image of science there is an asymmetry between human and material agency with respect to intentionality, i.e. plans and goals. Human agency has intentionality while material agency does not have intentionality. The extensions and the transformations scientists wish to bring about are historically embedded in existing culture.

Future aims and destinations are constructed "from existing culture [that] predisciplines the extended temporality of existing intentionality" (p. 19). Although the starting point of the work of scientists is plans and goals derived from the consideration of existing culture, the initial plans and goals are partial. There is no advanced way of knowing how devised artefacts, machines in the case of science, will behave and act since "the contours of material agency are never decisively known in advance" (p. 14). This behaviour is only revealed in real time of practice. As this behaviour is exhibited, plans and goals change based on the observed inadequate and non-expected performance of the new artefact. Thus

> Human intentions are bound up and intertwined (in many ways) with prior captures of material agency in the reciprocal tuning of machines and disciplined human performances. The world of intentionality is, then, constitutively engaged with the world of material agency even if the one cannot be substituted for the other (p. 20)

In the construction of a machine, for example, scientists will have the intention to construct a new machine in order to fulfil a specific function. Thereafter, they will adopt a passive role and monitor the workings of the machine. This is then the period when material agency actively manifests itself. The process of modelling is open-ended and because of the emergent nature of material agency, the intended way of the working of the machine is not reached. It is then the turn of human agency to revise some aspects of the construction of the machine and this process is what Pickering calls "tuning". Thereafter, human agency will then again be passive while the material agency performs. This "dance of agency", that is, this process where human agency is active while material agency is passive, followed by a reversal of roles where material agency is active and human agency passive, may be repeated. In this process of tuning, the goals and intentions of the scientist may change and the material form of the machine may change as well as the social relations of which it is part. This interaction is what Pickering calls the "mangle of practice". It is the "dialectic of resistance and accommodation where resistance denotes the failure to achieve an intended capture of agency in practice, and accommodation an active human strategy of response to resistance" (Pickering, 1995, p. 22).

Pickering explains the concept of temporal emergence as events happening in real time. When constructing a new machine, scientists are not able to predict what precise collection of parts will constitute a working machine. Also, scientists are not able to predict what its exact powers will be. In the construction of a new machine, these aspects will have to be found out in practice, as it happens, by going through the mangle.

The diagram below illustrates the interaction between human and material agency in real time and the reciprocal influences on each other. Every action has specific goals and intentions and these may be revised as a result of the outcomes of the workings of material agency.

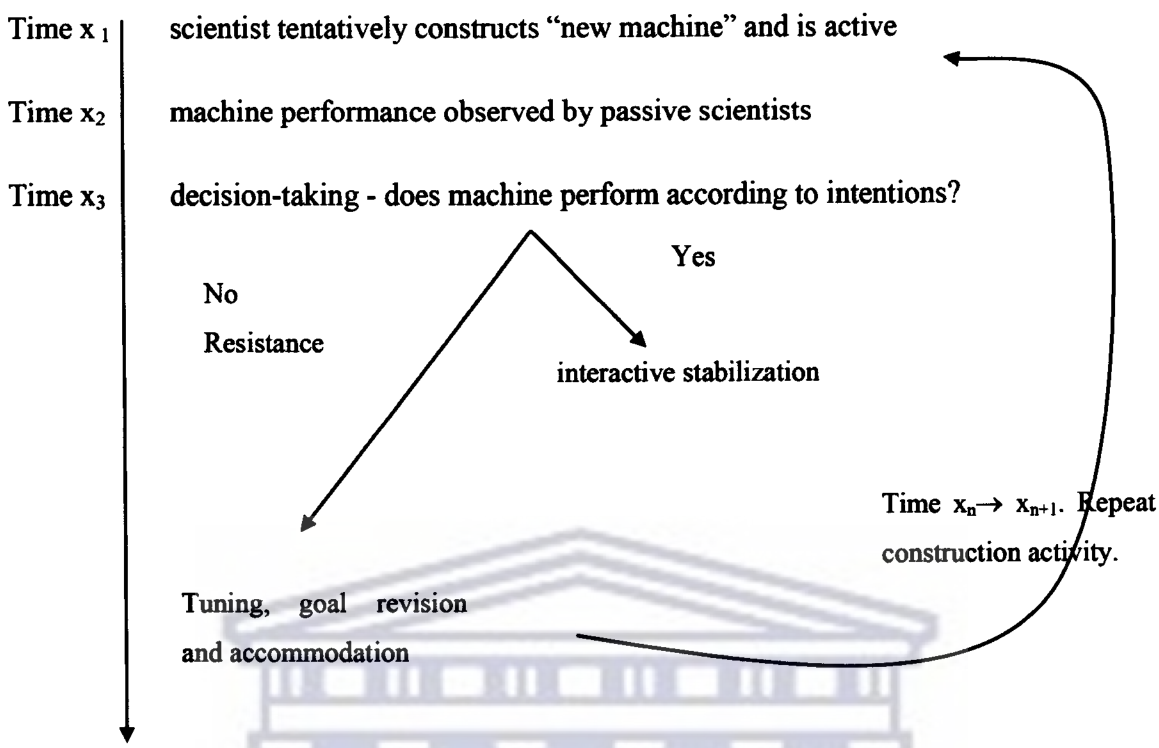


Figure 2: Representation of scientific practice based on Pickering's theory.

The mangle does not seek to only identify that the human and material agencies are at play in scientific practice, but rather to emphasize the "intertwining and reciprocal interdefinition of human and material agency" (p. 26). The distinctions of humanism and antihumanism are subverted within the performative idiom of scientific practice. Scientific practice seen in terms of the mangle now moves into a posthumanist space, a space where human actors and the non-human are inextricably entangled. "The world makes us in one and the same process as we make the world" (p. 26). Thus, instead of seeing scientific practice as separately consisting of activities of humans on the one hand and workings of machines on the other hand, it should rather be seen as the intertwining of the activities of humans and the workings of machines. This is what is termed "posthumanist", a level which is higher than how scientific practice had been perceived in the past, where the emphasis had been on human agency. The human and the material agencies are dependent on each other to produce new cultural artefacts.

The notion of machine construction, which is the visible material agency in scientific practice, is not existent in conceptual practice. How does one extend this performative idiom to conceptual practice and to Mathematics in particular?

Pickering proposes that resistance in conceptual practice within the mangle is now located in

> *disciplinary agency* – the sedimented, socially sustained routines of human agency that accompany conceptual structures as well as machines [and it plays] an analogous role in conceptual practice to that of material agency in material practice (p. 29).

Disciplinary agency is thus the counterpart to material agency in mathematical practice. It comprises the historically established, routinized and structured operational techniques that are applied and used in a mathematical domain. For example, if the square of a binomial has to be calculated as part of problem-solving, this process is done automatically, almost machinelike. This is an instance where the disciplinary agency is at play. Performing such techniques is independent of the goals and intentions of the human practitioners. This is the instance when the creativity of the scientist is passive and the disciplinary agency active. As is the case with material agency, "conceptual practice proceeds…through a process of modelling and new conceptual structures [are] modelled on their forebears" (p. 115).

The process starts when a new conceptual structure is modelled on existing structures. The three stages within any modelling sequence are bridging, transcription and filling. Bridging, or the construction of a bridgehead, is the preliminary fixation on a path of investigation to be pursued. It projects initial goals and intentions of the pursuit with allowances for such goals and intentions to be revised. In problem-solving in Mathematics, it could be any creative method in order to reach a solution. For example, if learners are asked to solve the quadratic equation $x^2 - 3x - 4 = 0$, the bridgeheads that they can construct can either be to factorise the left-hand side or to draw a graph of the function represented by the left-hand side.

Transcription is the use of established procedures from the old system to the new space established by the bridgehead. Once a method is decided on, the algebraic or algorithmic working in order to get to a solution can be classified as transcription. In the previous example, if the bridgehead chosen is to factorise the left-hand side, this would involve using existing algorithms. Those procedures would be transcriptions that are forced moves. Filling is creatively completing the new system to achieve the goals and objectives. Creative strategies to use whatever has been calculated in order to arrive at the solution are known as filling. Bridging and filling are free moves where scientists display choice and creativity, while transcriptions are forced moves by virtue of the discipline of established procedures.

The intertwining of disciplinary agency with its forced moves and human agency with its free moves in the real time construction of artefacts beyond what currently exist in the arsenal is the posthumanist, performative approach to mathematical practice.

The following are instances of how Pickering's theory has been used by researchers to analyse practices in mathematics classrooms. Brown and Redmond (2008) interpret Pickering's theory in terms of teachers' approaches in the classroom. If teachers teach by concentrating on the algorithms and the methods of doing Mathematics, they promote disciplinary agency. If they favour open-ended questions and encourage discussion and new ideas, they promote human agency. The students' movements from the disciplinary to the human agency are what constitute the dance of agency. Agency would then be the teacher's way of "being, seeing and responding" to learners (p. 107).

Boaler (2003) observed the different teaching approaches in Mathematics classrooms and found that in classes where students were given open-ended questions and were guided through the problem-solving process, students learnt to mingle the standard algorithms with their own thoughts when solving problems i.e. the dance of agency as *per* Pickering could be observed.

Grootenboer and Jorgensen (2009) compared the difference in the ways of working of research mathematicians and mathematics students. Research mathematicians are more collaborative while students are focused on the procedures and algorithms of the discipline. They assert that the learning of Mathematics should be like the practice of research mathematicians. When students switch from one way of working to the other, the dance of agency takes place.

In this study, problem-solving while working in collaborative groups is analyzed noting the episodes of bridging, transcription and filling, what the instances of resistances were and how these resistances were overcome by accommodation. The dance of agency, as described by Pickering, is thus highlighted in the analysis.

2.6 Concluding Summary

This chapter discussed what was meant by a problem and what was considered as problem-solving and thus set the tone for the type of problems which were investigated in this study. The definition for a mathematical problem employed in this study is Schoenfeld's definition which states that it is a problem that the learner was engaged in and one that the learner did not have a means to solve. A mathematical problem was thus seen as a relationship between the problem and the person engaging with the problem. The process of problem-solving was discussed and reference was made to the method advocated by Polya (1957) consisting of questions to guide solvers through the process. It has also been noted that problem-solving cannot be explained by mental structures of individuals alone, but that individuals should be placed in a particular context and cultural practice.

The distinction between cooperative group work and collaborative group work was discussed. Collaborative group work was seen as activities where the whole group engaged with the whole problem, as opposed to where there was a division of labour amongst group members. In this study, groups engaged in collaborative group work. Within studies of collaborative group work, research differed in terms of their unit of analysis, whether it was the individual within the group, the group itself, the activity or the context. This would depend on the theoretical frame of reference. Previous research was thus classified in terms of socio-constructivist, socio-cultural, shared cognition or situated perspectives.

Within socio-cultural theory specific terminology permeates the discourse and this had been discussed. Constructs like appropriation, artefacts, tools, mediation, meta-cognition and inscription had been defined and explained. How these constructs were manifested during group work will be discussed in the analysis of the group work.

The analytical framework was according to the mangle of practice as proposed by Pickering (1995) where he looked at scientific activity as moving into a posthumanist space i.e. where human actors and machines (disciplinary agency) were inextricably linked. This framework stemmed from the actor-network theory which in turn is an extension of the socio-cultural theory. The analysis of the data discussed the processes which the groups go through in solving the problems in terms of this mangle. This theoretical framework will shape the methodology used in order to answer the research questions.

CHAPTER 3

RESEARCH APPROACH

3.1 Introduction

This study analyzed group interaction while learners were doing problem-solving in small groups. The research was done in the form of a case study where groups of learners were engaged in Geometry problem-solving during the extended school day after normal school lessons. Socio-cultural theory underpinned this study and the discussion of the data highlighted how learners appropriated the cultural tools, especially the use of inscriptions, in order to solve Geometry problems. Pickering (1995) proposed a theory that scientific practice is a "dance of agency" which was the intertwining of human and material agency. In conceptual practice, which was what this study concentrated on, material agency was substituted with disciplinary agency. Therefore, the dialogues during group discussions were also analyzed by noting the occurrences of resistances and accommodations which make up this "dance of agency". In this chapter, the theoretical considerations regarding an ethnographic case study will be looked at. The data collection method, the ethical issues relevant for this study, the social context in which the study was conducted, the actual problems given to the learners and the methods for analysing the data are discussed.

3.2 Theoretical considerations

It is important to acknowledge that research is a human activity which is situated and presented within a particular set of discourses and conducted in a social context (Punch, 2009). A paradigm is our view of what we think about the world (Lincoln and Guba, 1985). According to Lincoln and Guba, our actions as researchers are situated within our world view.

Ethnography is the "study of social interactions, behaviours, and perceptions that occur within groups, teams, organizations and communities" (British Medical Journal, 2008, p. 1020). An ethnographic approach to data gathering had been used in this study. Ethnography means describing a culture and understanding a way of life as seen by its participants. Ethnography provides a research framework that allows for the "description of the routine, everyday, unquestioned, and taken-for-granted aspects of school and classroom life" (Hitchcock and Hughes, 1989, p. 55).

These would include the way learners related to one another during group work, their use of language, the uniform worn, the classroom environment, the seating arrangement, and the ringing of the bell to signify the end of a period or school day. In particular for this study, aspects like chatting about other issues not relevant to the problem at hand and how this impacted on learners' shared understanding of the problem, the code-switching and use of slang while discussing, their respectful nature in conversing with one another are all aspects that were present while discussing and played a role in the flow of the discussions.

The ethnographic approach to data collecting was preferred because this method was able to illuminate the social and cultural aspects in the interactions while the learners engaged in problem-solving. One was also able to make thorough and detailed analyses of the appropriation of tools and inscriptions while learners were doing problem-solving in small groups. According to Carlsen (2008), the motivation for using an ethnographic approach is a belief that this approach was generally successful for educational studies in Mathematics. It was best suited for understanding the subtleties of the appropriation process in Mathematics in comparison to other approaches. Therefore, the ethnographic approach was apt for the research questions of this study.

The term "ethnomethodology" was coined in the late 1950s by Harold Garfinkel. The prefix 'ethno' is used to indicate areas of indigenous practices of a community. 'Methodology' would therefore denote the study of those indigenous methodological practices. Thus ethnomethodology is the study of people's methods for conducting social practices. Ethnomethodology focuses on how people in a social setting make sense of their everyday social practices. Garfinkel claims that by studying the actual methods by which the social structures of society are made observable, all members of society, not only sociologists and philosophers, are doing sociology (Garfinkel, 1967). Ethnomethodology is a way to highlight the everyday taken-for-granted aspects of social life. Garfinkel, in his pioneering work, paid careful attention to practical actions of laboratory scientists and mathematicians. These 'studies of work' which focused on what people were doing when they are doing their jobs, involved close examination of the details of the works' practice. Under the banner of the Sociology of Scientific Knowledge (SSK), work is done to recover the specifics of some scientific and mathematical work. From the work of Pickering (1995), Livingston (1986) and other ethnomethodologists one can also study the produced artefacts to trace the "ethnomethods" used by others in their process of constructing an artefact.

This research is ethnomethodological since it focused on the detailed practices of what learners were doing when they were doing problem-solving in small groups and how they conducted problem-solving in the context of the classroom.

The case study method was preferred because it was more suited to describe the multiple realities encountered at any given site. Also, conclusions would be drawn ideographically (in terms of the particulars of the case). It also allowed for qualitative methods of research, which were preferred because they could take into account the many mutually-shaping factors and value patterns that might be encountered (Lincoln and Guba, 1985).

Stake (1995) distinguished between three types of case studies: intrinsic, instrumental and collective case studies. The intrinsic case study had as its purpose a better understanding of a particular case, the instrumental case study examined a case to give insight into an issue or to refine a theory and the collective case study was where a case study was extended over several cases to learn more about the phenomenon, population or general condition. This study was intrinsic, as well as instrumental, since the case study was undertaken in order to get an in-depth understanding of what learners did when they tackled Geometry problem-solving and how they appropriated the cultural tools at their disposal, and to gain insight into how a group interacted collectively to come up with a solution to the problem.

A case study is a thorough and comprehensive examination of a single case or a group of cases. It deals with the particular character and complexity of the case under scrutiny (Stake, 1995). This case study focused on three groups as they did geometry problem-solving in group work without the assistance of a teacher.

Punch (2009) identified four characteristics of case studies. Firstly, the case was a "bounded system" and these boundaries should be described (p. 120). In terms of the boundaries for this study, it investigated and analyzed the interactions of grade 12 learners involved in geometry problem-solving in small groups. Secondly, the case was a case of something and this needed to be identified. In this study, the case was an example of learners doing collaborative group work. Thirdly, the wholeness, unity and integrity of the case needed to be preserved but at the same time the focus of the research had to be made explicit. Since not everything about this case could be analyzed, the focus was on how learners appropriated the cultural tools at their disposal and their use of inscriptions in their interactions while solving Geometry problems.

The data were also analyzed in terms of instances of resistance and accommodation during problem-solving. The problem-solving process was modelled on existing structures and this modelling process was further broken down into bridging, transcription and filling moves (Pickering, 1995). Fourthly and finally, within a case study, multiple sources of data and collection methods were used. Data, in this study, were collected by means of observation, field notes, video and audio recordings, as well as from interviews with the learners. This case study, therefore, satisfied the four characteristics of case studies identified by Punch.

3.3 Data gathering

In this research the data was collected in the learners' educational setting, i.e. a classroom. All learners at this institution had an extended school day in which they are involved in academic work for an hour after normal lessons. These normally included doing study under supervision of a teacher or attending remedial or enrichment classes. Different venues were allocated to the grade 12 learners where they could study, complete homework assignments or projects, or discuss school work with peers without supervision during this hour. Most of the time they would work in informal groups to complete group work assignments or to complete homework assignments. The data collection was done during two of these sessions.

As mentioned before, data were collected by means of audio and video recordings of the small groups while they were engaged in problem-solving, teacher observation as well as from interviews with learners. The scribbled calculations learners made while doing the problems, and calculations and diagrams used by learners were included in the data. The grade 12 learners were allowed to work unsupervised, individually or in groups, according to their preference. It was quite common to see groups working on mathematical problems. It was during two of these sessions that the sample of groups was given Geometry problems by the researcher. In each session the groups worked on a different Geometry problem. Learners saw the problem for the first time at the start of the session.

Video recordings were used to collect the data because this method made possible multiple viewing of the interactions. It also allowed for the analysis of gestures, body language and any written work that learners have used in their discussions. It could also capture the use of instruments or tools used in problem-solving. The shortcomings of video recordings as listed by Carlsen (2008), included that the camera could not capture what an observer would see, the camera did have a point of view and that the camera did not cover context.

Therefore, this method was supplemented with observation and field notes by the researcher, as well as with later interviews with learners to verify observation and transcriptions.

The first round of data collection was done in May 2010 and the second round in August 2010. The sessions were approximately 30 and 90 minutes long respectively. Field notes were made by the researcher while the groups were being observed. During the first session, all three groups were in the same venue. All three groups were videotaped and, in addition, audio recordings of two of the groups' discussions were made as well. For the second session, each of the three groups was in a different venue with a novice cameraman. The researcher went from classroom to classroom to make observations, while the groups were discussing the problem.

The researcher is not fluent in isiXhosa, the learners' home language and the language that was mostly used in the discussions. The discussions had to be transcribed by home-language speakers and then translated into English. These were done by two ex-learners who transcribed three recordings each. The transcripts were typed and the translations were audio-recorded. These translations were then typed by the researcher. These recordings were then listened to repeatedly by the researcher for the analysis to be done. Also, after the transcriptions were completed, the researcher asked groups to peruse the translation while listening to the audio recording to confirm that the translation captured what they had meant. This turned out to be very time-consuming and thus the groups did not peruse all of the translations thoroughly. In addition, a debriefing session was held where the researcher discussed the different ways of obtaining a solution with the learners, so that they could reach closure on the work and not be left without knowing whether their solution was appropriate. Also, a brief semi-structured interview was held with the groups to determine their feelings on the group work activity.

3.4 Ethical Issues

Four ethical principles are listed in Carlsen (2008): harm to participants, lack of informed consent, an invasion of privacy and whether deception was involved. Every effort has been made to ensure that none of these principles has been dishonoured. The learners were informed of the purpose and scope of the study and they volunteered to be participants in the study.

The Western Cape Education Department granted permission for this study to be conducted. A letter requesting consent was sent to the parents of the learners involved in the study. This letter explained the purpose and details of the study and all parents signed the letter giving permission for their child to participate in the study. The study was conducted at school in a familiar setting and the researcher is their Mathematics teacher, thus the anxiety of being placed in an unfamiliar setting with an unknown researcher was minimized. It is acknowledged, though, that one cannot completely eliminate the effects of the external recording devices on the participants, more so if it is not a common occurrence in the classroom setting. The names of the participants would not be mentioned in the study and the participants were informed that they could withdraw from the study at any time should they so wish. Thus, in the second round of problem-solving, in two of the groups a member had been replaced with someone else. The persons involved in the translation of the recordings have been requested to keep the information confidential and an agreement has been signed in this regard. Thus, it can be seen that ethical principles were taken into consideration to ensure that the participants and their parents or guardians were fully informed of the scope and purpose of the study and it was ensured that no harm would come to participants as a result of their participation in the study.

The Geometry content covered in the given problems is examinable in paper 3 of the national Mathematics school-leaving examination, albeit not in the same format. As mentioned earlier, Geometry is examined in the optional paper 3. The result obtained by the learner in this paper does not affect his total mark (paper 1 plus paper 2 plus the school based assessment mark) for Mathematics at the end of the year, but is shown as a separate result on the learner's certificate. In the final external examination two compulsory papers are written. Both papers are out of 150 marks and are written in three hours. Paper 1 examines Algebra and Calculus while paper 2 examines Coordinate geometry, Transformation Geometry, Trigonometry and Data Handling. The optional paper 3 is written in two hours and is out of 100 marks. This paper examines Probability, Data Handling, Recursive Sequences and Geometry. All of the learners involved in the study were registered for this optional paper.

3.5 Validity, Reliability and Relevance

When tests are administered and the same results are obtained repeatedly, then the test is said to be reliable. Validity refers to the extent to which a test measures that which it was intended to measure.

Golafshani (2003) argued that these definitions for reliability and validity as they were used for quantitative research were viewed differently by qualitative researchers. Validity and reliability in quantitative research referred to the credibility of the research while in qualitative research they referred to the ability and effort of the researcher. He stated that the terms are viewed separately in quantitative studies but in qualitative studies terms that embodied both concepts were used. He looked at the different ways scholars had defined reliability and validity in the qualitative framework and concluded that researchers would define validity according to their own perceptions within their choice of paradigm assumptions. The terms used for reliability and validity in the qualitative framework include terms like quality, rigour, credibility, transferability and trustworthiness (Golafshami, 2003).

Creswell and Miller (2000) stated that there were a number of ways in which researchers could establish validity in qualitative studies. These included member checking, triangulation, thick description, peer reviews and external audits. The choice of the validity processes depended on the lens the researchers chose to validate their studies, i.e. the researcher, the participants or external people, and the researchers' paradigm assumptions, i.e. post-positivist, constructivist or critical paradigms.

Triangulations is a "validity procedure where researchers search for convergence among multiple and different sources of information to form themes or categories in a study" (Creswell & Miller, 2000, p. 126). In this study, for triangulation, the different sources of information in this study were video recordings, audio recordings, field notes of observations, scribbled notes of participants as well as interviews with participants.

In order to ensure that valid conclusions were drawn from the data in this study, deliberate steps were taken. Debriefings with the participants in the study were held to check on whether the translations were correctly done and also to check whether the interpretations of the dialogues were what had been intended by the participants. This process involved replaying the videos and/or audio recordings while reading through the transcriptions. Also, during these sessions, questions of clarification were asked by the researcher about sections that were ambiguous. This process is called member checking by Creswell and Miller (2000).

In order to establish the validity procedure of external auditing, the data was also be checked by two professors of Mathematics education to assess the validity of conclusions drawn from the data.

Relevance concerns how relevant the study is for practice. The relevance of this case study in South Africa is high, since emphasis is placed on group work in the new curriculum and any study that contributes to elucidating quality understanding of group work is relevant within this context. The focus on Geometry problem-solving is important since Geometry is, from 2012, included as an examinable section of Mathematics.

The aim of this case study was neither to generalize nor to extend its findings to other similar cases. According to Punch (2009), there are two cases where the objective of the case study would not be to generalize. These are the intrinsic case study and the instance of the negative case. As indicated previously, this case is an intrinsic case study, since the aim is to investigate in detail the particulars of this specific case.

Punch (2009) also argued that a case study, although not generalizable, had important functions in qualitative research on three fronts. Firstly, it was what we could learn from studying a particular case, in its own right. Also, an in-depth study was the only way in which insight into a persistently problematic or new research area can be obtained. Thirdly, the case study could be used in combination with other research approaches like surveys. For example, if the case study is conducted ahead of the survey it could give direction to the survey, which would not have been possible without the information gained from the case study. The rationale for this case study was to learn about this particular case and not to generalize its findings.

3.6 The social context

The learners all reside in Khayelitsha, a township approximately 30 km outside Cape Town, and most came from disadvantaged backgrounds. The school that the learners attended was a fairly new school, established in 1999, and therefore the classifications of the old system do not apply. Prior to 1994, schools in South Africa were classified in racial terms. Department of Education (DOE) schools were the schools for white learners, Department of Education and Culture (DEC) schools serviced coloured learners and Department of Education and Training (DET) schools serviced black learners.

Historically, the DOE schools fared relatively better than the DET and DEC schools, with the DET schools faring the worst academically.

The learners involved in the study were accepted at the school in question in grade 10 after going through a selection procedure. One of the strategies used in selecting suitable learners was to have a teaching session where learners would be taught a particular Mathematics topic. They would then be given some exercise to do for homework, which would be discussed at the next session. They would then be given a written test on that topic. A similar procedure is followed for English. Learners who do well in these tests were possible candidates. The marks obtained by learners in their grade 9 year were also taken into consideration. Thus, learners accepted at the institution were believed to have the ability to master the content of the curriculum of Mathematics, Physical Science and Information Technology. Because these learners came from impoverished backgrounds, they would not have had the opportunity to study at ex- model C schools because of the high school fees that those schools charge, unless they were given a bursary. Ex-model C schools are generally the historically white schools with adequate resources and better qualified teachers and where learners have a greater opportunity to excel. The pass rate of learners at these schools is much higher than those of township schools. Bloch (2009) cites a study done by Nick Taylor which used Mathematics higher grade passes at grade 12 level to indicate the differences in the performances of schools. The study showed that two-thirds of the higher grade Mathematics passes is produced by just 7% of the schools (mostly ex-model C schools) while 79% of the schools produced 15% of the passes. The ex-model C schools have a higher overall pass rate as well. In 2007 less than 2% of white learners failed while 39% of black learners failed (Bloch, 2009). While it was known that not all black learners attend township schools and similarly all white learners do not go to privileged schools, these statistics paint a picture of the contrast between performances at the advantaged versus the disadvantaged schools.

For learners in Khayelitsha, attending an ex- model C school would also require learners to travel out of their community, have their schooling in a different setting and then having to travel back to their own community. The school in question offered learners the opportunity to have facilities conducive to learning within their own cultural setting. It is a well-resourced school, which has smaller classes than other schools in the learners' communities and which prides itself on, *inter alia*, instilling a culture of learning in its learners and placing high importance on regular attendance, punctuality, quality teaching and delivery of the curriculum, the motivation of learners and the importance of good academic results.

The learners at the school generally perform well in the final examinations in Mathematics. The Mathematics department at the school tries to be innovative in teaching and learning approaches and encourages group work in classrooms. As previously mentioned, Euclidean Geometry was a section of the work that was examined in the optional third paper and not all the learners were registered for paper 3. The content of this paper was taught after normal lessons and on Saturdays.

Three groups of learners were a sample taken from a class of grade 12 learners who did group work in Mathematics. The learners were taken as a convenience sample. They were grade 12 learners who were taught by the researcher and who indicated a willingness to participate in the study. They chose who they wished to work with within their groups. The learners all spoke isiXhosa as their home language and were taught in English. While most of the discussions were in isiXhosa there were some dialogues that were only in English. The teacher was able to understand very little in isiXhosa and was not able to explain Mathematics in isiXhosa. Each group is heterogeneous with respect to academic ability, with the majority of learners performing above average in class assessments. The categorization of the learners' abilities is based on their performance in class assessments.

Learners were told that they were could withdraw from the project at any time. There were thus changes to the groups for the second round of data collection. In group 2 YH was replaced by YM and in group 3 QM was replaced by NB. Both of them were absent on that day.

The following table gives the composition of the groups for problem 1:

Table 2: Composition of groups for problem 1

Group 1

Learner	Male/Female	Age	Mathematics ability
ZM	Male	18	Average
NB	Male	17	Above average
SM	Male	18	Average

Group 2

Learner	Male/Female	Age	Mathematics ability
NS	Female	17	Average
AN	Male	18	Above average
YH	Male	18	Average

Group 3

Learner	Male/Female	Age	Mathematics ability
AM	Female	16	Above average
AD	Female	17	Average
QM	Female	17	Average

The following gives the composition of the groups for problem 2:

Table 3: Composition of groups for problem 2

Group 1(unchanged)

Learner	Male/Female	Age	Mathematics ability
ZM	Male	18	Average
NB	Male	17	Above average
SM	Male	18	Average

Group 2 (YM replaced YH who was absent on the day)

Learner	Male/Female	Age	Mathematics ability
NS	Female	17	Average
AN	Male	18	Above average
YM	Male	18	Average

Group 3 (NB replaced QM who was absent on the day)

Learner	Male/Female	Age	Mathematics ability
AM	Female	16	Above average
AD	Female	17	Average
NB	Female	17	Average

3.7 The selected problems

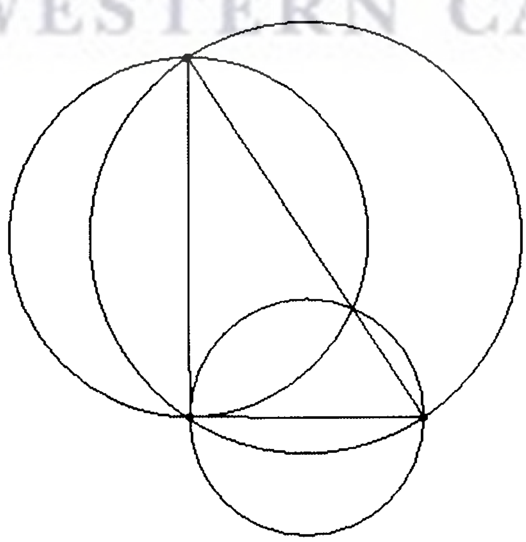
The problems chosen satisfied the definition of a problem (Schoenfeld, 1985) for the learners. None of them had seen the problem before and they also did not have readily available algorithms to solve the problems. Therefore, the problems necessitated group interaction in order to obtain a solution. The problems were also within the scope of the learners' content knowledge and academic ability.

The first problem was one of the daily problems given at an Association for Mathematics Educators in South Africa (AMESA) congress in 2009. Delegates at the congress solve the problems as part of a daily competition. The problem was chosen to be part of the study because it would definitely need collaboration in order to be solved for groups doing the problem for the first time. It also required logical thinking and it was different to the standard type of textbook Geometry problems. The problem is open to various methods of solution. Learners could do an investigation using numerical values or they could do abstract algebraic manipulations. Thus, the problem was made accessible to the learners, whatever their mathematical ability.

Problem 1

Circles are drawn with the sides of a right triangle as diameters. If the area of the triangle is 36cm², find the total area of the shaded regions.

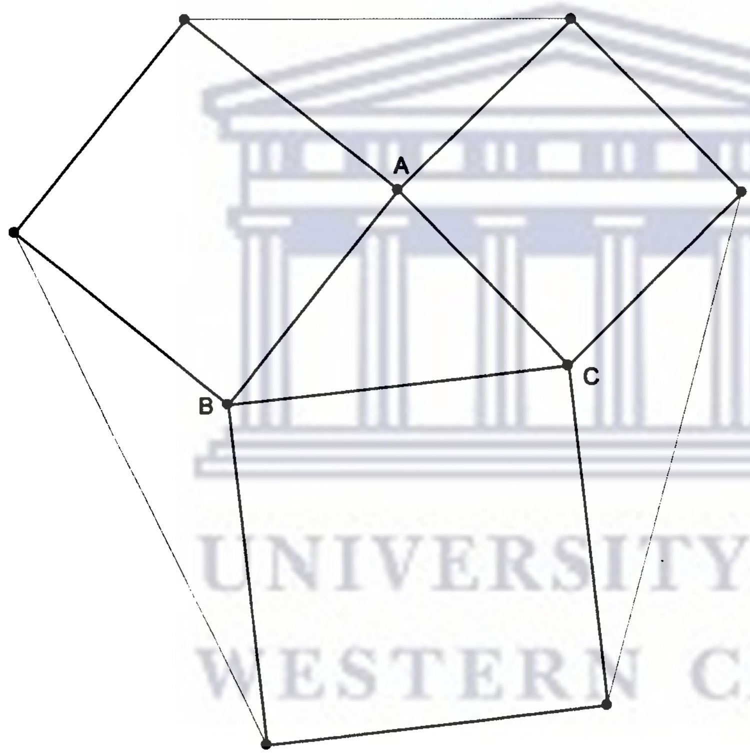
(Daily problem (29 June) AMESA congress, 2009)



The second problem was selected because of the opportunity to make use of the *Geometer's Sketchpad*. Learners had to do the construction manually and could then choose to use the software to verify their findings.

Problem 2

Construct any triangle ABC. Now construct a square on each of the sides of the triangle ABC. That is on AB, BC and AC. Next, join the vertex of a square with the one adjacent to it, forming three new triangles. Investigate the relationship between the areas of these triangles and the area of the original triangle ABC.



3.8 Data Analysis

All of the recorded sessions were translated from isiXhosa to English by repeatedly listening to the recordings and capturing all utterances. The translations in the transcripts were not necessarily literal but conveyed the meaning of what was said. Discussions were also held with the groups in order to verify that the transcripts were true reflections of what had transpired in the groups and also to clarify some of the issues that had not been clear to the researcher.

Learners often engaged in code-switching where they would switch from isiXhosa to English. Extensive use is made of colloquialisms like "yabo" (an abridged version of "Uyabona?" in isiXhosa. (Do you see?)), "le weyi" (this thing), "ja" (Afrikaans for yes). Learners also tend to use the isiXhosa subject concords with English words, for example, ii-angles, i-area as well as some Afrikaans words like le-plek (this place). These transcriptions were analyzed in terms of how the learners appropriate the cultural tools (calculators and computer software) and their use of inscriptions (diagrams).

As mentioned before, the study also analyzed the process of problem-solving in terms of what was called the "mangle of practice" which sees scientific practice as a dialectic of resistance and accommodation and a 'dance of human and material agency' (Pickering, 1995. p. 22). The dialogues were categorized in episodes classified as resistances, bridging, transcription and filling. In addition to the analysis of audio and video transcripts, the analysis included analysis of field notes, and learners' written work while they were engaged in problem-solving.

In this chapter the method for analyzing group interaction while learners were engaged in collaborative group work was discussed. This was qualitative research using a case study to gather data in order to answer the research questions. Three groups of grade 12 learners, each consisting of three learners, were videotaped while doing Geometry problems on two separate occasions. The social context of the school and the learners was discussed. Data were analyzed using a socio-cultural perspective noting the appropriation of tools and inscriptions and how these helped with the shared understanding within the group. Analysis of data also entailed identifying episodes of resistances, bridging, transcription and filling as explained by Pickering (1995).

CHAPTER 4

ANALYSIS OF THE DATA

4.1 Analysis of group work

The analysis of group work assumed that scientific practice was a process of modelling (Pickering, 1995). Conceptual practice, therefore, was modelled based on structures that had gone before. This process of modelling was further decomposed into bridging, transcription and filling. Thus, modelling had the characteristic of intertwining resistance and accommodation to overcome the resistances. According to Pickering (1995) all scientific practices follow these processes. This analysis highlighted how the above constructs came to the fore during the interaction of the learners with the problems.

In this analysis episodes were categorized as bridging, transcription and filling events, as well as resistances. An episode would include all the consecutive verbal exchanges focussing on that aspect. A new episode would start when the verbal exchanges shift towards a different aspect. An episode of resistance occurred when the working of the problem did not go as intended or when the group did not have the tools to further the solution. Accommodation to these resistances included all attempts to overcome these resistances and to work towards a solution. Bridging occurred when the group initiated a method or strategy in order to solve the problem. Transcription happened when existing algorithms and formulae were used to pursue that strategy. These algorithms and formulae were the ones used in the base model i.e. on which the existing practice was modelled. Filling occurred when creative methods were used that were different from previous models. In some cases, the distinctions between the episodes were not clear-cut, since one could identify a bridgehead, as well as transcription in the same utterance.

Next, the analysis focused on the use of inscriptions and tools and the roles that these played in the group's shared understanding of the problem. The use of, for example, the scientific calculator, mathematical instruments and computer software were analyzed in terms of how these aided or did not aid the group in reaching a solution. The inscriptions used, like diagrams, were similarly discussed.

As mentioned in the previous chapter, two different Geometry problems were given to each of the three groups. The first problem concerned circles and triangles and required that learners think creatively about which geometrical figures the sketch consisted of. The second problem dealt with triangles and squares and needed some construction (manually or by using *Geometer's sketchpad*) and measurement in order to make deductions. In this section, each problem was analyzed and interpreted, using the above framework in terms of the interactions within each group. The analysis consisted of the following steps:

- The transcriptions of the audio and video recordings and the translation thereof. These were done by two Xhosa-speaking ex-learners who transcribed and translated three recordings each.
- Extensive listening to the audio tapes and reading of the translations of the transcriptions by the researcher.
- From the transcriptions episodes of bridging, transcription and filling were selected. Excerpts that demonstrated these steps were analyzed and discussed in terms of how these helped in progressing (or not) to the solution.
- The use of tools and inscriptions by the learners was analyzed and discussed.

The following symbols were used in the transcriptions:

// means simultaneous talk.

( ) short pause

... long pause

[ ] explanatory notes by researcher

The utterances of the learners were numbered consecutively. In order to keep anonymity, the initials of learners were used. The utterances in isiXhosa and their translations in English have both been given in the excerpts.

4.2 Analysis of problem 1

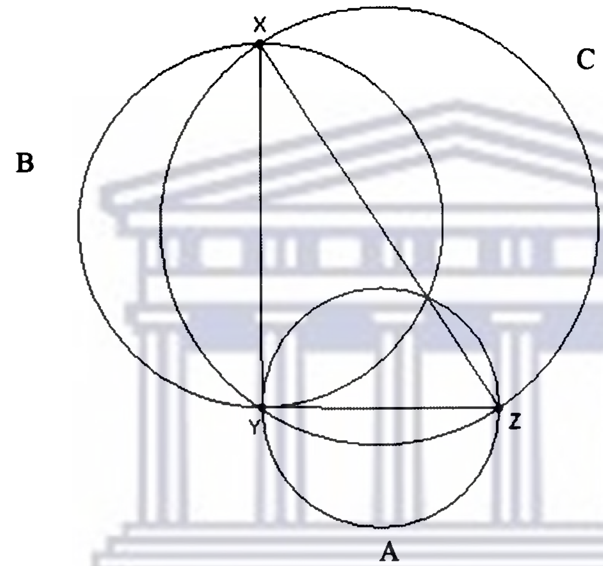
4.2.1 Group 1.

This group consisted of three boys, two of whom scored average on test scores and one who scored above average.

While doing the problem, the group labelled the different circles. The smallest circle was labelled A, the middle one B and the biggest circle was labelled C. The group referred to the sides of the right-angled triangle by pointing to them on the sketch. Therefore, for practical purposes, the sides of the triangle were named XY, YZ and XZ.

The following sketch indicates the labels used by group 1.

Sketch



The excerpts below demonstrate the conceptual practice during problem-solving. It demonstrates the bridging, transcription and filling moves as the group tried to make sense of the problem.

The context of the excerpt.

The way this group organised themselves was to allow one person at a time to explain their approach or strategy while the rest asked questions for clarification or in order to state disagreements. The strategy that the group used to tackle the problem was to first read the question and as they read the question they referred to the sketch to make sure that they understood the problem. This excerpt was taken very early in the session. During the first part of the discussion (turns 1 – 52) the group clarified what the problem required of them and identified the triangle, the right angle, diameters and radii for each of the circles.

Excerpt 1

53. SM: So in order mos man, i-base mos man ngu-12 surely, ngu-12 times 6. Then i-reason laweyi iyodwa ngu-6.
So in order man, surely, the base is 12, you see. It's 12 times 6. Then the reason for that is that this thing alone is 6. [YZ]
54. ZM: Eyiphi?
Which one?
55. SM: Le, um-ignore lowa u-6. I-half ka-12 ngubani? Ngu-6 times ngu-6 ngubani?
This one, [YZ] ignore 6. What is the half of 12? It is 6, times 6, is how much?
56. SM:// 36
57. ZM:// 36
58. SM: So which is i-area. So, i-base mos man yile-plek.
So this is the area. So this part is the base.
59. ZM: Ewe, it is the half of it, ngu-6.
Yes, it is the half of it, it is 6.
60. SM: Nantsi i-triangle.
Here is the triangle.
61. ZM: Yi-baseleya ngu-12, SM
This base is 12, SM.
62. SM: Eyiphi? Lendawo, ne?
Which one? This part? [XY] [Asking questions by way of explanation.]
63. ZM: Ja.
Yes.
64. SM: So i-half yale ndawo ngubani? Ngu -6.
So what is the half of this part? [XY] It is 6.
65. ZM: Ngu-6.
It is 6.
66. SM: So, ngu-6 apha, ngu-6 apha.
So it is 6 here and it is 6 here. [radii, when diameter XY is halved]
67. NB: Ja
Yes
68. SM: So le-plek ngu two times lena mos, man?
So this part is two times this part, isn't it? [XY = 2 YZ]
69. ZM: So le-plek ( ) ngubani i-area ye-circle kanene?
So this part ( ), what is the area of the circle, by the way?

A bridgehead was established when SM said (turn 53): "So in order mos man, i-base mos man ngu-12 surely, ngu-12 times 6. Then i-reason laweyi yodwa ngu-6". **(So in order man, surely, the base is 12, you see. It's 12 times 6. Then the reason for that is that this thing [YZ] alone is 6).**

The information given in the problem was used to find the lengths of the two sides of the triangle that were the diameters of those two circles and this was used as the starting point. They did this by assigning values (turn 53) to the sides, XY = 12 cm and YZ = 6 cm, and this was done using transcription. These values were calculated using the formula for the area of a triangle: $A = \frac{1}{2}\, base \times height$. It was given in the problem that the area of the triangle is 36 cm^2. The group assumed that the value of the base, therefore, had to be 12 cm and height 6 cm, since these values would give an area of 36 cm^2 (turns 55 - 58). This was a transcription move because the group used pre-existing knowledge regarding the area of a triangle. It was also what Pickering classifies as 'disciplinary agency' at work. The discipline determined how the area of a triangle was calculated. The group carried out the algebraic manipulation but this was governed by the mathematical discipline that prescribed how it should be calculated. SM's method of arriving at the values 12 and 6 for the base and height respectively was based on the sketch. The values decided on were plausible within the context of the sketch. XY looked twice as long as YZ and therefore values had to be assigned where the one was double the other one. Evidence of this way of thinking is in the question "what is the half of 12?"(turn 55) He answered his own question, "It is 6, times 6, is how much?" This then gave the value of the area of the triangle. The rest of the discussion (turns 58 - 67) was to verify their answer by referring to the sketch.

The question in turn 69 constituted a filling move. The learners were using the lengths of the sides that they had calculated to determine the areas of the circles. This was a creative move to see whether they could come closer to obtaining the solution to the problem. The verbal exchanges that followed were about establishing precisely what the radius of each of the circles was (turns 75 - 124). This creative move, which was a filling move, lay the basis for more transcription moves in which the areas of all the circles and the length of the hypotenuse were eventually found (turns 125 - 175). The areas of circles A, B and C were 9π, 36π and 45π respectively and the length of the hypotenuse was $6\sqrt{5}$.

Excerpt 2 contained an example of a bridging move that followed on from this information. In turn 179, the bridging move was to subtract the area of circle B from the area of the bigger circle C. The belief was that this would then result in finding the area of the shaded region of that circle (circle B). The transcription move that followed from this was a basic subtraction of areas of the circles.

A resistance then followed from this bridging process, as illustrated in the next excerpt. However, the resistance did not stem from the fact that the answer did not represent the shaded region.

Excerpt 2

176. ZM: Yile weyi yonke le, SM.
It's this entire thing, SM. [referring to circle C with area 45π]
177. SM: Yi-area yale-circle yonke le.
It's the area of the entire circle.
178. ZM: Ja
Yes
179. SM: And then xa ndi-get the i-area yale-circle yonke ndi-minuse le-area, phi? Kule-weyi yonke.
Then when I get the area of this entire circle [circle C] I will minus this area. [circle B with area 36π]Where? From this entire thing.
191. SM: Izandinika eyiphi indawo? Izandinika le-ndawo surely kaloku.
Which part will this give you? Surely it will give you this part. [Indicating the shaded part of circle A)
192. ZM: Le? Ja.
This one? Yes.

In excerpt 3, turn 254 illustrates a basic subtraction algorithm that is a transcription move. SM was convinced that the shaded area is a quarter of the area of circle B. However, he divided the difference of the two areas by 4, indicating that he meant that the area of circle B is represented by the difference between the two areas previously calculated. This was in fact incorrect as the area of circle B was 36π as calculated before. This assumption was not disputed by the rest of the group. What was disputed was the notion that the area of the shaded region of circle B was a quarter of the area of circle B. Even though the group agreed that this could not be a quarter, they still went ahead and assumed that it was a quarter when they did the calculations. This approach of finding the area of the shaded region of circle B could not be used to find the shaded area of the smallest circle A and this once more led to a resistance.

Excerpt 3

254. SM: Yabo? So xa uthabatha, xa uminuse lena kushiyeka lo 9π
So u-timezange-quarter, i-timezange-quarter?
You see? So if you would take, if you would subtract this one 9π is left. So you multiply with the quarter, do you multiply by a quarter?
255. ZM: Uyifumene phi ukuba yi-quarter, SM?
Where did you get the quarter, SM?
256. SM: Surely yi-quarter man, ZM, le jonga yi-half le.
Surely this is a quarter ZM, look it's half of this. [looking at circle B and Stating that the shaded region is half of the semi-circle]
257. ZM: Uyifumene phi ukuba yi-quarter le?
Where did you find out that this is a quarter?
258. SM: Khawume, yihalf le, ZM, xa ufaka enye i-half apha kuzophuma leweyi straight. Uphinde uthi suleweyi, yabo?
Wait, this is the half, ZM, when you put in another half here it will result into this. And then you do this like this, you see?
259. ZM: You can't be so sure kaloku, SM.
You can't be so sure, SM.
260. SM: Ja, andinokwazi ukuba-sure, ja, ndiyayivuma lo reasoning…
Yes, I can't be sure, yes, I agree with that reasoning…

The group considered another approach dealing with squaring the areas of the circles that also resulted in resistance because it did not give the intended answer. Further, a bridging move that involved joining X and Y to a point W on the circumference of the circle so that ZYXW formed a rectangle was also pursued. NB then tried to determine the length of the diagonal of this rectangle which was the diameter of the big circle, C.

Another bridgehead was constructed when NB wanted to determine the diameter of the circle C, which was the hypotenuse of the right-angled triangle (see excerpt 4). The other group members appropriated the values that they have already calculated (turns 520, 521 and 523) to find this diameter. NB then tried to explain to them that the length, XZ, was obtained by using assumed values and that these values could have been any values as long as they had a product of 36. This was a bridgehead because this indicated a shift in the way of thinking about the problem. The other members had not thought about the problem in that way. This would involve using variables instead of actual numbers and would have implications for the transcription moves to be used.

Excerpt 4

517. NB:// Isimoko seso, ngubani le-line?
That is the problem. What is the length of this line? [XZ]
518. ZM: Ngu..., ye mfondini, sizothi le squared minus le squared is equal to…
It ... my friend, we will say this squared minus this squared is equal to…
519. NB: Ngubani ezi?
What are these? [XY and YZ]
520. ZM:// Ngu-12 no-6.
It is 12 and 6.
521. SM:// Ngu-12 no-6 kaloku ntangam.
It is 12 and 6, my friend.
522. NB: Azikho ezinye i-values onomultiplaye ngazo apha nalapha?
Are there no other values that you could multiply with here [XY] and here [YZ]?
523. ZM: Ngu-6 no 6.
It is 6 and 6.
524. SM: Ngu-12, kufunwa ntoni apha ntangam? Kodwa u-3 yaphuma, uyaphuma u-2?
It is 12, what is needed here, my brother? But it is 3, do we get it [the answer], do you get 2?
525. NB: Ingangu-4 lo.
This could be 4.
526. SM: Ibengu ( )
And then it is ( )
527. NB: Inganguthree lo.
It could be three.
528. SM: Ha-a tshuyisani!
Huh-uh do it!

However, there was resistance when one considers NB's use of variables for the lengths. He assigned the variables x and y to XY and YZ respectively, but he did not get the opportunity to follow through on his method owing to time constraints and the other members' insistence on working with values. However, had the group continued with this method, it would have been possible for them to reach a solution.

NB appropriated terminology from computer programming, soft coding and hard coding (see excerpt 5). Soft coding is where variables are used and the solution is not dependent on the values used. With hard coding fixed values are used and the solution is dependent on the values chosen. He was encouraging his group members to use variables in order to find a general way of finding the areas without relying on values. Later on in the discussion he emphasized that the group should be able to justify answers: "We have to present it in court" (turn 591).

Excerpt 5

558. NB: Ndifuna thina si-soft, sibe-soft coding, size neformula instead of coming with u-36. Of which u-miss angajika athi "this is not drawn to scale". Yiyo ntoke ndisithi masingafakini values apha. Ma-size neformula ethini? Ezotananob'umiss uthe hayi mamelani ngu-4 lo ngu-3 lo, because 4 times 3 ngu-12".
I want us to soft, to be soft coding, to have a formula instead of coming up with 36. Of which miss we will then say "this is not drawn to scale". That's why I'm saying let's not substitute values here. Let's come up with a formula that will be the same even if miss says " no, listen this is 4 and 3, because 4 times 3 is equal to 12".
565. ZM: But andi-understandi yiformula le ndiyithethayo.
But I don't understand, what I'm saying it's a formula which I am referring to. [referring to his use of the theorem of Pythagoras]
566. NB: Shaded region sawuyifumana ngeyiphi iformula? What formula can we use to find the shaded region le ndawo using esi-structure? Masithi leweyi leya khouyi-ringayo uba ( )
Which formula are we going to use to find the shaded region? What formula can we use to find the shaded region, this part using this structure? Let's say what you are saying is ( )
567. ZM: Ok, sizokufumana i-area yaleweyi kaloku le.
Ok, we will find the area of this thing then.

This group did not find accommodations to the resistances and in the end did not get to a solution to the problem. The solution to the problem could have been approached from two perspectives. On the one hand an inductive approach could have been used where values were assigned to the different sides of the right-angled triangle. Other different values could then have been assigned to the sides to see whether a pattern emerged and thus a generalization could be made. On the other hand variables could have been used to denote the lengths of the sides. This would have involved an algebraic solution, using geometrical theorems. The two approaches for finding the solution are presented as follows:

Solutions to the problem.

To follow through on the approach started by SM, and using an inductive approach, a solution would be as follows:

Let smallest circle = A, the middle circle = B and the biggest circle = C.

Since area of ΔXYZ is 36 cm^2

Let YZ = 6 cm

and XY = 12 cm

Therefore the hypotenuse XZ of the right-angled triangle is: $XY^2 = 12^2 + 6^2$

$= 144 + 36$

$= \sqrt{180}$

$= 6\sqrt{5}$

These sides are the diameters of the circles:

Thus the radius of circle A = 3 cm;

The radius of the circle B = 6 cm

and radius of circle C = $3\sqrt{5}$

Area of circle A would then be: $A = \pi r^2$

$= \pi(3)^2$

$= 9\pi$

Area of circle B: $A = \pi r^2$

$= \pi(6)^2$

$= 36\pi$

Area of circle C: $A = \pi r^2$

$= \pi(3\sqrt{5})^2$

$= 45\pi$

Area of the shaded area = area semicircle A + area semicircle B + area triangle XYZ – area semicircle C

$$= \frac{9\pi}{2} + \frac{36\pi}{2} + 36 - \frac{45\pi}{2}$$

$$= 36 \text{ cm}^2$$

Thus the total area of the shaded region is equal to the area of the triangle.

Different values need to be substituted for the sides of the triangle for a pattern to emerge and for generalizations to be made.

An alternative method using variables, using the variables that NB assigned to the sides:

Let YZ = x and XY = z

Then $XZ^2 = x^2 + z^2$

$XZ = \sqrt{x^2 + z^2}$

Thus radius of circle A = $\frac{x}{2}$ cm

radius of the circle B = $\frac{z}{2}$ cm

and radius of circle C = $\frac{\sqrt{x^2+z^2}}{2}$

Area of circle A would then be: $A = \pi r^2$

$= \pi(\frac{x}{2})^2$

$= \frac{x^2}{4}\pi$

Area of circle B: $A = \pi r^2$

$= \pi(\frac{z}{2})^2$

$= \frac{z^2}{4}\pi$

Area of circle C: $A = \pi r^2$

$= (\frac{\sqrt{x^2+z^2}}{2})^2\pi$

$= \frac{x^2+z^2}{4}\pi$

Area of the shaded area = area semicircle A + area semicircle B + area triangle XYZ – area semicircle C $= \frac{x^2}{8}\pi + \frac{z^2}{8}\pi + 36 - \frac{x^2+z^2}{8}\pi$

$= 36$ cm^2

Thus the total area of the shaded region is the same as the area of the triangle.

Discussion.

SM (turn 53) assigned values to the two sides of the right-angled triangle based on the given area of the triangle. This was a bridging move. One should note that this was a creative move on the part of the learner and that he had no idea what this move would bring about or the direction the path would take using this method. The only way of knowing what would emerge was to actually do the working and to find out, i.e. to go through the mangle. This is what Pickering (1995) called "temporal emergence".

What emerged during this session of problem-solving by this collaborative small group is the way the process of scientific practice as theorised by Pickering unfolded. The steps that conceptual practice followed, as well as the interplay of human and disciplinary agency, are illustrated in the episodes highlighted. This process, with reference to figure 2, is as follows:

T x_1: The group determined that the sides of XY and YZ were 12 and 6 respectively based on the information given in the problem.

Tx_2: The group then determined the length of the hypotenuse, using the theorem of Pythagoras. This was disciplinary agency at work.

T x_3: The disciplinary agency (machine) did not perform according to intention. The wrong answer was obtained and there was thus resistance.

Tuning was therefore done by reflecting on what they had done. They realised their mistake and determined the correct length for the hypotenuse.

This information allowed the group to establish another bridgehead with the decision to calculate the areas of the circles and thus the cycle was repeated.

Disciplinary agency was again at work in the calculation of the areas of the circle. The human agency was thus passive.

The disciplinary agency performed according to intention – the areas of the circles were calculated correctly.

The group then decided to use this information to determine the area of the shaded regions. This could be regarded as a filling move.

They subtracted the area of circle B (36π) from the area of circle C (45π) to get an answer of 9π – disciplinary agency.

There was resistance in that the group did not know how to use this answer. The accommodation to this resistance was to consider the shaded area of the middle circle as a quarter of the area of the circle. They calculated this to be $\frac{9}{4}\pi$. However, there was resistance to this answer because the group did not agree that this was the area of the shaded region.

Accommodation to the resistance was the drawing of a chord parallel to XY and a chord parallel to YZ to form a rectangle. This could be regarded as a bridgehead.

Attempts to find accommodation to the resistance included using variables for the sides of the triangle and also using different values for the lengths of the sides.

This group failed to see how the shaded areas could be derived by subtracting the area of semicircle C from the total area of the other two semicircles and the triangle. In terms of Pickering's terminology, they could not find accommodations to the resistances. They were on the right track and one might think that, given extended time, they might have come to the correct solution.

The approach that SM initiated eventually enabled the group to calculate the areas of the three circles, which was a suitable strategy to get to a solution. His approach was to assign specific values to the diameters of the two smaller circles based on the area of the triangle which was given as 36 cm^2. Since the diameters of the circles were the sides of a right-angled triangle, they assumed that the diameter of circle B was 12 and the diameter of circle A was 6. These two diameters would then be the base and height respectively of the right-angled triangle. They used these values to calculate the hypotenuse of the right-angled triangle, which was also the diameter of circle C. These values allowed them to calculate areas for the three circles. However, this strategy assumed that the values chosen were the only values for the sides and they did not consider that there could be more values which would also give that area for the triangle.

The resistance that this group encountered was that they could not see how they could use the areas of the circles to get to an answer to the problem. They first followed an inductive approach, where they looked at specific values for the diameters of the circle.

These values were chosen because the one diameter looked like half of the other one. This kind of reasoning was again seen when SM stated that the area of the shaded part of the middle-sized circle was a quarter of the whole part. When questioned about whether he was sure that that was a quarter of the circle and he agreed that he could not be sure, but he still went ahead and calculated the area of the shaded part to be $\frac{9}{4}\pi$. This reasoning demonstrated the dependency on the sketch to find values for the sides of the triangle. Because these values were the only values that seemed, to the group, to fit the lengths of the sides of the triangle as given in the sketch, they were taken as the only possible answers. Thus, they looked at only one set of values and did not explore other sets of values in order to determine a pattern so that they could deduce a solution.

NB attempted a deductive approach by substituting variables for the diameters. He tried to explain to the group that values could not be assumed because they looked like those lengths on the diagram and he reflected on what the teacher had told them in class. Unfortunately, the rest of the group was not able to help him explore that method, which could have, if pursued, resulted in a solution to the problem.

This group clearly worked collectively with the shared objective of finding a solution to the problem. One person would be given an opportunity to explain his approach to the others. The others would give their attention to the one explaining and ask questions for clarification. They were free to disagree. Clearly, they were all trying to get a shared sense of the problem and were mostly working collectively on the problem and not as individuals. Individual group members had their own unique style of getting their ideas across. For example, SM (turn 55) explained his method for finding the sides of the triangle by asking questions of the others in the group. "What is the half of 12? It is 6, multiplied by 6? What is the answer?" He clearly knew the answer but this was his way of getting the group to understand his method of arriving at the lengths of the sides. This was an example of group collaboration. The members of the group were all equals in terms of their responses to one another and in terms of their content knowledge with respect to the problem. While discussing the problem, in the group context, the zone of proximal development was bridged.

As a result of group collaboration, one group member (ZM) could identify what the diameters in the sketch were, as well as identifying the one diameter as the base of the triangle and the other diameter as the height of the triangle, which he initially could not do.

In excerpt 5, NB tried to let the others see the lengths of the sides in terms of variables. He used the term "soft coding" to de-emphasize the importance of assigning values to the lengths of sides. Clearly, from the dialogue, the other group members did not perceive the lengths in this way. This excerpt therefore demonstrated an instance where the zone of proximal development was bridged. The other two group members understood the difference in working with concrete values, as opposed to working with abstract variables. They were also able to see the problem from a different perspective.

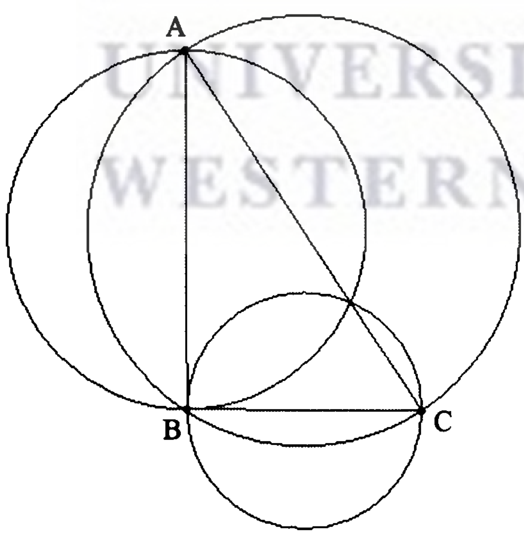
4.2.2 Group 2.

The group consists of two boys and a girl, two average and one above average.

This group labelled the vertices of the triangle A, B and C as indicated in the sketch.

Problem 1

Sketch



The following excerpt illustrates how group 2 initiates the problem-solving process.

Excerpt 6

7. YH: Mamela, jonga, jonga, yima, yibone leweyi ngoluhlobo ngqina lam. Ukusuka apha kule plek uzekanje, one thing neh, uphinde uthathe lena ukusuka apha, kanje okuza apha uthathe ngokuba le yi-part yale. Uyabona ukuba yiweyi epheleleyo then iphinde ibeyi half i-le ibeyihalf yale.
Listen, look, look, wait, see this thing like this, my friend. From here to here, it's one thing hey, and then you then take this, from here, like this to here and take it as a part of this. You see that it's a whole thing and then it's also a half, this becomes the half of this.
8. AN: Uyazelaphi ukuba yi-half?
How do you know that it's a half?
9. NS: // Xauyijongile
When you look at it.
10. YH: // Jonga, jonga le-plek le () like le plek kanje… yabo le-plek le. Then Ungayijonganga la-plek ujonge lena kanje… le inkulu apha on top of… kanje if…
Look, look at this part () like this part like this… you see this part. Then without looking at this part you look at this one like this… this big one here on top of it… like this, if…
11. NS: If the area of the triangle is 36 (reading the question) asinokwazi in a way like sisebenzise la-36 kwinto yethu like to find…
If the area of the triangle is 36, (reading the question) can't we in a way like use that 36 in our thinking like to find…
12. YH: Yabo jonga uphinde nakweli cala, le andiyazi ukuba uzoyiso yifumana njani. It will be like this1, 2, 3 yabo?
You see, look, you do the same for this side, I don't know how you are going to find this one. It will be like this, 1, 2, 3, you see?
13. NS: Uthini, YH?
What are you saying YM?
14. YH: **If it's like this.**
15. NS: Oh 1, 2, 3 ibeyi- $\frac{1}{3}$.
Oh 1, 2, 3 and then it becomes $\frac{1}{3}$.
16. YH: **It's $\frac{1}{3}$ and this one is $\frac{1}{4}$.**

Context of the excerpt.

This excerpt was taken from very early in the session. The discussion started with YH saying that the shaded area of the middle-sized circle (with diameter AB) is half of the area of the semicircle (turn 7) and he explained that this was so because it looked like half of the semi-circle (turn 9). He then looked at the smallest circle (with diameter BC) and reduced the area of the shaded area to a fraction of the circle as well.

He argued that the three regions in the smallest circle could be divided in the ratio 1:2:3 with the unshaded semicircle 3 parts, the shaded region 2 parts and the unshaded section of that semicircle equal to 1 part. The shaded region would then be a third of the area of the small circle and the shaded area of the other circle would be a quarter of that circle (turn 16). This is considered a bridging episode. The shaded area (of each circle) is considered a fraction of that whole circle and therefore its area can be calculated if the area of the circle is known. This paved the way for an approach to solve the problem.

In the next part of the discussion the group used the information that the area of the triangle was 36 cm^2 and used variables to determine an expression for the height (h or AB) in terms of b, $h = \frac{72}{b}$. The radius of that circle was thus $r = \frac{72}{2b}$. Before they agreed on this expression for r, there was a discussion on how to divide $\frac{72}{b}$ by 2, because YH did the calculation differently. This indicated a resistance. This occurred because two different answers were obtained by the group from the same transcription move. YH's method to calculate the radius was as follows:

$$\frac{\frac{72}{b}}{2} = 72 \div \frac{b}{2}$$

$$= 72 \times \frac{2}{b}$$

$$= \frac{144}{b}$$

The group managed to show YH the correct way of dividing by fractions, thus overcoming the resistance.

Excerpt 7 illustrated the filling episode when the areas of the circles were calculated using the expressions found. In this episode the group attempted to find the area of the circle in terms of b (turn 115). This was a filling move. Their reasoning was that since the radius of the circle is $\frac{72}{2b}$, the area of the circle could be found in terms of b. The group also assumed, by looking at the diagram, as had group 1, that the area of the shaded part of that circle was a quarter of the area of the whole circle by looking at the diagram. However, the answer that they got for this area ($\frac{5184\pi}{4b^2}$) - which is correct - did not seem right to them and this was an episode of resistance for the group.

Excerpt 7

117. YH: **Find the area in terms of b and we are not dealing with the circle. We are not dealing with the whole circle, we are dealing with i-quarter**
118. NS: Sidilisha with i-quarter yayo. Izobangu-quarter times
We are dealing with a quarter, it will be quarter multiply
119. YH: Ngubani? Yi-quarter le?
What is it? Is this a quarter?
120. NS: Yi-quarter. Ewe…
Yes, it is a quarter…
121. YH: **So you can find the answer in terms of b?...**
122. NS: Ok, so uthini ke?
Ok. So what are you saying then?
123. YH: **Find the answer in terms of b**
124. NS: In terms of b. Yibani niqhobekeka kaloku.
In terms of b. Carry on so long.
125. AN: Yintoni eyenzwayo ngoku?
What are we doing now?
126. YH: **72^2, $4b^2$, 5^2**
127. NS: Ye bethuna masibaleni lento.
People, let's calculate this.
128. YH: **Where did you get the h?**
129. NS: Uh. What did I do? Oh, sorry. Uxolweni bethuna ndim lo wenzenje.I'm supposed to substitute this? Yilena mos neh?
Uh, what did I do? Sorry people, I'm the one who's done this. I'm supposed to substitute this? It's this one, isn't it?
130. YH: Ja, then u-finde i-square root.
Yes, and then you find the square root.
131. NS: Ndithini YH, ndithi 72^2 okanye ndithi 72×2?
What should I say YH, should I say 72^2 or say 72×2
132. YH: Kodwa kukho lo-*b*, ayuzukwazi ukwenzeka.
But there is this *b*, it cannot happen.
133. NS//: Izokwazi.
It will.
134. AN//: If besizazi_ezi values.
If we knew the values.
135. YH: U-pi. U-pi mos ngu-22 over 7.Apha zizokuyibeka in terms of pi. So sizothi one...
Pi. Pi is 22 over 7. Here we are going to put it in terms of pi, so you will say one…
136. NS:// Ewe.
Yes.
137. YH: **over 4**
138. NS: YH, ndithi 72×2 okanye ndithi 72^2?
YH, should I say 72×2 or say 72^2?
139. YH: **72^2 over 4**
140. NS: Yho! Umbonile u-72^2 ukubangubani?
Yho![exclamation]. Did you see what 72^2 is?
141. AN: **5000**
142. YH: **5000? You must be kidding me?**

143. NS: Nantsike bethuna ingxubakaxaka!
Here is a serious problem guys!
144. AN:// I-area yeziweyi.
The area of these things.
145. NS:// Uzothi 5184 over $4b^2$, [times] π? Nants ibethuna yi-ntoni ingxaki.
You will say 5184 over $4b^2$, [times] π? Here it is guys, what is the problem?

NS, who was busy doing the calculations, was not happy with the answer she was getting and apologized for 'doing nonsense'. She checked three times with YH (turns 129, 131, 138) whether they were using the correct algorithm for finding the area of the circle, "should I say 72^2 or say 72×2". The answer that was obtained by doing the transcription move – finding the area by using the formula – was not what they thought it should be. It was thought to be too big a number to be correct. The reason for the confusion was the incorrect method that YH had used when dividing by fractions. Because the square of 72 was such a big number, 2 times 72 seemed plausible. The group agreed on the square of 72 and accepted the area as calculated at that stage. However, later in the discussion, the focus returned to this calculation (turns 248 - 260) when the area was calculated and verified again.

At this stage the discussion was around finding the actual value of the area of this circle. The group was reasoning that they needed the value of *b* for this.

Excerpt 8

156. YH: **How can we use that value? Oh, here's the value of b. Here is the value of b.**
157. NS: Sizosubstitutha lento phakula lantuza?
Are we going to substitute that thing there in that thing?
158. YH: Iphinda izaninye i-variable u-h.
It gives us another variable h.
159. AN: Iphinde i-substitutha i-value of b, iphinde i-substitutha i-value of b.
And then again substitute the value of b, and then again substitute the value of b.
160. NS: **Yho!**
161. YH: **How can we find the value of b? We use that 36.**
162. NS: Simthini u-36?
What should we do with 36?
163. YH: **Oh, I don't know.**

The group wanted the actual area for the circle with AB as diameter because the shaded area of that circle, according to their understanding, was a quarter of the area. They thus needed to find a value for b. This was a filling move because they were using what they had calculated before in order to determine a solution to the problem. They suggested using the expression for b which had previously been calculated ($b = \frac{72}{b}$) by substituting it into the expression for the area but realized that this will give them another variable h (turn 158). If they then substituted the expression for h, they would have an expression in b again. This was an occurrence of resistance. They did not know how to proceed and their transcription moves did not allow them to see a possible solution. An accommodation to this resistance was to look at the information given in the problem and to suggest that the 36 cm^2 should be used (turn 161). However, another instance of resistance occurred since they had no idea how to use the 36 in order to find an actual value for b.

A strategy employed by NS to overcome this resistance was to reflect on what they had done thus far and whether there were not mistakes in their calculations. She questioned again whether they should have squared 72 or should have multiplied by 2 (turn 174). Again, she was assured by the other two group members that they were correct in squaring. She did this again (turns 248-260) where she verified that they had calculated the area of the middle circle (with diameter AB) correctly to be equal to $\frac{5184\pi}{4b^2}$.

Another bridgehead is established when YH inquired about the relationship between the two circles that had shaded parts (turn 207). When NS replied that she could not see any link, besides the fact that the diameters formed part of the same triangle, they abandoned this train of thought. They failed to see that the outer semicircles of the two circles which contain the shaded parts and the triangle as well as the outer semicircle of the biggest triangle make up the whole figure. This recognition would have possibly led them to a solution to the problem. However, they stated that "[t]here is no use to calculate the area of this one if we are not going to get it"(turn 211). They wanted to calculate the actual value of the area and did not consider working with variables.

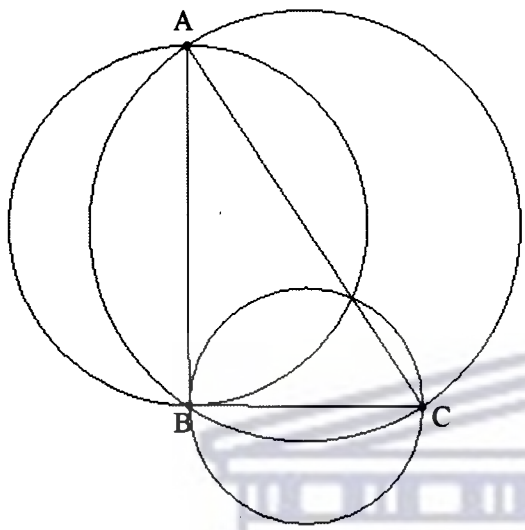
Excerpt 9

207. YH: Yintoni elinkisha i-area yalena neyalena...?
What is the link between the area of this one and that one...?
208. NS: Yintoni into elinkisha i-area yale?
What is the link between the area of this one...?
209. YH: neyale nale?
and of this one?
210. NS: Andiyiboni mna ngaphandle kokuba yitriangle eyi-one le.
I don't see it except for the fact that it's the same triangle.
211.YH: There is no use ukuba, there is no use ukuba si-calculate i-area yale ukuba ayizophuma.
It is no use. There is no use to calculate the area of this one if we not going to get it.
212. NS: Eyalena?
Of this one?
213. YH: Asikabinaye u-h, masilindeni u-h.
We don't have h yet, let's wait till we get h.

From their reasoning at the beginning, the group believed that the area of the shaded part of the small triangle was a third of the area of the small triangle. They thus wanted to calculate a third of the area of the small circle, in order to find the shaded region of that circle. However, they became confused at this stage because they had calculated a third of the middle circle. Furthermore, YH still had problems with the transcription move of multiplying and dividing with fractions. They agree that the shaded area was a third of the circle and yet they divided by $\frac{1}{3}$ instead of multiplying by $\frac{1}{3}$. As a result of time spent on the discussion to clarify this confusion, this group unfortunately did not arrive at a solution to the problem.

Solution to the problem.

A possible solution using the group's starting point is as follows:



Diameter of the middle circle: Let AB = h

Area of triangle: $\frac{1}{2}bh = 36\ cm^2$

$$bh = 72$$

$$h = \frac{72}{b}$$

Radius is thus $\frac{h}{2}$ which is equal to $\frac{72}{2b}$

and BC = b

Area of the middle circle (diameter AB) = $\pi(\frac{72}{2b})^2$

$$= \frac{5184\pi}{4b^2}$$

Area of the smallest circle (diameter BC) $= \pi r^2$

$$= \pi(\frac{b}{2})^2$$

$$= \frac{b^2\pi}{4}$$

Length of the hypotenuse: $AC^2 = h^2 + b^2$

$$AC = \sqrt{h^2 + b^2}$$

Radius of big circle (diameter AC) = $\frac{\sqrt{h^2+b^2}}{2}$

Area of big circle (diameter AC) = $\pi(\frac{\sqrt{h^2+b^2}}{2})^2$

$$= \frac{(h^2+b^2)\pi}{4}$$

Area of the shaded parts:

area of semicircle with diameter AB + area of semicircle with diameter BC + area of triangle ABC – area of semicircle AC

$$= \frac{5184\pi}{8b^2} + \frac{b^2\pi}{8} + 36 - \frac{(h^2+b^2)\pi}{8}$$

$$= \frac{5184\pi + b^4\pi + 288b^2 - h^2b^2\pi - b^4\pi}{8b^2}$$

$$= \frac{5184\pi + 288b^2 - h^2b^2\pi}{8b^2}$$

$$= \frac{5184\pi + 288b^2 - h^2b^2\pi}{8b^2}$$

Substitute $h = \frac{72}{b}$ in equation

$$= \frac{5184\pi + 288b^2 - (\frac{72}{b})^2 b^2\pi}{8b^2}$$

$$= \frac{5184\pi + 288b^2 - 5184\pi}{8b^2}$$

$$= \frac{288b^2}{8b^2}$$

$= 36$ cm^2

Discussion.

The process of the conceptual practice of this group, in terms of figure 2, can be summarised as follows:

Tx_1: The construction of a bridgehead was established when the group deduced that the area of the smallest circle was divided in the ratio 1:2:3, where the unshaded semi-circle was 3 parts, the shaded section was 2 parts and the unshaded section of the semi-circle was 1 part, i.e. the shaded section was $\frac{1}{3}$ of the semi-circle.

Tx_2: There was resistance because they could not find actual values for the areas of the circles.

Tx_3: Accommodation to this resistance was to try and find the lengths of the sides of the triangle.

This was considered as another bridgehead. The group used variables in order to find the lengths of the sides of the triangle. Disciplinary agency was at work when the group calculates the values of *b* and *h* in terms of variables by using the formula of the area of a circle. Resistance occurred, on the one hand, because they could not find the actual lengths of *b* and *h*. On the other hand, there was resistance because of how the calculations were done. YH did his calculation as follows:

$$r = \frac{\frac{72}{b}}{2}$$

$$= \frac{72}{1} \times \frac{2}{b}$$

$$= \frac{144}{b}$$

The other group members did the calculation as follows:

$$r = \frac{\frac{72}{b}}{2}$$

$$= \frac{72}{b} \times \frac{1}{2}$$

$$= \frac{72}{2b}$$

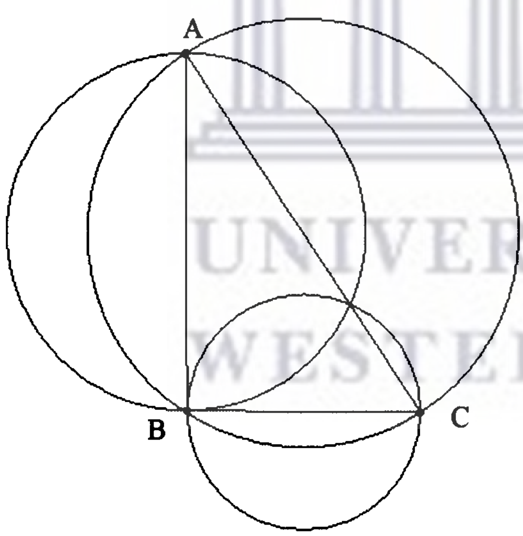
The group settled for the latter answer - thus accommodation to the resistance.

A filling move was where the group suggested that this expression for the radius should be used to find the area of the circle in terms of b. The argument was that since the shaded area is $\frac{1}{4}$ of the area of the circle, one could thus determine the shaded area of the middle circle in terms of b. The transcription move to find the area led to resistance because they could not believe the answer and thought that it was too big a number for the area.

Accommodation to this resistance was to reconsider YH's method of calculating the radius. This did not help either and in the end no accommodation to the resistance was achieved.

4.2.3 Group 3.

This group consisted of three girls, one scoring above average and the other two scoring average marks in assessments.



Excerpt 10

12. AM: So kengoku, ne, le i-area … funeka si-finde onke la ama-side.
So now, this area…we must find all the sides.
13. QM: Masifune onke lama-sides.
Let us find all these sides.
14. AD: Mos ukuba singa-finda lama-sides then izokuba lula ukuba kengoku si-finde eziya .
If we are able to find these sides (of the triangle), then it would be easy to find these ones (the area of the circles).
15. AM:// **How?**
16. QM:// **How?**
17. AM: Andithi for i-diametre yethu… wenza lanto ithi pi r squared?….I-area.
Don't you use the pi r squared method for the diameter? The area.
18. QM: Asina radius!!
We don't have the radius.
19. AD: Asinakwazi ukuthi…(interrupted)
Can't we…
20. AM: Kaloku i-radius uba sifumene eli-side and then sili-divide nge-half izokuba yinantsika i-radius yalena //
Listen we will get the radius if we find this side and divide it in half. That will be the radius of this one.
21. QM: Sizokuyifumana njani kengoku eyeli-side?
How are we going to get it for this side.
22. AM: Sinikwe i-area qha apha yalenantsika…yale triangle or right angled.
We are only given the area here for this thing…for this triangle or right angled.

Context of the excerpt.

This episode occurred early in the session. The group had just read the problem to try to make sense of the problem. They clearly (from their utterances) did not have a ready method to solve the problem.

In turns 12 and 13, AM and QM realised that they needed to find the lengths of the sides of the right-angled triangle as a starting point. This would help them find the areas of the circle: "If we are able to find these sides, then it will be easy to find these areas"(turn 14). This was considered a bridgehead. It gave direction to the problem-solving process and it was the starting point for the group. The lengths of the sides of the triangle would be halved to give the radii of the respective circles. They also realised that they had to somehow use the given area to find the lengths of the sides, but they did not know how to make use of this information to find the lengths of the sides.

Later on in the discussion, the group identified on the sketch which side of the right-angled triangle they would regard as the height and which side would be the base, in order to help them find the lengths of the sides, using the given area. They also (similar to group 2) labelled the vertices of the triangle A, B and C.

The way this group tried to solve the problem was to firstly to note that $\frac{1}{2}$ base x height = 36

so
$$\frac{1}{2}b.h = 36$$

$$bh = 72$$

$$b = \frac{72}{h} \text{ and } h = \frac{72}{b}$$

Thereafter they substituted $b = \frac{72}{h}$ into the formula $A = \frac{1}{2}bh$:

$$36 = \frac{1}{2}\left(\frac{72}{h}\right)h$$

$$72 = \left(\frac{72}{h}\right)h$$

This was a transcription move. However, the group made a mistake and they simplified:

$$\left(\frac{72}{h}\right)h = \frac{72}{h^2}$$

They thus have
$$72 = \frac{72}{h^2}$$

$$72h^2 = 72$$

$$h^2 = 1$$

$$h = 1$$

They thereafter substituted these values that had been calculated, 1 for the base and 72 for the height. Using these values the radius of the middle circle would be 36. They then calculated the area of a circle with radius 36 which was 4071,5. Given the scale of the drawing (the fact that the area of the triangle was 36), this value was not plausible. This thus constituted a resistance. It did not make sense for the values for the base and height to be 72 and 1 respectively. The difference between the areas of the two circles was too big.

Excerpt 11

184. AM: Ayenzi-sense, i-area engaka nyhani?
This does not make sense, an area this big?
185. QM: I-area engaka nyhani, ibizoba worse ukuba ibingu 72. Ok, bhala.
The area is so big. It is going to worse when it is 72. Ok, write.
186. AM: Hayi, masifakeni u-1 apha.
No, let's put 1 here.
187. QM: Hayi bethunana, uyaqonda ukuba izoba ngakanani?
No people, do you know how much it is going to be?
(all laughing)
188. QM: Mmmh,ayikho possible le nto.
This is not possible.
190. AD: Hayi ayikho-possible loo nto. Masizameni elinye icebo.
No, that is not possible. Let's try another method.
191. QM: Hayi bethuna, sense njani? Izoba ngu-36 ne?
No people, how are we going to do this? It's going to be 36 hey?
192. AM: Hayi andiqondi ukuba yi-formula e-right lena
No, I don't think this is the right formula

The way they tried to overcome this resistance was to redo their calculation and they still arrived at the same answer. In order to justify this answer they argued that the drawing was not to scale and that these values for the areas could possibly be acceptable. They did not consider using other values because they calculated the value of h (albeit wrongly) to be 1. After giving it some thought, the group agreed that the values used could not be correct. They had to find another method to solve the problem. This group could not come to a meaningful shared understanding of how to proceed in finding a solution to the problem.

Their use of actual values for the sides was similar to the approach by group 1. If they had therefore investigated other values as well, they could have recognised a pattern and then made generalizations.

The group also attempted to use the trigonometric definitions (another bridgehead) to find the lengths of the sides of the right-angled triangle. This also constituted an episode of resistance because they did not have the sizes of the angles and could therefore not calculate the sides. One of the members wanted to assume that it was an isosceles triangle and that the angles were 45 degrees each. This method was abandoned.

This group also assumed, as had the other groups, that the shaded part of the middle circle was half of the area of the semi-circle and if the area of the semi-circle was found, then the shaded area can be found. They could not follow through on this idea because they failed to find the lengths of the sides.

The solution based on their starting point of writing b in terms of h would be similar to that of group 2. If they had pursued the notion of working with numbers, the solution would be similar to that discussed under group 1.

Discussion.

The interpretation of the group's conceptual practice based on figure 2 was as follows:

Tx_1: A bridgehead was established when the group decided to calculate the lengths of the sides of the triangle.

Tx_2: They then calculated b in terms of h, and h in terms of b using transcription. The group incorrectly calculated that the value of $h = 1$.

Tx_3: Using this value for h, resistance occurred when the group calculated the area of the circle. The value was too large to be plausible.

Accommodation to this resistance was to consider trigonometric ratios. This method was abandoned because of insufficient information.

Another attempt at overcoming the resistance was to simplify matters by assuming that the right-angled triangle was isosceles. This method was also abandoned because group members pointed out that the triangle was not isosceles.

This group could not find accommodations to the resistances and did not arrive at a solution to the problem.

4.3 Analysis of problem 2

This problem required construction of geometrical figures, either manually or by using the *Geometer's Sketchpad*. The three groups followed similar methods in solving the problem. The discussion of the solutions, as well as the use of tools and inscriptions, would be a combined discussion because of these similarities.

4.3.1 Group 1.

The group consisted of the same members as for problem 1. They were very light-hearted while doing the problem and made jokes with lots of laughter as they proceeded with the problem.

Group 1 started by members each constructing their own triangle manually by following the instructions in the problem. These were all transcription moves. They later switched by looking at one person's construction in order to answer the question posed. This triangle constructed happened to be an equilateral triangle and they admitted to having chosen an easy one. The squares constructed on the sides of the triangle were therefore all equal. In order to find the area of the original triangle, they used the formula $A=\frac{1}{2}$ base x height. They therefore had to determine the height of this triangle. They had a choice of measuring the height or to calculate the height using the theorem of Pythagoras. They chose using the theorem of Pythagoras because "[they] need[ed] to be accurate" (turn 151). Each side was 4 cm long, the height was thus $\sqrt{12}$ cm and the area 6,9282 cm^2, which the group rounded off to 7 cm^2. These were all transcription moves.

A bridgehead was established (see excerpt 12) when SM suggested that any of the newly formed triangles can be taken at random (since they were all congruent) and its area calculated (turn 267). This area would then be compared with the original triangle and the ratio would thus be determined. He also suggested that the areas of the other triangles formed could then be deduced from this ratio. ZM suggested that they needed to draw additional constructions in order to verify their conclusions and the other group member, NB, recommended that they do the other construction using the software. This could be regarded as a bridgehead since it paved the way to a solution to the problem. This also indicated that they knew that only one example would not be enough to generalise.

Excerpt 12

267. SM: **If we could then take these other triangles, choose any triangle to those newly formed ones at random and calculate it's area and compare it's area with the area of this one and then use the ratio to determine the other triangles' areas.**
268. ZM: **Find the ratios.**
269. SM: **But then we will then compare what's the relationship between the squares and the triangles formed.**
270. NB: Yeah, maan.
Yes, man.
271. SM: And our problem will be finished. No problem and then we can talk.
272. ZM: **We will do another thing again.**
273. SM: Ja, ja, we draw another thing to prove yabo ifacts ukuba ok.
Yes, yes, we will draw another thing to prove that our facts are ok.
274. NB: Ja, we can go to la machine.
Yes, we can go to that machine.
275. SM: Ja, kwicomputer kengoku.
Yes, to the computer then.

In order to determine the area of one of the newly-formed triangles, they used exactly the same method as for the original triangle, $A = \frac{1}{2}$ base x height. They measured the unknown side with a ruler and found it to be equal to 7 cm. They also measured the height and they found it to be equal to 2 cm. The area of that triangle was therefore also equal to 7 cm^2.

Thereafter, they did the construction using *Geometer's Sketchpad* and they did exactly the same construction as they had done manually, in order to be "sure of this one of ours first" (turn 409). The measurements provided by *Geometer's Sketchpad* corresponded to their manual calculations. Unfortunately, they were not sufficiently competent to use the feature of Geometer's Sketchpad for calculating areas of triangles and resorted to do manual calculations to find these. They came to the same conclusion that the areas of the triangles were equal and that the ratio of the newly formed triangles to the original triangle was 1:1.

They then tried the suggestion of NB, who said that if squares are formed using the new triangle's sides (7 cm), then the relationship between the squares could be obtained and that could be the same as the relationship between the triangles. They found this ratio to be 16: 49 and then they abandoned this method. Their conclusion was that the ratio of the area of the original triangle and the areas of the newly formed triangles is 1:1.

This group looked at only one example and based their conclusion on this one example. Although they had said at the beginning that they needed to draw more triangles in order to verify their conjecture, they never got round to doing that. What the use of the software programme could offer them was also not capitalised. Instead of doing more constructions, which would have been done much quickly on the computer, they only did the same construction using the computer.

In addition, the group did not look at algebraic manipulations in order to solve the problem and used concrete values and lengths throughout. Also, they did not look at other ways of determining the area of a triangle to verify that they have done the calculations correctly.

The frivolousness of the group during this session could have hampered the looking at other methods and alternative calculations. The group members had been distracted by small talk and jokes.

4.3.2 Group 2.

One of the members of this group was replaced.

NS established a bridgehead when she suggested that all of them draw different triangles so that they could see the relationship in all the triangles (turn 1). The method suggested implied that they would find the respective areas of their triangles in their constructions and then compare each one with the original triangle and see if a pattern emerged. They did this manually by using mathematical instruments. They spent a good portion of the session doing the construction and making sure that the construction was correct.

After they had completed their construction they measured first the sizes of the angles of the squares to see if they had been drawn correctly, and secondly, the lengths of the sides to verify that these were equal. These were transcription moves, based on their knowledge of the properties of squares.

After verifying that her sketch was in order, NS measured the sides of her triangle. The measurements are 35 mm, 49 mm and 35 mm. All of them labelled their sketches with the lengths of the sides that they have measured. In addition, they measured the angles of the triangles and wrote down these values on the sketch. They then verified that the angles for each triangle added up to 180 degrees.

They looked at the lengths of the sides only and applied the formula $A = \frac{1}{2}$ base x height to all of the triangles in order to find the area, even the triangles which were not right-angled triangles. This caused an episode of resistance because they had expected the areas of the triangles to be equal after the first area that they calculated was found to be equal to the original one. Fortunately, they realised their mistake and calculated the areas of the remaining two triangles using the area rule. However, even these calculations did not render equal areas and this also constituted a resistance. They compromised to conclude that since the areas are almost equal, the ratio was 1:1.

Thereafter, they worked on the computer programme to verify their conjecture.

In all groups there was a great deal of irrelevant small talk and this distracted the members of the groups from the issues under discussion.

Solutions to the problem.

The solution to this problem could have been obtained using an inductive or a deductive approach. All three groups started with constructing the required sketch manually and then proceeded to manually measure the lengths of the sides and the sizes of the angles as constructed. Each of these unique sketches would have produced different measurements. The areas of the original triangle and the newly-formed triangles were then calculated and the areas then compared. It should have been concluded that the areas were equal and that the ratio is 1:1. This deduction could have been made only if enough examples had been constructed to make generalizations.

The following algebraic method could have been used to determine the solution:

In original ABC: Let AB = x, AC = y and BC = z

The area of ABC can be written as: $A = \frac{1}{2} xy \sin A$

or $A = \frac{1}{2} xz \sin B$

or $A = \frac{1}{2} yz \sin C$

The square on AB would each have side of length x, the square on AC, a side of length y and the square on BC a side of length z.

The areas of the newly-formed triangles could be calculated as follows:

Area triangle 1 = $\frac{1}{2}$ xy sin (180 – A)

= $\frac{1}{2}$ xy sin A (which is equal to the area of the original triangle)

Area triangle 2 = $\frac{1}{2}$ xz sin (180 – B)

= $\frac{1}{2}$xz sin B (equal to the area of the original triangle)

Area triangle 3 = $\frac{1}{2}$ yz sin (180 – C)

= $\frac{1}{2}$yz sin C (equal to the area of the original triangle)

This clearly shows that the area of the original triangle was equal to the area of each newly formed triangle and the ratio was thus 1:1.

The key to solving this problem was the recognition that each new triangle had an angle that was supplementary to an angle in the original triangle. If the area rule were used, the sine of an angle (A) and the sine of its supplementary angle (180 - A) would be equal and thus the areas of the two triangles would be equal.

Discussion.

In all the groups there was a pre-occupation with solving the problem numerically. Group 1 started by assigning numerical values to the sides of the triangle. Group 2 and 3 started with variables but their ultimate aim was to find numerical values for those variables. This preoccupation with numbers resulted in the groups using just one or two examples and they made generalizations based on these cases. In addition, the groups did not consider any other possible ways of doing the problem.

None of the groups attempted to state the general case for this problem. During my observations I concluded that the groups thought it sufficient to use one or two examples and that generalizations could then be made based on those results.

Although all three groups arrived at the correct answer, they based their answer on insufficient information or as in the case of group 3, on incorrect answers.

Group 3 had different answers for the areas of the triangles but since they were "almost equal" they concluded that the ratio was 1:1.

The analysis of the group work by the different groups across the two problems was done using the analytical framework developed by Pickering. It was found that the conceptual practice demonstrated by the groups while doing group work fitted well into the framework. Bridging, transcription and filling moves could clearly be identified. These illustrated the cases where human agency was active and where disciplinary agency was active and how these synergised to create the 'dance of agency'.

The analysis should now be linked to the research questions regarding the appropriation of tools and inscriptions, how learners interact with each other and the subject matter and the structure of the problem-solving process while doing problem-solving.

CHAPTER 5

DISCUSSION AND CONCLUSION

5.1 Introduction

The aim of this study was to investigate how learners appropriate physical and intellectual tools while doing Geometry problem-solving in small groups, as well as what learners do when they attempt problem-solving. The analysis focused on what three groups of learners did while solving two geometrical problems by analysing the conversations, their use of tools and their interaction with one another and with the problems. This study gave insight into the practice of the groups that took place during problem-solving and how collaborating in groups provided opportunity or not for learners to undergo the whole process of problem-solving as practitioners of Mathematics. In both problems and across all groups one could identify the conceptual practice based on the model outlined by Pickering (1995) as detailed in the previous chapter.

Some general comments about observations made during this study are:

- Across all the groups there was a tendency to solve the problems numerically. This hampered finding a general solution. The groups thought that they had to come up with a single solution to the problem. This is exemplified by them just doing one construction in problem 2 and then concluding that the relationship between the original triangle and the constructed triangles was that they have the same area, in **all cases**.
- While doing problem 1, all groups were focused on the problem and all discussions were about the problem. However, when the groups were doing problem 2, they were engaging in small talk and were easily distracted from the problem. The reasons for this could be that the teacher was not in the venue all the time. Each group was recorded in a different venue to reduce the noise of the other groups and the teacher moved from group to group to observe. Another reason could be that they were being recorded by a fellow learner.
- In all cases the learners made use of code-switching where they mixed English mathematical terms with their home language isiXhosa. When conversing in isiXhosa, they would also make use of slang. The mix of Xhosa with English and Afrikaans is the way they communicate with one another during informal conversations.

- What is worth noting is the respectful manner in which learners addressed their fellow group members, especially the males addressing other males.

In terms of the focus of this research, the use of inscriptions and tools by the groups while solving the problems and how these assisted (or not) with the discussions shall now be discussed.

5.2 The use of inscriptions

Inscriptions are also called "devices for seeing" (Suchman and Trigg, 1993, p. 145). Major importance is placed on inscriptions in ethnomethodological studies because "the work of scientific inquiry comprises an emergent interaction between scientists and their materials" (Suchman and Trigg, 1993, p. 146). Inscriptions play a mediating role in externalizing and clarifying ideas.

In group 1, the diagram given in the problem 1 was used to name the circles to make it easier to identify. This helped in the discussions because constant reference was made to the circles in terms of the new notations. All groups labelled the sides of the triangle in their own way. Therefore, the sketch played a major role in the discussions, since the lengths of the sides and the diameters and radii of the circles were always referred to. Throughout the discussions all group members were intently focused on the diagram and members were always pointing to the diagram to clarify their statements. Gesturing, by pointing to the circles, lines, triangles and shaded areas, played a major part in the discussions and the different calculations.

The diagram was also used to clarify some misconceptions, for example, when ZM was confused about the radius and the diameter in the sketch, the other group members used the diagram to clarify this issue. In the post discussion, when ZM was asked about this, he indicated that the terminology had confused him.

In addition, the diagram negatively influenced the problem-solving process. It appeared on the diagram (problem 1) as if the diameter of the middle circle was twice the size of the diameter of the smallest circle in the diagram. The learners were given the area of the triangle as 36 cm^2 and therefore, when the groups assigned values to the sides by trial and error, they automatically assumed that one side had to be 12 cm and the other 6 cm. Learners made exclusive use of the diagram to justify and verify their choice. Group 1 did not consider, in their verbalizations, any other possible lengths for the sides.

The diagram confirmed their choice because XY looked as if it were double the length of YZ. Group 3, through incorrect calculations, used the measurements 1 cm and 72 cm. The members ultimately rejected this because the diagram did not support the huge difference between the areas of the circles calculated using these measurements. In addition, group 3 concluded that in the smaller circle, the area of the shaded part was a third of the area of the circle.

The same reasoning was observed when it was assumed that the area of the shaded part of the middle circle B was a quarter of the area of the whole circle. To the learners, the area of the shaded part of the middle B circle looked as if it could be a quarter of the area of that circle. All three groups assumed that the shaded part of the middle-sized circle is one quarter of the area of that circle. They pursued this line of reasoning and this was one of the reasons why the groups could not reach a solution. It would seem that the learners believed that diagrams in Geometry problems were drawn to scale and that they could rely on the naked eye to judge whether the length of a side was half of the other one or whether the area of one region was a particular fraction of another area.

The diagram as given in problem 1 was thus used as an inscription in the following ways in order to do problem-solving:

- Labelling of circles and sides
- Pointing to the diagram when referring to any part thereof
- Points of reference: making sure that everyone was clear on what was referred to on the diagram
- Justification of choice of values for sides of triangle and areas of circles

Another way inscriptions were used by group 1 was to draw another right-angled triangle in the biggest circle adjacent to the given one in order to form a rectangle. They thus added lines to the diagram in their attempt to reach a solution. NB said that his diagram was all messed up, because he added construction lines to his diagram (see figure 3).

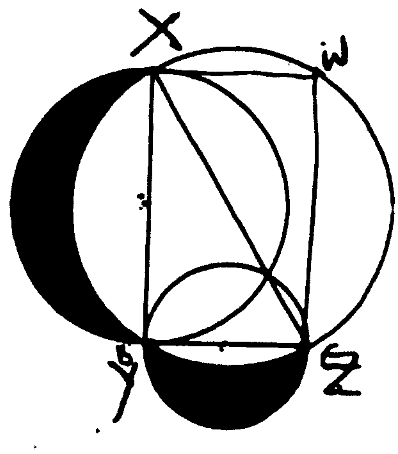


Figure 3: An example of inscription

A third way in which inscriptions were used by groups was to write down their calculations so that the rest of the group could follow their reasoning. When group 2 members explained to YH the correct way to divide fractions, they did these calculations on paper to emphasize the difference between the two methods. In addition, all the groups wrote down some of the calculations they had used to calculate the area of the circles and the length of the hypotenuse of the triangle.

For the second problem, the groups had to construct a geometrical figure by following the information given in the problem. All groups initially did the construction manually. Their calculations were then based on their constructions. These constructions thus formed the focal point of their discussions. They were required to verify that the figures were squares, measure the lengths of the sides, measure the angles and calculate the areas of the triangles. Therefore, extensive use of the diagram was required. For an example see figure 4.

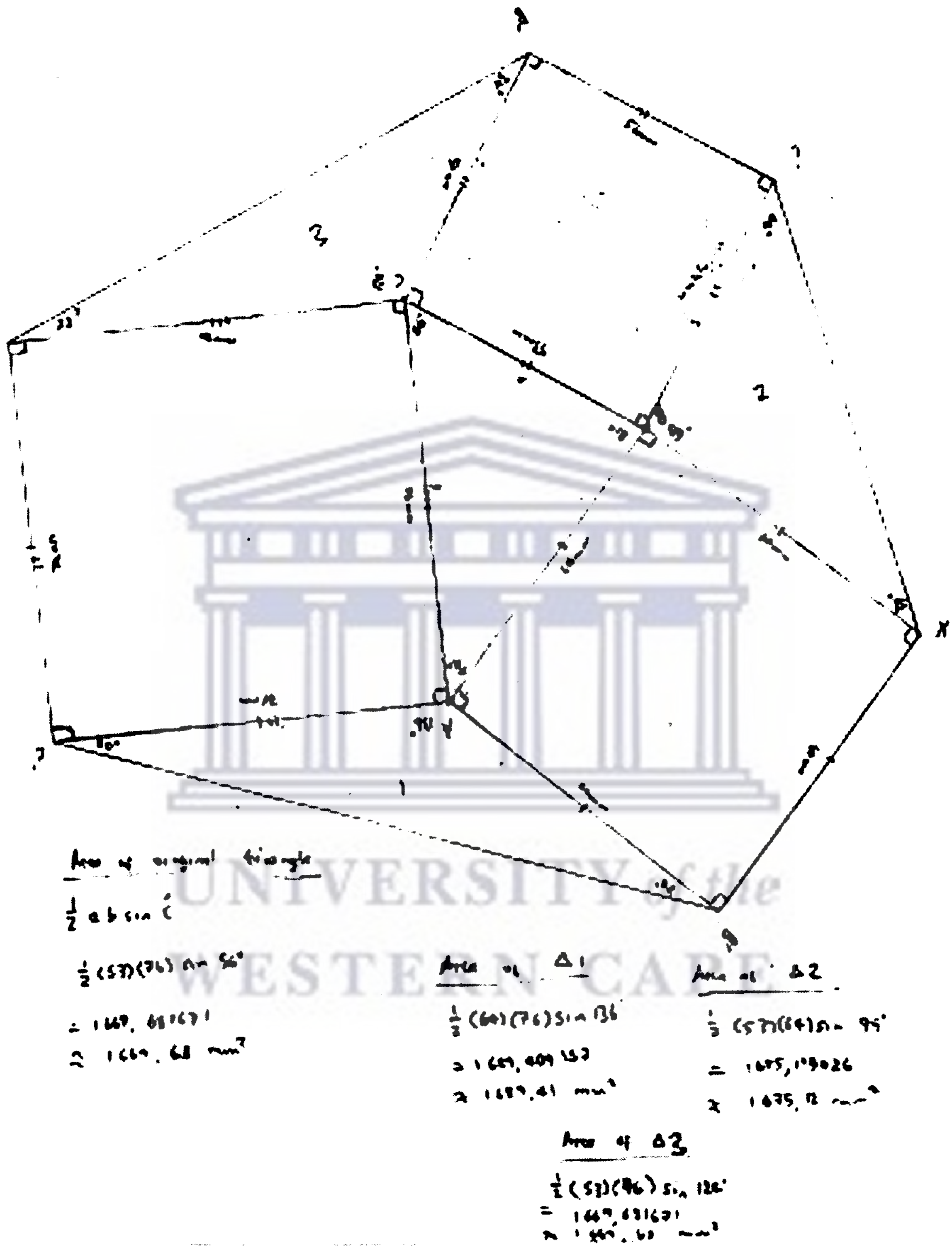


Figure 4: Example of written work

Furthermore, all the groups wrote down their calculations and conclusions for problem 2.

5.3 The use of tools

The calculator was used by all groups for calculating the area of the circle and the length of the hypotenuse in problem 1 and to calculate the areas of the triangles in problem 2.

When group 1 had to calculate the length of the hypotenuse, assuming that the other two sides were 12 and 6 respectively, they initially did the calculations incorrectly without a calculator. When they added 124 (this was incorrect) and 36, the answer obtained was 150. From the utterances it is clear that they added this mentally. "124, it's 130, it's 140, isn't it?" (143). First the 6 was added and thereafter groups of 10 were added, but this method produced an incorrect answer. After using the calculator, they obtained the correct answer.

For problem 2, two of the groups used *Geometer's sketchpad*, the dynamic Geometry software, to do the construction. This allowed them to replicate their manual construction or to make other constructions. Group 1 replicated their manual construction to verify their conclusion. Group 3 did a construction with different side lengths from their manual construction to confirm their conclusion. They were able to use the software efficiently to do the construction and to do the measurements of the lengths of the sides and the sizes of the angles. They were, however, not sufficiently proficient to use the software to determine the areas of the triangle and had to resort to doing those calculations manually. This did not appear to be a problem for the groups.

These two groups did not capitalize on the capacity of the software to vary the lengths of the sides (by using, for example, the animation feature). This would have given them numerous different constructions in a couple of minutes and the measurements for the areas of the triangles would have remained constant, allowing them to make generalizations regarding the relationship between the areas of the triangles. The learners knew how to use the feature but because they thought that one example was enough to be able to make a generalization, they did not think about using it. In retrospect, the wording of the question instructed the learners to start the construction with any triangle and to investigate the relationship between the area of the original triangle and the other angles formed in that construction. Since English was not their mother tongue, this could have been interpreted literally by the learners as having to do only one construction.

Therefore, learners appropriated physical tools as well as intellectual tools while doing Geometry problem-solving in collaborative small groups.

take it very seriously. This explains the playfulness of the groups whilst they were doing the second problem. A way around this dilemma is to make this a regular way of teaching and learning in the classroom, so that learners can come to expect collaborative group work as part of their learning experience.

Another solution would be to satisfy the need of learners to be rewarded in the form of marks by assessing the group activity. However, this opens up the question of how groups are assessed.

This way of allowing learners to experience the problem-solving process in groups is very time consuming, given the large volume of syllabus content to be covered during normal class time. At the school where the study was undertaken the learners have an extra study hour every day. A suggestion could be to have learners do problem-solving, say, once a week after school so that normal class time can be used to cover the syllabus. This should be planned so that all learners are exposed to this collaborative group work problem-solving experience.

What would be done differently if this study is to be replicated was that learners would be given more time to complete both problems. The groups could not complete problem 1, owing to time constraints. The learners had to leave at the end of the school day because of their transport arrangements. Another limitation to the research was that while recording problem 1, two video cameras and 2 voice recorders were used while the groups were all seated in the same room. One group felt that the discussions from the other groups were a disturbance. To minimize noise, the groups should be in different venues and video cameras should be used for all groups. This was done when the groups were recorded for the second problem. The limitation experienced with the second problem was that the researcher could not observe all the groups at the same time.

Although group work is done regularly in the classroom, this was the first time that the teacher (as researcher) would not intervene in the discussions and try to help learners on the right track. This experience was strange because it was difficult not to follow one's instinct to intervene and guide learners. Fortunately, the mantle of teacher could be assumed after the research and input could be given to learners regarding the different strategies that could have been used to solve the problems. The debriefing sessions were important for that purpose and also for contextualising their input in the whole research project.

5.6 Significance of the study

The form of group work done in the study is very different from a typical traditional Geometry lesson which is usually taught by explaining the wording of a geometrical theorem, followed by the proof of the theorem. Learners usually learn these theorems by memorising them. Examples of Geometry problems would then be explained by the teacher and learners would be given exercises where they would have to apply the theorem. Typically the exercises would increase in degree of difficulty with the easier ones at the beginning of the exercise and the more advanced problems towards the end of the exercise. These would also be of the type that learners can expect in an examination. Most traditional Mathematics textbooks also follow this order.

In this study the content knowledge and skills needed to solve the problems were very basic and learners were expected to do the problems without the assistance of the teacher so that one could observe how they would use the tools at their disposal to solve the problems in collaborative groups. The problems were also not the routine problems that one would be able to solve by directly applying a theorem or formula. The collaborative problem-solving skills of the group were put to the test while they were doing these problems. It was interesting to note that the groups collaborated to do the problem in the absence of one clear method. There was no one in any of the groups who had seen the problems before. In addition, no one could see the solution at first glance. In tackling problem 1, for example, the groups tried out different methods (they established bridgeheads, in Pickering's terminology), to see whether any would lead them to a solution.

One might think that while there is a place for the teaching of Geometry in the typical traditional way outlined above, more of this type of collaborative group work should be made possible in the Geometry classroom. The learners would then be able to engage with the materials and to do Mathematics as mathematicians do. They would be exposed to the conceptual practice that mathematicians and other scientists experience, as opposed to being presented with the final product of that conceptual practice. They would be able to experience the whole process of problem-solving and in so doing learn strategies and skills to solve problems which they might encounter for the first time in their examinations and more importantly, they would have problem-solving skills that they could apply to many other aspects of their daily lives.

It is widely accepted that the teaching and assessment that happens in schools are moulded in terms of the types of questions found in external examinations. It is acknowledged that in this study the researcher had the freedom to choose her own problems and those problems were not typical of problems asked in the examinations.

These were questions which could be used as investigations and which could not typically be solved in a short space of time, as could examination-type questions. With the new Curriculum and Assessment Policy Statement (CAPS) teachers are guided in terms of the types of questions that will be examined and those types will be given preference in the teaching of Geometry. The significance of this study was to highlight the practice of collaborative groups and that learners should be given the opportunity to experience the whole process of problem-solving. The strategy would be to look for problems that allow learners to go through the complete process of problem-solving while still doing an examination type problem or to adapt the examination type problems to allow for the problem-solving process to happen.

5.7 Suggestions for further research

One could consider monitoring the practice of learners in collaborative small groups over a longer period of time, say over two years, in order to have a clearer indication of the impact on learning and to assess whether their problem-solving strategies had been refined and enhanced when learners were faced with problems which they did not know how to solve.

The issue of group assessment has been mentioned earlier. How does one ascertain whether each group member has contributed in the discourse, albeit that our unit of analysis is the group? How does one assess the group and then assign individual marks to group members? Does the group get a mark and does this mean that each individual group member gets the same mark or can the group among themselves rate individual group members depending on their contribution? How will this fit in with the CAPS document which only makes provision for individual assessment? This could be an area for further research.

5.8 Concluding remarks

A study of this nature contributed in exposing the practice that happens when learners are engaged in problem-solving in small groups. Group work is important for this kind of interaction to take place.

The study confirmed that learners appropriate the tools at their disposal in the solving of problems in small groups. The groups used intellectual tools (inscriptions, written calculations) and physical tools (calculators, Geometry software) in order to find a solution to the problems. Within their groups there was interaction among the learners while the problem was being solved, and their engagement with the problems resulted in various solution paths. The structure of the conceptual practice in the solving of problems was found to be in accordance with Pickering's theory of the mangle of practice.

In South Africa group work is promoted as a means to improve the quality of learning. This study proposed that collaborative group work should form an integral part of Mathematics-learning in general and Geometry-learning in particular.

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